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SPRING COLLEGE ON PLASMA PHYSICS

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NUMERICAL SOLUTION OF THE VLASOV EQUATION

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CANADA

Numerical Solution of the Vlasov Equation

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- A method for solving the Vlasov equation in configuration space is described
- It treats the space (x) and velocity (v) directions separately (splitting schemes or fractional steps) and produces a scheme of second order in the time-step Δt.
- Spline interpolation methods are used to solve the hyperbolic terms of the splitted equation.
- Result obtained solving one and two dimensional Vlasov equation are presented.

The Splitting Scheme

- We want to solve the dimensionless system:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - E\frac{\partial}{\partial v} f(x,v,t) = 0 \qquad (1a)$$

$$\frac{\partial E}{\partial x} = 1 - \int_{-\infty}^{\infty} f dv \qquad (1b)$$

where $f \equiv f(x,v,t)$ for periodic boundary conditions in x. We split up the equation, and solve the hyperbolic system

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0 \qquad (2a)$$

for a half time step, and then hyperbolic system

$$\frac{\partial f}{\partial t} + E(x,t)\frac{\partial f}{\partial v} = 0 \qquad (2b)$$

for the second half time step. The result should be an approximation to the solution of Eqs.(1).

- A solution for Eqs(2) can be written down (assuming $E(x,t)$ to be known) at a time Δt: 3

$$f(x, v, \Delta t) = f(x - v\Delta t, v, 0) \qquad (3a)$$

$$f(x, v, \Delta t) = f(x, v + E(\bar{x}, \tau)\Delta t, 0) \qquad (3b)$$

$$x - v\Delta t \leq \bar{x} \leq x \qquad 0 \leq \tau \leq \Delta t$$

Eq. (3a) is an exact solution of Eq. (2a). It represents a shift of the distribution function by an amount $v_i \Delta t$.
In Eq. (3b) an appropriate electric field has to be chosen in order to make the solution consistent with Eq. (2b), by shifting the velocity space by an amount $E(\bar{x}, \tau)\Delta t$.

- We have reduced the integration of the Vlasov equation to two successive shift operation
- The shift operations are to be effected using cubic splines.

- To obtain a scheme which is second order in Δt, consider the characteristics of Eq. (1a): 4

$$\frac{dv}{dt} = -E(x,t) \qquad (4a)$$

$$\frac{dx}{dt} = v \qquad (4b)$$

Their solution is equivalent to the solution of Eq. (1a). We write for the solution of Eqs(4) a time-symmetric scheme which, by virtue of its symmetry, is of second order in Δt.

$$v^{n+1} = v^n - E^{n+\frac{1}{2}}(\bar{x})\Delta t$$

$$x^{n+1} = x^n + \tfrac{1}{2}(v^{n+1} + v^n)\Delta t$$

We rewrite these equations, characterizing the beginning and the end of a piece of trajectory by "b" and "e" respectively:

$$v^n(b) = v^{n+1}(e) + E^{n+\frac{1}{2}}(\bar{x})\Delta t \quad (5a)$$

$$x^n(b) = x^{n+1}(e) - \left[v^{n+1}(e) + \frac{1}{2}E^{n+\frac{1}{2}}(\bar{x})\Delta t\right]\Delta t \quad (5b)$$

The position $\bar{x}$ where the electric field has to be evaluated is given by $\bar{x} = x - v\frac{\Delta t}{2} + O(\Delta t^2)$ for a second order scheme in Δt.

- Let us now consider the following sequence of shifting of the distribution function to advance à time-step Δt:

$$f^*(x,v) = f^n\left(x - v\frac{\Delta t}{2}, v\right) \quad (6a)$$

$$f^{**}(x,v) = f^*\left(x, v + E(\bar{x})\Delta t\right) \quad (6b)$$

$$f^{n+1}(x,v) = f^{**}\left(x - v\frac{\Delta t}{2}, v\right) \quad (6c)$$

By substituting successively from Eqs (6a) and (6b) in Eq. (6c) we obtain:

$$f^{n+1}(x,v) = f^n\left(x - \Delta t\left(v + \frac{1}{2}E(\bar{x})\Delta t\right), v + E(\bar{x})\Delta t\right) \quad (7)$$

with $\bar{x} = x - v\Delta t/2$.

A comparison of the arguments of f on the right hand side of Eq. (7) shows that they are identical with the right hand sides of Eq. (5), and this completes the proof that the scheme is second order in Δt. The electric field in Eq. (6b) is obtained from the distribution $f^*(x,v)$ after the shift in Eq. (6a).

- The interpolation

Solving Eq. (2a) is done by replacing the value of f at (x_j, v_i) by the values at $(x_j - v_i\Delta t, v_i)$. These values are calculated using cubic splines interpolation.

We write the interpolated values

$$\tilde{f}_i = f(x_i + \delta\Delta x) \qquad 0 \leq \delta \leq 1$$

Note that all interpolated values $\tilde{f}_i$ are equidistant;
i.e Δx is independent of the index i (uniform mesh).

Use Taylor expansion:

$$\tilde{f}_i = f(x_i + \delta\Delta x) = f_i + p_i\,\delta\Delta x + \frac{1}{2} s_i (\delta\Delta x)^2 + g_i (\delta\Delta x)^3$$

Together with cubic spline relations:

$$p_{i-1} + 4p_i + p_{i+1} = \frac{3}{\Delta x}(f_{i+1} - f_{i-1})$$

$$s_{i-1} + 4s_i + s_{i+1} = \frac{6}{\Delta x^2}(f_{i-1} - 2f_i + f_{i+1})$$

$$g_{i-1} + 4g_i + g_{i+1} = \frac{1}{\Delta x^3}(-f_{i-1} + 3f_i - 3f_{i+1} + f_{i+2})$$

We get:

$$\tilde{f}_{i-1} + 4\tilde{f}_i + \tilde{f}_{i+1} = Af_{i-1} + Bf_i + Cf_{i+1} + Df_{i+2}$$

