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NONLINEAR DISPERSIVE EQUATION FOR

ELLIPTICALLY POLARIZED ALFVÉN WAVES

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ABSTRACT

A nonlinear evolution equation for elliptically polarized Alfvén waves in a hot plasma is simply derived using Lagrangian formalism. This vector equation reduces to the derivative nonlinear Schrödinger equation for parallel propagating circularly polarized waves. For sufficiently oblique propagation, it reduces to the KdV equation for the oblique fast Alfvén wave, and to an apparently new evolution equation for obliquely propagating slow Alfvén waves.

derive contains the others as limiting cases.

The equations governing one-dimensional waves propagating along the x-axis in a quasineutral two-fluid plasma with finite ion-inertia dispersion are:

$$\frac{d\rho}{dt} + \rho \frac{\partial V_x}{\partial x} = 0 \quad , \tag{1}$$

$$\frac{dV_x}{dt} + \frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{B_\perp^2}{8\pi} + P\right) \quad , \qquad P\rho^{-\gamma} = \text{const.} \tag{2}$$

$$\frac{d\vec{V}_\perp}{dt} = \frac{B_x}{4\pi\rho}\frac{\partial \vec{B}_\perp}{\partial x} \quad , \tag{3}$$

$$\frac{d\vec{B}_\perp}{dt} = B_x \frac{\partial \vec{V}_\perp}{\partial x} - \vec{B}_\perp \frac{\partial V_x}{\partial x} - \frac{\partial}{\partial x}\left(\frac{cB_x}{4\pi n e}\hat{e}_x \times \frac{\partial \vec{B}_\perp}{\partial x}\right) \quad , \tag{4}$$

where $d/dt = (\partial/\partial t + V_x\, \partial/\partial_x)$, $\vec{B}_\perp = (B_y, B_z)$, $\vec{V}_\perp = (V_y, V_z)$ and ρ, $\vec{V}$, n are the mass density, velocity and particle density. γ is the ratio of specific heats, which is assumed to be the same for electrons and ions. The ion-inertia term in (4) stems from the dispersive Ohm's law. Electron inertia has been neglected since it unnecessarily complicates the algebra without materially changing the results. However, for transverse or almost transverse propagation, finite electron inertia cannot be neglected.

By introducing Lagrange variables (t_o, x_o) (1)-(4) can be reduced to:

$$\frac{\partial^2 U}{\partial \zeta^2} + \frac{\partial^2}{\partial \xi^2}\left(\frac{B_\perp^2}{2} + P_o U^{-\gamma}\right) = 0 \tag{5}$$

and

$$\frac{\partial^2}{\partial \zeta^2}(U\vec{B}_\perp) - \frac{\partial^2 \vec{B}_\perp}{\partial \xi^2} + \lambda\, \frac{\partial^3(\hat{e}_x \times \vec{B}_\perp)}{\partial \xi^2 \partial \zeta} = 0 \quad , \tag{6}$$

where $U = \rho_o/\rho$, $\zeta = V_A t_o \cos\theta$, $\xi = \int dx_o = \rho_o^{-1}\int \rho dx$, $\lambda = V_A/\Omega_i$, $V_A^2 = B_o^2/4\pi\rho_o$, $P_o = C_o^2/(V_A^2 \cos^2\theta)$, $\Omega_i = eB_o/m_i c$, $C_o = [\kappa(T_e + T_i)/m_i]^{1/2}$, $\vec{B}_o = B_o(\cos\theta, \sin\theta, 0)$ and $B_\perp$ is normalized to $B_o \cos\tilde{\theta}$. For $C_o = 0$, these equations reduce to those of Karpman[10] except for the electron inertia term which we have neglected. The coupled set of equations (5) and (6) describes the full nonlinear evolution of obliquely propagating fast, intermediate, and slow waves with ion inertia dispersion. When dispersion is neglected, (5) and (6) describe nonlinear MHD waves. We can recover from (5) and (6) most of the published fluid theory results concerning the weakly nonlinear Alfvén waves in various limiting cases.