If $\hat{f}_i = f(x_i - \delta\Delta x)$

we have

$$\hat{f}_{i-1} - 4\hat{f}_i + \hat{f}_{i+1} = Af_{i+1} + Bf_i + Cf_{i-1} + Df_{i-2}$$

$$A = (1-\delta)^3$$

$$B = 4 - 3\delta^2(2-\delta)$$

$$C = 4 - 3(1-\delta)^2(1+\delta)$$

$$D = \delta^3$$

- The technique can be readily generalized to a two or three dimensional Vlasov Equation:

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} - \vec{E}\cdot\frac{\partial f}{\partial \vec{v}} = 0$$

$$\nabla\cdot\vec{E} = 1 - \int_{-\infty}^{\infty} f\, d\vec{v}$$

We split and solve

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} = 0$$

$$\frac{\partial f}{\partial t} + \vec{E}\cdot\frac{\partial f}{\partial \vec{v}} = 0$$

We use the following sequence of shifts

$$f^{*}(\vec{r},\vec{v}) = f^{n}(\vec{r} - \vec{v}\tfrac{\Delta t}{2}, \vec{v})$$

$$f^{**}(\vec{r},\vec{v}) = f^{*}(\vec{r}, \vec{v} - \vec{E}(\vec{r})\Delta t)$$

$$f^{n+1}(\vec{r},\vec{v}) = f^{**}(\vec{r} - \vec{v}\tfrac{\Delta t}{2}, \vec{v})$$

- Nonlinear Evolution of a Monochromatic Wave in a one-dimensional Vlasov Plasma.

Initial Condition $f = f_0(1+\alpha\cos kx)$

$k = 2\pi/L$ $\quad f_0$ Maxwellian

$N = 256$ in space

$M = 128\times 2 = 256$ in velocity space.

$\alpha = 0.1$

Fig. 1 shows the spatially averaged distribution function.

Fig. 2 shows the time evolution of the total electric energy

Fig. 3 shows the time evolution of the amplitude of the mode excited with $k = 0.3$

The Final solution is a B.G.K. structure.

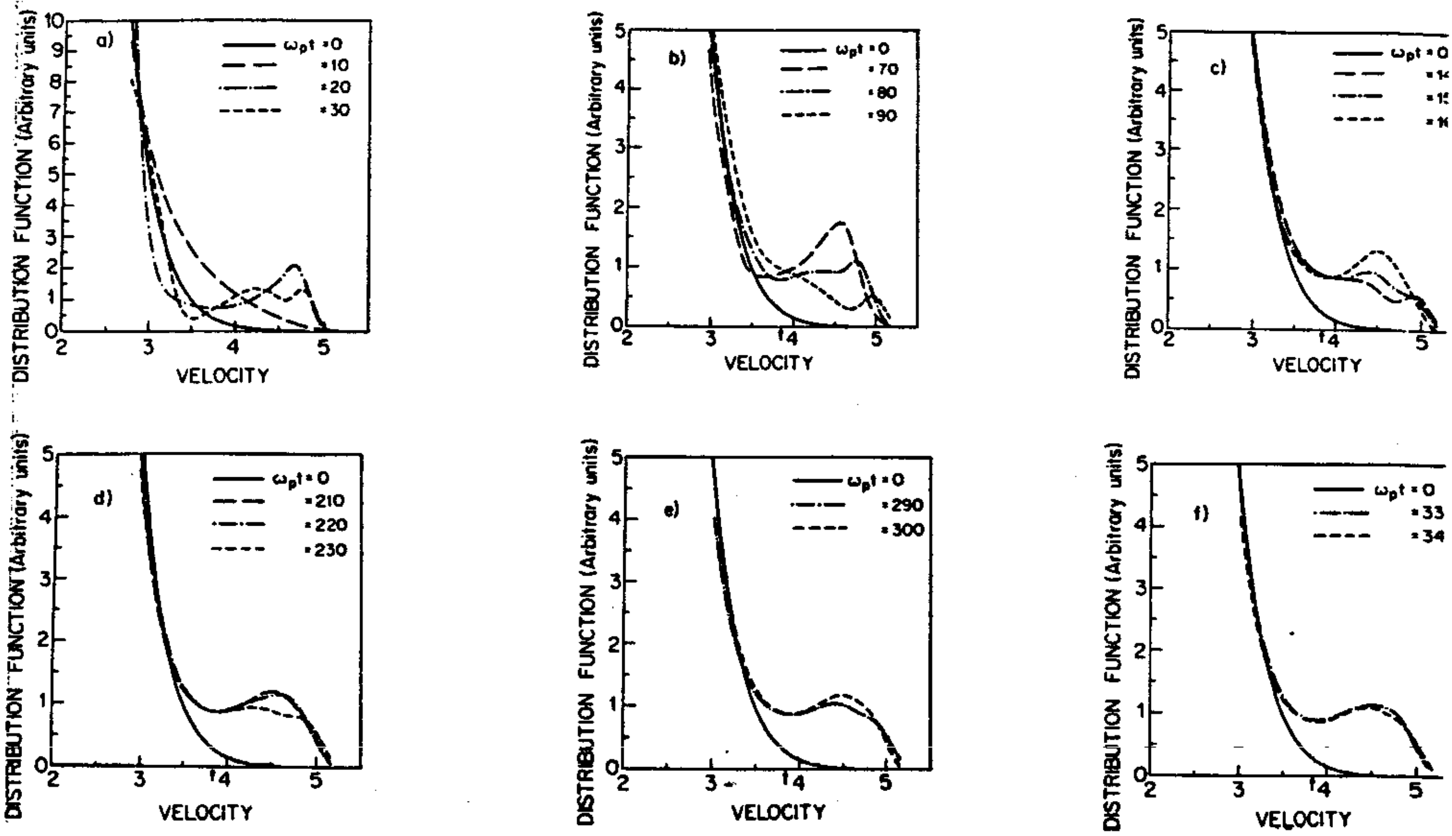


FIG. 1. (a)–(f) Temporal behavior of the spatially-averaged distribution function. The phase velocity of the initially excited wave is marked by an arrow. ~~The units of the vertical scale are the same as in Table I~~.

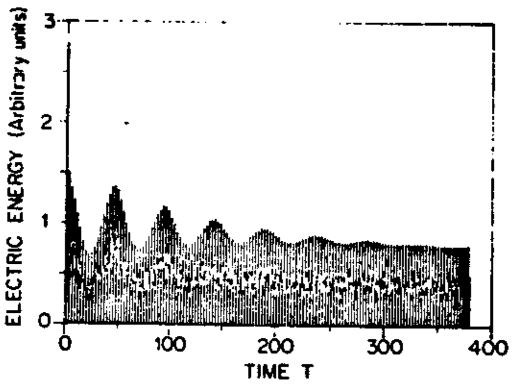


FIG. 2. Time evolution of the total electric energy (arbitrary units).

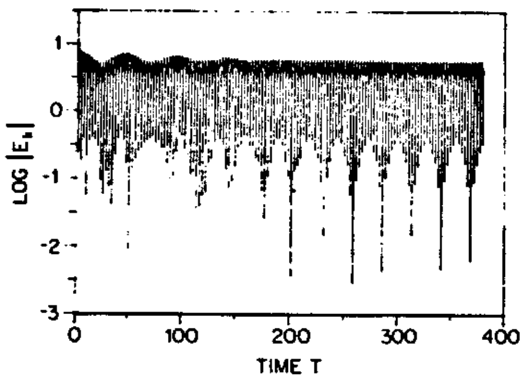


FIG. 3. Time evolution of the amplitude of the electric fiel(d) (on a logarithmic scale) for the fundamental mode $k = 0.3$.

- Nonlinear Evolution of the Bump on Tail instability

$$f(v) = \frac{n_p}{\sqrt{2\pi}}\, e^{-v^2/2} + \frac{n_b}{\sqrt{2\pi}}\, e^{-(v-v_d)^2/v_t^2}$$

$n_p = 0.9 \quad n_b = 0.1 \quad v_t = 0.5 \quad v_d = 4.5$

Initial perturbation $f = f(v)(1+\epsilon \cos kx)$

$\epsilon = 0.04 \quad L = 20\pi \quad n = 3 \quad k = \frac{2\pi n}{L} = 0.3$

Fig 1. Time history of the spatially averaged distribution function

Fig. 2 Time evolution of the total Electric Energy

Fig. 3 Time evolution of the Fourier mode initially excited.

Final equilibrium is a B. G. K. structure.

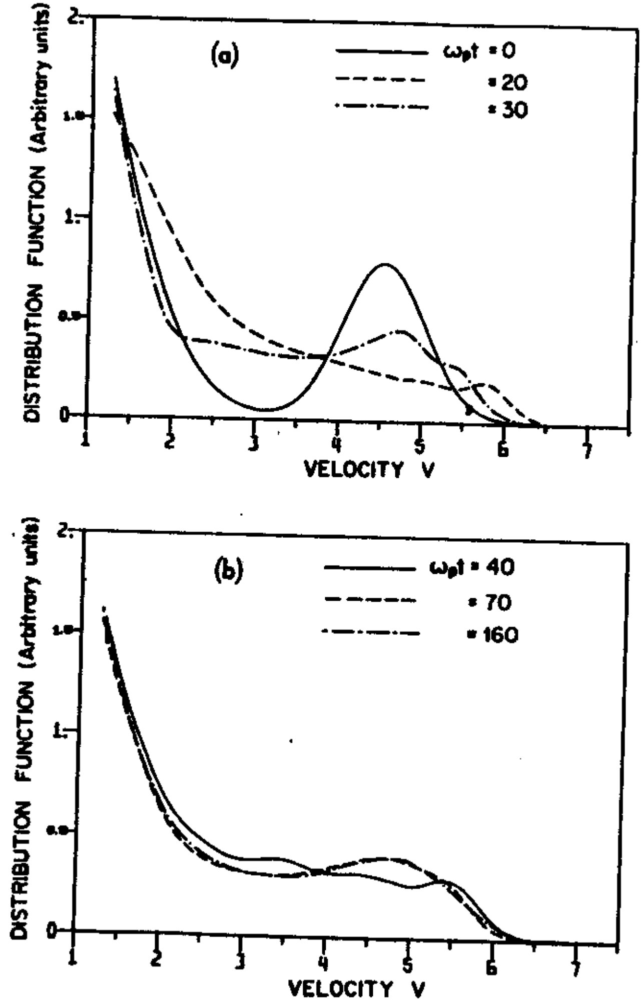


FIG. 1. Spatially averaged distribution function plotted against velocity for different times t.

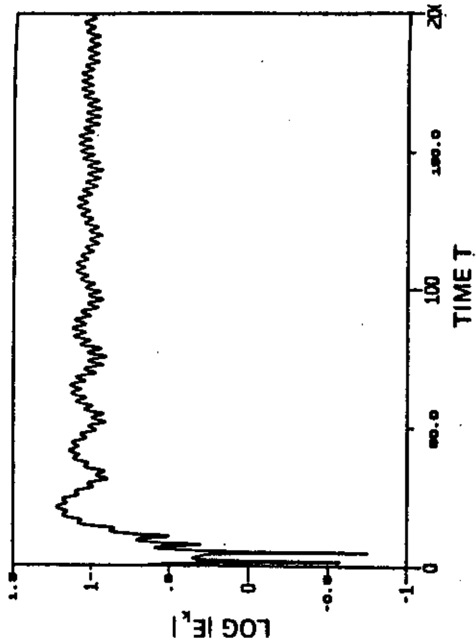


FIG. 3. Time evolution of the Fourier mode with n = 3, (… = 0.3).