To make our point, we look for stationary solutions of (5) and (6) of the form $f(\xi-\zeta, \epsilon\zeta)$; ϵ here indicates a slow time variation. On using the boundary conditions, $U = 1$, $B_y = B_o \sin\theta$, $B_z = \partial U/\partial\eta = \partial\vec{B}_\perp/\partial\eta = 0$ at $\eta \equiv \epsilon(\xi-\zeta) = \infty$, (6) gives

$$U \doteq 1 - (1-\beta)^{-1}\left(\Theta\, \tilde{B}_y - \frac{\tilde{B}_\perp^2}{2}\right) \quad , \tag{7}$$

where $\beta = \gamma C_o^2/(V_A^2 \cos^2\theta)$, $\Theta = B_{yo}/B_x = \tan\theta$ and $\tilde{B}_y = B_y - B_{yo}$, $\tilde{B}_z = B_z$. (The case $\beta = 1$ should be treated separately, since in obtaining (7), we assumed that $|(U-1)| \ll \beta^{-1}$, which is not a good approximation for $\beta = 1$.) By substituting (7) into (6), and taking the slow variations of only the linear terms, we obtain the following 2-vector equation

$$\frac{\partial \vec{b}_\perp}{\partial \tau} + \frac{1}{4(1-\beta)}\frac{\partial}{\partial \eta}\left[\vec{b}_\perp (b_\perp^2 - \Theta^2)\right] + \frac{\lambda}{2}\frac{\partial^2}{\partial \eta^2}(\hat{e}_x \times \vec{b}_\perp) = 0 \quad , \tag{8}$$

where $\vec{b}_\perp = (0, \tilde{B}_y+\theta, \tilde{B}_z)$. Now by defining $b_\pm = b_y \pm i b_z$, (8) can be rewritten as the following two differential equations,

$$\frac{\partial b_\pm}{\partial \zeta} + \frac{1}{4\,(1-\beta)} \frac{\partial}{\partial \eta} \left[|b_\pm|^2 b_\pm - \theta^2 b_\pm \right] \pm \frac{i\lambda}{2} \frac{\partial^2 b_\pm}{\partial \eta^2} = 0 \quad ; \tag{9}$$

which have the form of DNLS equations. For $\theta = 0$, (9) gives the evolution of purely right hand (-) and left hand (+) circularly polarized Alfvén waves[6-9]. For $\theta \neq 0$, each component of (8) resembles a DNLS equation corresponding to nonvanishing boundary conditions (i.e., $B_{\perp\infty} \neq 0$)[8,11] but the physics involved is quite different since (8) is not restricted to circular polarization. To make this point more transparent, we write (5)-(7) in a nonsymmetric form, namely

$$\frac{\partial \tilde{B}_y}{\partial \zeta} + \frac{1}{2\,(1-\beta)} \frac{\partial}{\partial \eta} \left[\tilde{B}_y \left(\theta^2 + \frac{3}{2}\,\theta \tilde{B}_y + \frac{1}{2} \tilde{B}_\perp^2 \right) + \frac{\theta}{2} \tilde{B}_z^2 \right]$$

$$- \frac{\lambda}{2} \frac{\partial^2 \tilde{B}_z}{\partial \eta^2} = 0 \tag{10}$$

and

$$\frac{\partial \tilde{B}_z}{\partial \zeta} + \frac{1}{2\,(1-\beta)} \frac{\partial}{\partial \eta} \left[\tilde{B}_z \left(\tilde{B}_z \left(\theta \tilde{B}_y + \frac{1}{2} \tilde{B}_\perp^2 \right) \right) \right] + \frac{\lambda}{2} \frac{\partial^2 \tilde{B}_y}{\partial \eta^2} = 0 \quad . \tag{11}$$

For $\theta \gg \tilde{B}_y$, $\tilde{B}_z$, according to (10), $\partial/\partial\zeta \sim (\theta^2/2)\,(1-\beta)^{-1}$, which when substituted in (11) immediately leads to:

$$\tilde{B}_z = \frac{\lambda\,(1-\beta)}{\theta^2} \frac{\partial \tilde{B}_y}{\partial \eta} \tag{12a}$$

and

$$\frac{\partial \tilde{B}_y}{\partial \tau} + \frac{3\theta}{2} \tilde{B}_y \frac{\partial \tilde{B}_y}{\partial \eta} - \frac{\lambda^2\,(1-\beta)}{2\theta^2} \frac{\partial^3 \tilde{B}_y}{\partial \eta^3} = 0 \quad , \tag{12b}$$

where $\partial/\partial\tau = (\partial/\partial\zeta + \theta^2/2)$. This is the well known KdV equation, but it is worth noting that the wave is elliptically polarized and not linearly polarized, since $\tilde{B}_z \neq 0$. For a nondispersive medium, (12b) indeed leads to the linearly polarized fast MHD wave which moves with the phase speed, $V_p = V_A\,(1 + k^2\lambda^2/2\theta^2)$, where V_p is obtained from the linearized version of (10) and (11) assuming $(k\lambda/\theta)^2 \ll 1$. However, (12a) clearly shows that, for the fast mode, $\tilde{B}_z$ is not negligible even for weak dispersion.

The soliton solution of the KdV equation (12b) is:

$$\tilde{B}_y\,(\bar{\zeta}) = \frac{2M}{\theta}\,\mathrm{Sech}^2 \left(\frac{\theta}{\lambda} \left[\frac{M}{2\,(1-\beta)} \right]^{1/2} \bar{\zeta} \right) \quad , \tag{13}$$

where $\bar{\zeta} = (\eta - M\tau)$, M is the speed of the nonlinear right hand polarized (fast) wave. Since M and $(1-\beta)$ have to have the same sign, for $\beta < 1$, Eq. (18) yields a compressive forward propagating solitary wave whereas for $\beta > 1$ it gives a rarefactive backward propagating wave.[14] These correspond to fast $(\beta<1)$ and slow $(\beta>1)$ MHD waves in the MHD limit.

If we define the ellipticity by

$$P = \frac{|\text{minor axis of the oval made by } \tilde{B}_\perp|}{|\text{major axis of the oval made by } \tilde{B}_\perp|} \quad ,$$

then according to Eqs. (12a) and (13),

$$P \sim \frac{M^2}{\theta^2} \left[(1-\beta)/M \right]^{1/2} \quad . \tag{14}$$

Note that the ellipticity is independent of the strength of dispersion and is the same for forward and backward propagating waves.

The slow mode $(\tilde{B}_y \sim 0)$, according to the linearized solutions of (10) and (11), has the phase speed $V_s = V_A\,(1-k^2\lambda^2/2\theta^2)\cos\theta$. So for θ sufficiently large, $V_s << V_A$, so that and the nonlinear coupling between the slow and the fast Alfvén modes should be rather weak, which is reflected by (12a,b). On the contrary, for parallel and quasiparallel propagation, the nonlinear coupling is an essential feature, since the two modes propagate almost with the same speed and hence interact with each other for a much longer time.

For $\tilde{B}_y << \tilde{B}_z$, we see from (10) and (11), neglecting the slow variation of $\tilde{B}_y$ for θ sufficiently large $\left(\partial/\partial\zeta << (\theta^2/2)\,\partial/\partial\eta\right)$,

$$\tilde{B}_y \approx \frac{\lambda(1-\beta)}{\theta^2}\frac{\partial\tilde{B}_z}{\partial\eta} - \frac{\tilde{B}_z^{\,2}}{2\theta} \qquad , \qquad (15)$$

which when substituted into Eq. (11) gives a linear differential equation,

$$\frac{\partial\tilde{B}_z}{\partial\zeta} + \frac{\lambda^2\,(1-\beta)}{2\theta^2}\frac{\partial^3\tilde{B}_z}{\partial\eta^3} = 0 \qquad , \qquad (16)$$