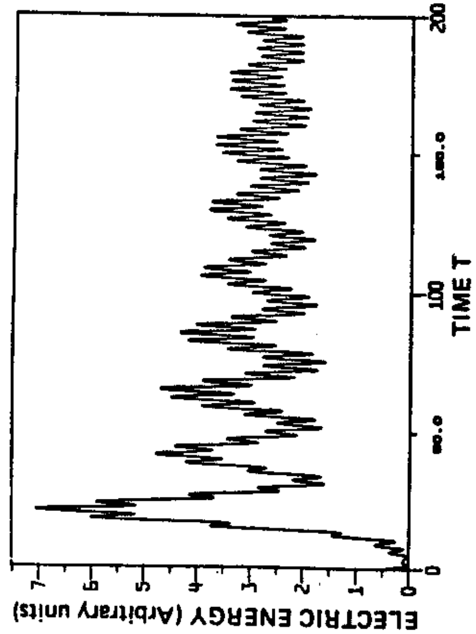


FIG. 2. Time evolution of the total electric energy.

- Nonlinear evolution of BGK structure.

We take as initial value a BGK equilibrium

$$f(\varepsilon) = \frac{\mu}{\sqrt{2\pi}} \frac{2-2\xi}{3-2\xi} \left(1 + \frac{\varepsilon}{1-\xi}\right) e^{-\varepsilon}$$

ε = total energy $= \frac{v^2}{2} + \phi(x)$

$\phi(x)$ solution of

$$\frac{d^2\phi}{dx^2} + \mu \frac{3-2\xi+2\phi}{3-2\xi} e^{-\phi} - 1 = 0$$

at $x=0$ $\phi=0$ $\frac{d\phi}{dx}=0$

$\xi = 0.9$ $\mu = 0.22$

The space period is 14.71. We take four vortices with $L = 4 \times 14.71$, to which we apply an initial perturbation. The sequence of figures show the coalescence of the four vortices into a single vortex. Those figures with two figures show the results obtained from a second code which are added for comparison

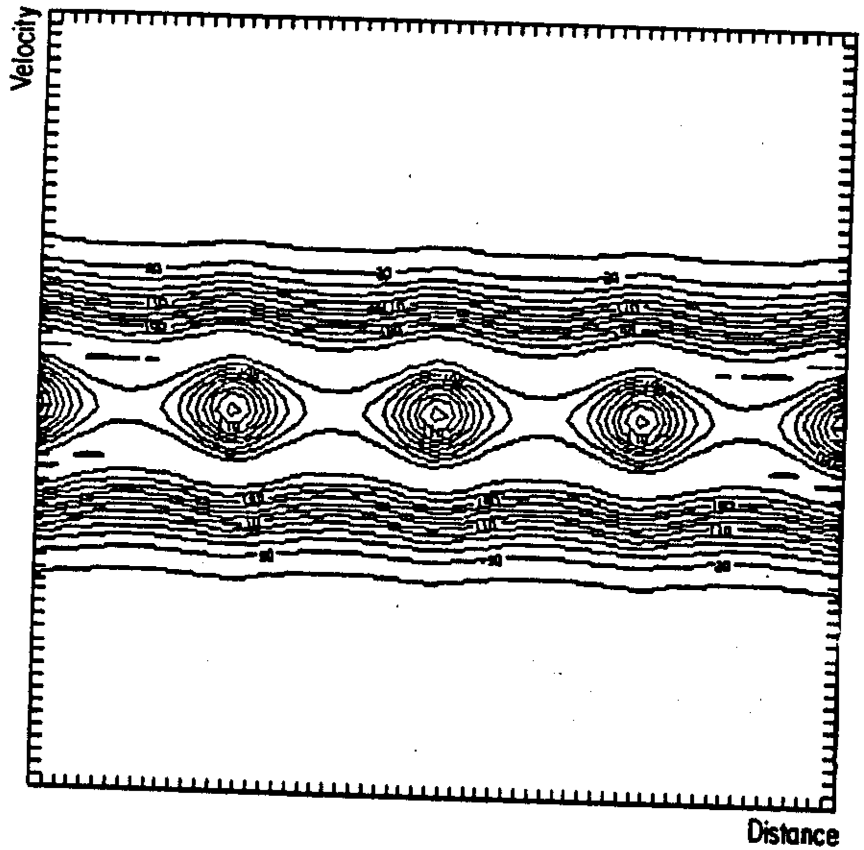
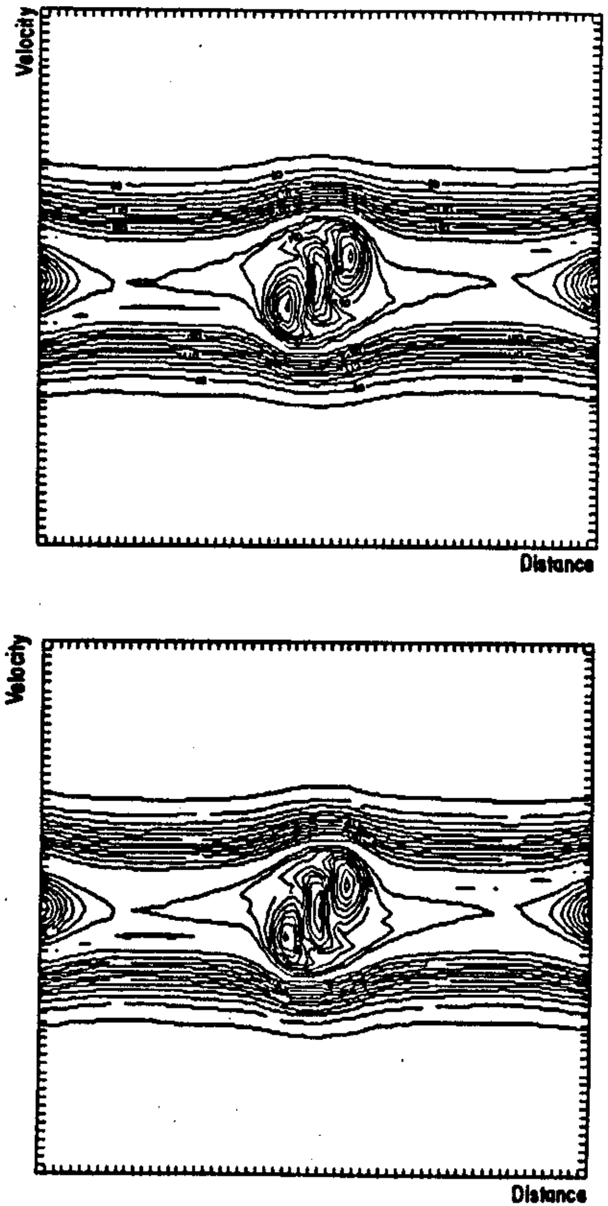


Figure - 1

$t = 0$



a)

b)

Figure - 2 $t = 100\,\omega_{pe}^{-1}$

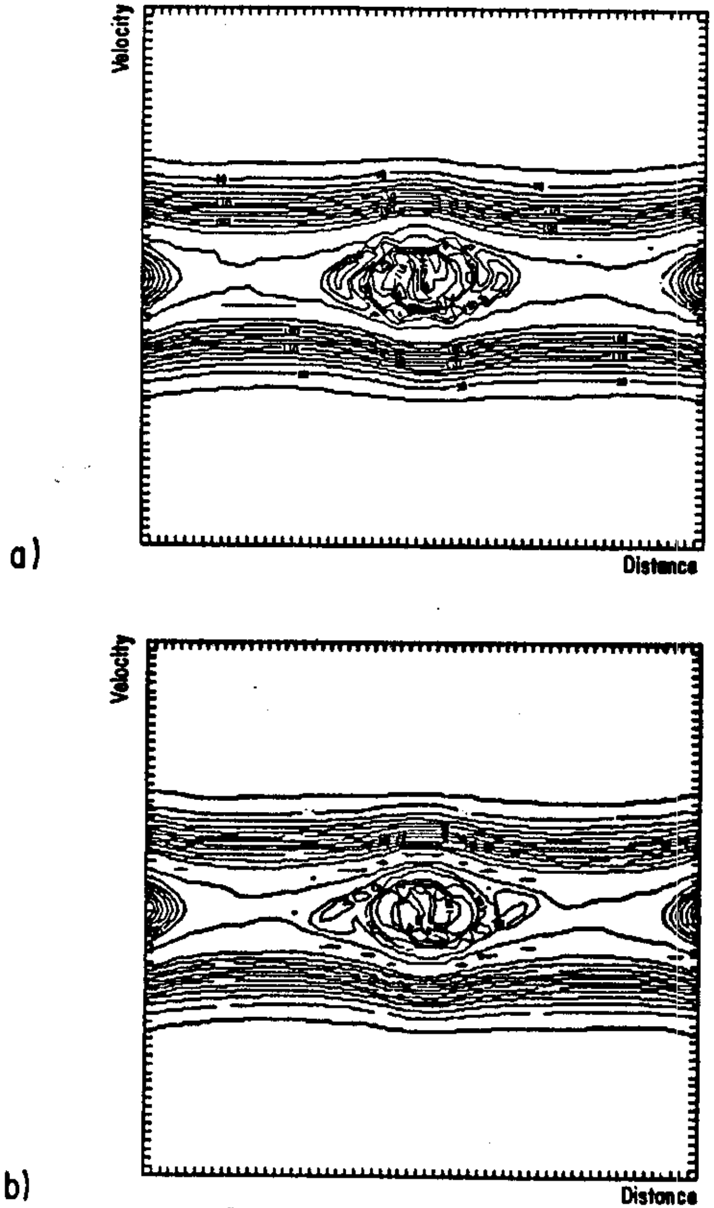


Figure - 3 $t = 200\,\omega_{pe}^{-1}$

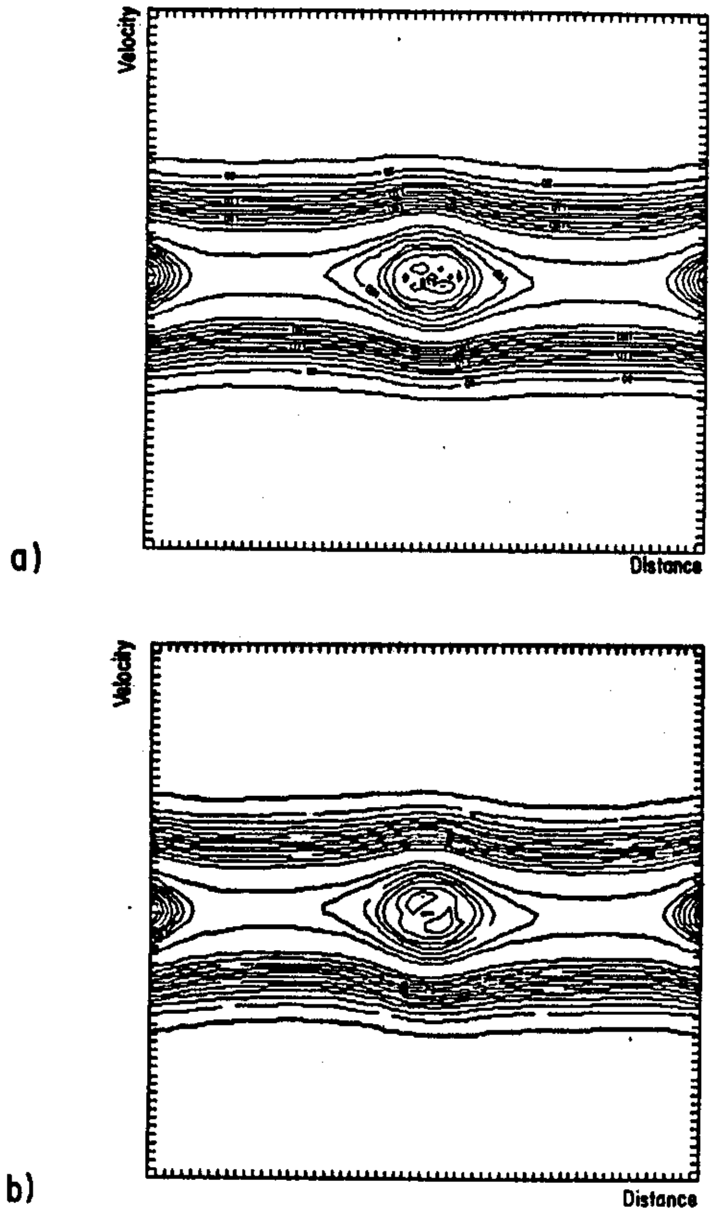
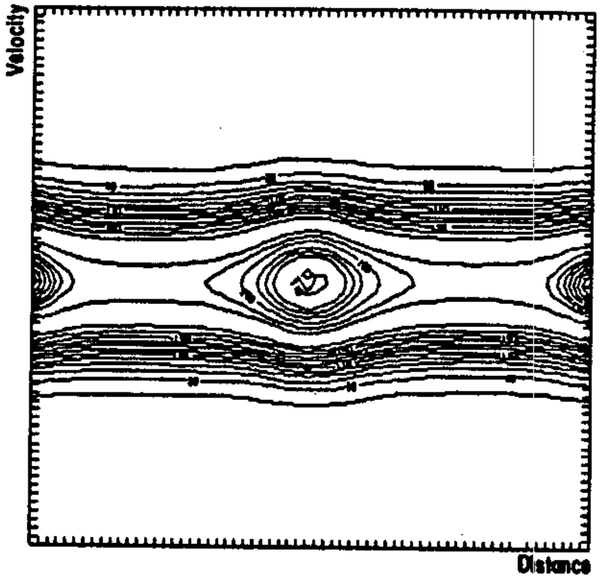
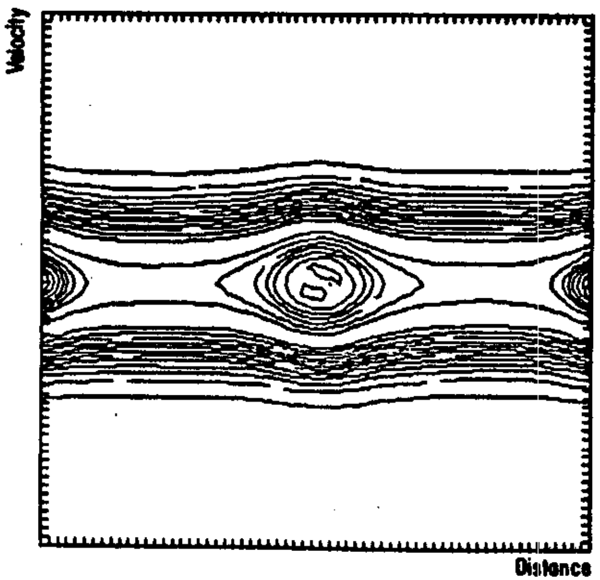


Figure - 4 $t = 400\,\omega_{pe}^{-1}$

7



a)



b)