which confirms that the slow Alfven wave does not steepen to second order. However, since according to (15) $\tilde{B}_y \neq 0$, like the fast mode, the slow mode is also elliptically polarized. Similar conclusions were drawn by Kawahara[12]. However, if θ is so large that $\theta\tilde{B}_y >> \tilde{B}_z^{\,2}$, then from (15), $\tilde{B}_y = \left[\lambda\,(1-\beta)/\theta^2\right]\,\partial\tilde{B}_z/\partial\eta$ and (11) reduces to

$$\frac{\partial\tilde{B}_z}{\partial\zeta} + \frac{\lambda}{4\theta}\frac{\partial^2\tilde{B}_z^{\,2}}{\partial\eta^2} + \frac{\lambda^2\,(1-\beta)}{2\theta^2}\frac{\partial^3\tilde{B}_z}{\partial\eta^3} = 0 \qquad . \qquad (17)$$

This is a modified version of the KdV equation in which the nonlinear term depends on the dispersive scalelength. It is not the modified KdV equation discussed by Kakutani and Kawahara[12,13], which has cubic nonlinearity and which describes the evolution of the density and not of B_z as in (17).

So far we have discussed the case with boundary condition $B_{\perp\infty} \equiv B_\perp(\eta\to\infty) = B_0 \sin\theta$ but (12a,b) easily incorporate more general boundary conditions. In fact we can even leave $b_{\perp\infty}$ unspecified; in this case a reduction of (5) and (6), similar to the one that yielded (8), gives

$$\frac{\partial\vec{b}_\perp}{\partial\zeta} + \frac{1}{4(1-\beta)}\frac{\partial}{\partial\eta}\left[\vec{b}_\perp\left(b_\perp^2 - b_{\perp\infty}^2\right)\right] + \frac{\lambda}{2}\frac{\partial^2}{\partial\eta^2}\left(\hat{e}_x \times \vec{b}_\perp\right) = 0 \qquad , \qquad (18)$$

where $\vec{b}_\perp$ is the same as in (8). For right hand and left hand polarized wave, (18) simply reduces to

$$\frac{\partial b_\pm}{\partial\zeta} + \frac{1}{4\,(1-\beta)}\frac{\partial}{\partial\eta}\left[b_\pm\left(|b_\pm|^2 - |b_{\pm\infty}|^2\right)\right] \pm \frac{i\lambda}{2}\frac{\partial^2 b_\pm}{\partial\eta^2} = 0 \qquad . \qquad (19)$$

These equations can treat outgoing plane wave boundary conditions for which Ichikawa et al.[9] obtained spiky solitons for the case of parallel propagation. In deriving (9) and (19), we have not made any harmonic expansion of the field variables. Consequently, unlike the nonlinear Schrödinger equation, which describes the modulation of the amplitude only, these equations give the modulation of the complex field, i.e., of both the amplitude and the phase.

In summary, we emphasize that equations (5) and (6) express all the information contained in one-dimensional, two-fluid theory with finite ion inertia dispersion. Finite electron inertia dispersion can easily be added. Expressing two-fluid theory in Lagrangian form permits a straightforward and transparent derivation of the model equations for weakly nonlinear dispersive waves. When the "quasi-static" approximation (7) is valid, the sound mode is decoupled from the two Alfvén modes, and it is possible to obtain a "Vector DNLS" equation, (8), which describes two nonlinearly coupled Alfvén waves which propagate in the same direction at oblique angles to the average or upstream magnetic field direction. The vector DNLS equation reduces to the ordinary DNLS equation for circularly polarized, parallel propagating Alfvén waves for either vanishing or nonvanishing wave amplitude at infinity. At sufficiently oblique propagation angles, the Vector DNLS equation reduces to a KdV equation, (12a,b), for an elliptically polarized fast mode wave (when $\beta<1$), and an apparently new nonlinear equation (17) for an elliptically polarized intermediate mode wave whose nonlinear term depends upon the dispersion. Subsequent research will report on analytical and numerical solutions of the Vector DNLS equation.

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