Figure - 5 $t = 800\ \omega_{pe}^{-1}$

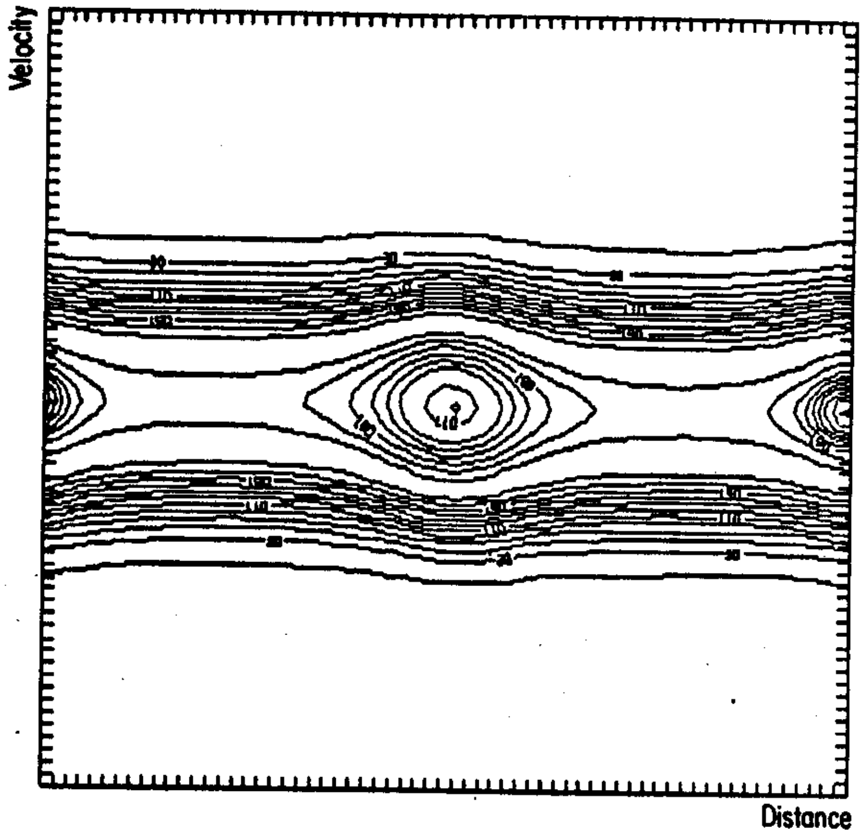


Figure - 6 $t = 2100\ \omega_{pe}^{-1}$

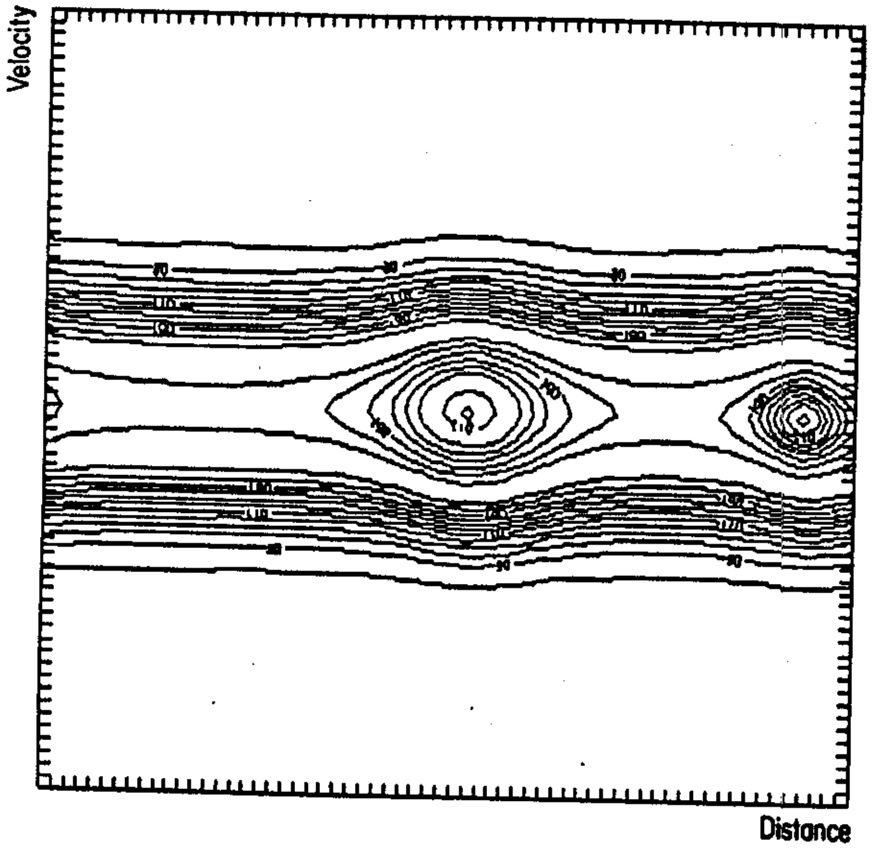


Figure - 7 $t = 2400\ \omega_{pe}^{-1}$

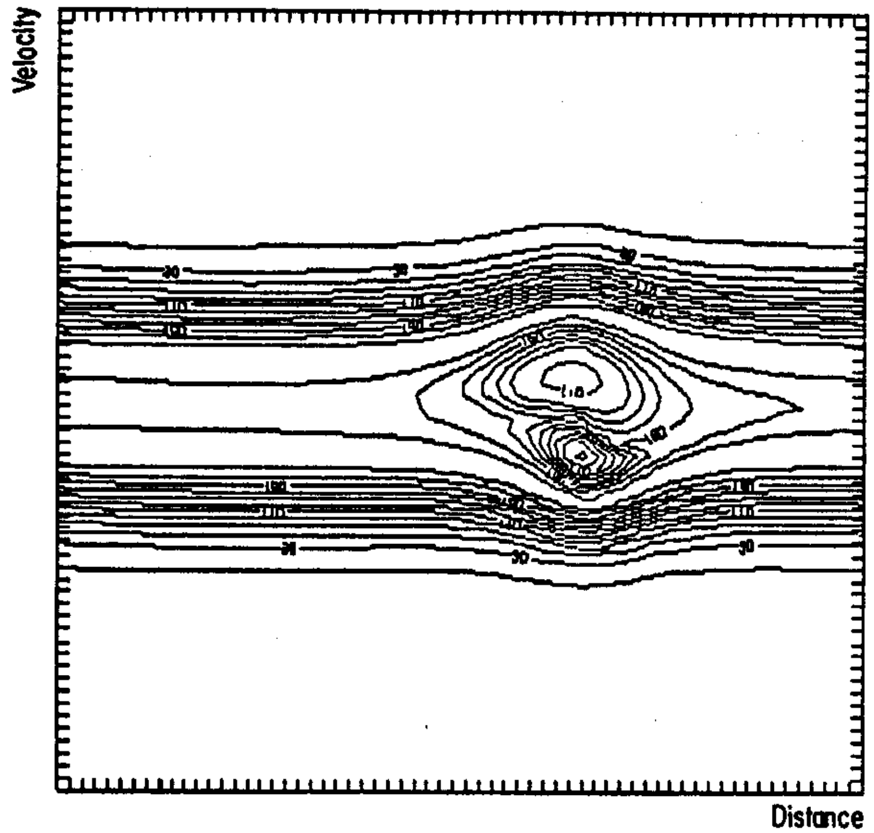


Figure - 8 $t = 2520\ \omega_{pe}^{-1}$

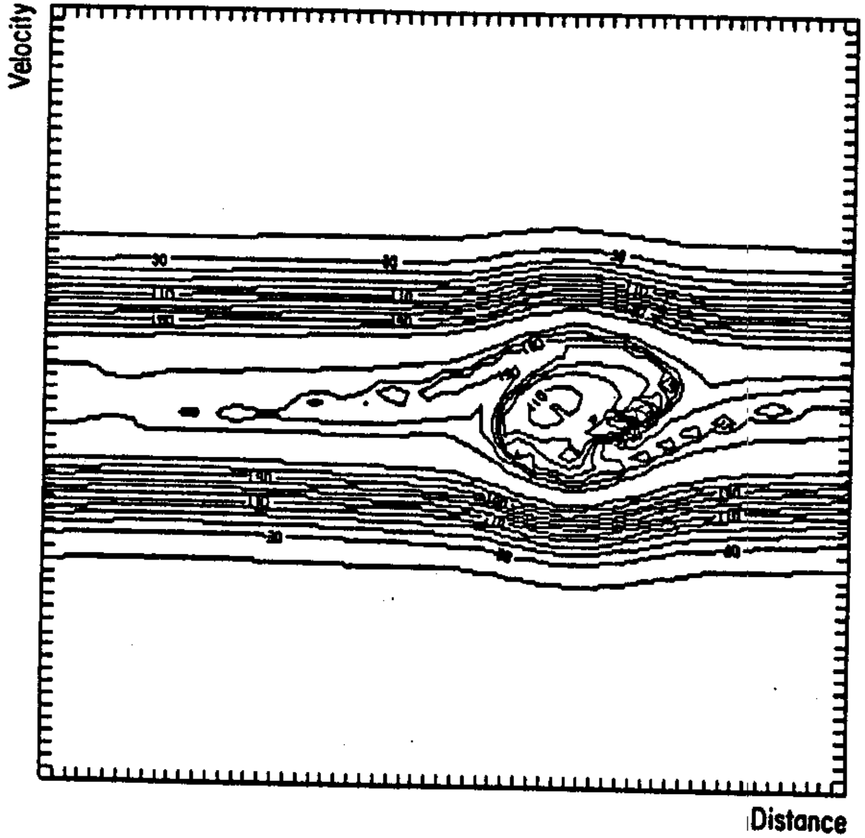


Figure - 9 $t = 2560\ \omega_{pe}^{-1}$

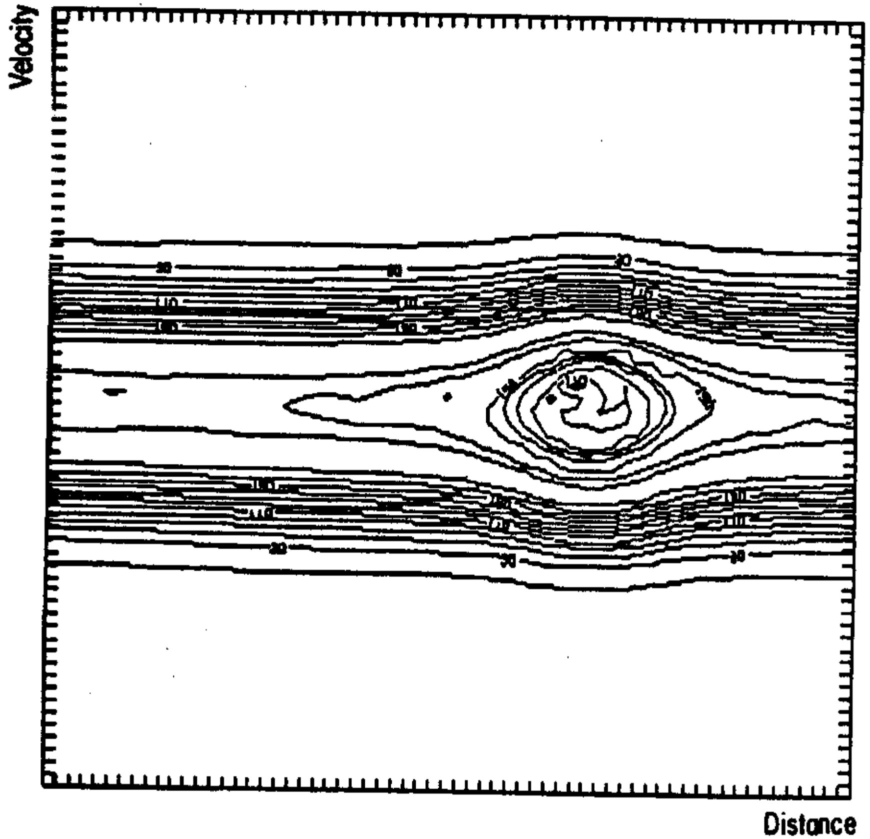


Figure - 10 $t = 2700\ \omega_{pe}^{-1}$

- Numerical solution of the Two-Dimensional Vlasov equation: 27

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} - E_x \frac{\partial f}{\partial v_x} - E_y \frac{\partial f}{\partial v_y} = 0$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 1 - \iint_{-\infty}^{\infty} f \, dv_x \, dv_y$$

Initial equilibrium distribution consists of two drifting Maxwellian beams:

$$f_0(\vec{v}) = \frac{0.5}{\pi} e^{-(\vec{v} - v_1 \vec{e}_x)^2/2} + \frac{0.5}{\pi} e^{-(\vec{v} + v_1 \vec{e}_x)^2/2}$$

$N = 16$ points in each spatial direction

$M = 32$ points in each velocity direction

$\Delta t = \frac{1}{8}$

Initial perturbation in x and y such that all modes with $0 < L_y k_y / 2\pi < 3$ and $0 < k_x L_x / 2\pi < 3$ (except for the modes $k_x = k_y = 0$) have the same energy

The figure shows the evolution of the total electric energy.

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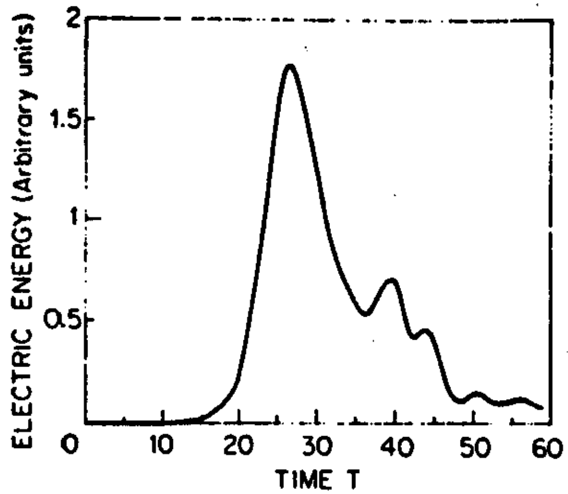

.. Time evolution of the total electric energy for the two-dimensional electrostatic two-stream instability.