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ABSTRACT

A generalised energy principle is devised for arbitrary magnetic confinement schemes having anisotropic plasma pressure that are two dimensional or three dimensional in a unified treatment with the requirement that the rotational transform be finite. The minimisation of this energy principle is demonstrated to yield geometric components of the MHD force balance relation that vanish in the equilibrium state. An iterative procedure is applied to advance the Fourier amplitudes of the inverse geometric coordinates and of a poloidal angle renormalisation parameter until a minimum energy state is reached. An application to an ELMO Snaky Torus configuration in the helically symmetric limit is carried out. The conditions that a hot electron layer induces a local magnetic well that extends to the plasma edge are determined.

INTRODUCTION

Inverse Fourier moments mangnetohydrodynamic (MHD) equilibrium codes have become extremely useful tools for the generation of MHD equilibria with finite rotational transform.[1-3] They are particularly useful for the computation of quantities that need to be evaluated on the flux surfaces such as the coefficients for transport and stability analysis. This is because the calculations are carried out in anon-orthogonal magnetic flux coordinate system (ρ,θ,ϕ) where ρ is the variable that identifies the flux contours, and θ and ϕ represent angular variables. Inaccuracies introduced by the interpolation from a

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geometric grid to the flux surfaces do not thus arise in these coordinates.

In this work, we introduce a generalised energy principle and demonstrate that its minimisation yields MHD equilibria with anisotropic plasma pressure for magnetic confinement schemes that are two dimensional (2D) and three dimensional (3D). These serve as models for devices with auxiliary heating systems, particularly those that rely on an energetic trapped population to provide plasma stability such as the ELMO Snaky Torus. We consider a pressure tensor that has only diagonal elements and we invoke the vanishing of the MHD force balance relation along the magnetic field lines to show that it reduces to

$$\underline{F} = -\frac{\partial p_\parallel}{\partial \rho}\underline{\nabla}\rho + \underline{K}\times\underline{B} \qquad (1)$$

where we define the effective current density as $\underline{K} = \underline{\nabla}\times(\sigma\underline{B})$ and the anisotropy factor is $\sigma = 1/\mu_0 + (p_\perp - p_\parallel)/B^2$. To obtain Eq. (1), we have made use of Ampère's Law. Also, the parallel pressure $p_\parallel$ and the perpendicular pressure $p_\perp$ are functions of ρ and B. In 3D, in principle, the pressures can vary from field line to field line on a flux surface as well, but for simplicity such a dependence is ignored here.

MAGNETIC FIELD GEOMETRY

Introducing the magnetic flux coordinate system (ρ,θ,ϕ) with condition $B\cdot\nabla\rho = 0$ and noting that $\nabla\cdot B = 0$ implies that the B field can be expressed in contravariant representation as

$$\underline{B} = \underline{\nabla}\phi\times\underline{\nabla}\Psi + \underline{\nabla}\Phi\times\underline{\nabla}\theta_* \qquad (2)$$

where $2\pi\Psi(\rho)$ and $2\pi\Phi(\rho)$ are the poloidal and toroidal magnetic fluxes, respectively. The variable θ_* is that particular choice of poloidal angle that straightens the magnetic field lines. It is related to the optimal poloidal angle θ of the equilibrium calculation by $\theta_* = \theta+\lambda(\rho,\theta,\phi)$, where λ is a periodic renormalisation parameter.[3] As a result, the magnetic field components in the contravariant representation are

$$B^{\phi} = \frac{\Phi'}{\sqrt{g}} \left(1 + \frac{\partial\lambda}{\partial\theta}\right) \tag{3}$$

and

$$B^{\theta} = \frac{\Phi'}{\sqrt{g}} \left(\iota - \frac{\partial\lambda}{\partial\phi}\right) \tag{4}$$

where $\iota(\rho) \equiv \Psi'/\Phi'$ and primes indicate the derivative of a flux surface quantity with respect to ρ. For 3D and axisymmetric systems, $\sqrt{g}$ denotes the Jacobian of the transformation from the cylindrical coordinates (R,ϕ,Z) to the flux coordinates. R is the distance of any point in the plasma to the major axis, Z is its distance from the mid-plane, and ϕ is the geometric toroidal angle. For systems with helical symmetry, $\sqrt{g}$ denotes the Jacobian of the transformation from the rotating Cartesian coordinates (X,Y,ϕ) to the flux coordinates.[4,5] Here, X and Y have the same interpretation as R and Z, respectively, with the axis of symmetry corresponding to the major axis and $\phi = hz$. The helical pitch is h and z is the distance along the axis of symmetry.

ENERGY MINIMISATION

The total energy of the system is defined as

$$W = \iiint d^3x \left[\frac{B^2}{2\mu_0} + \frac{p_\parallel}{(\Gamma - 1)}\right] \tag{5}$$

where the parallel pressure is expressed as

$$p_\parallel(\rho,B) = M(\rho)\left(\frac{d\Phi}{d\rho}\right)^{\Gamma} \frac{1+p(\rho,B)}{\langle 1+p(\rho,B)\rangle^{\Gamma}} \tag{6}$$

The adiabatic index Γ is chosen as 5/3 which makes W positive definite and thus guarantees that the minimum energy state corresponds to an MHD equilibrium. The energetic species contribution to the pressure $p(\rho,B)$ is the source of the anisotropy, and $\langle p(\rho,B)\rangle \equiv \iint d\theta d\phi \sqrt{g} p(\rho,B)$ denotes its flux surface average. A variation of W is carried out with respect to an artificial time parameter t. The invariants of the problem are the mass function M, the magnetic flux functions Ψ and Φ, and the flux coordinates ρ,θ, and ϕ.

For a fixed boundary calculation in the 3D case, we obtain

$$\frac{dW}{dt} = - \iiint d\rho d\theta d\phi F_R \frac{\partial R}{\partial t} - \iiint d\rho d\theta d\phi F_Z \frac{\partial Z}{\partial t} - \iiint d\rho d\theta d\phi F_\lambda \frac{\partial\lambda}{\partial t} \tag{7}$$

where

$$F_R = \frac{\partial}{\partial\theta}\left[\sigma\sqrt{g}B^{\theta}\left(B^{\theta}\frac{\partial R}{\partial\theta} + B^{\phi}\frac{\partial R}{\partial\phi}\right) - R\frac{\partial Z}{\partial\rho}\left(p_\perp + \frac{B^2}{2\mu_0}\right)\right] + \frac{\partial}{\partial\phi}\left[\sigma\sqrt{g}B^{\phi}\left(B^{\theta}\frac{\partial R}{\partial\theta} + B^{\phi}\frac{\partial R}{\partial\phi}\right)\right] + \frac{\partial}{\partial\rho}\left[R\frac{\partial Z}{\partial\theta}\left(p_\perp + \frac{B^2}{2\mu_0}\right)\right] + \frac{\sqrt{g}}{R}\left[p_\perp + \frac{B^2}{2\mu_0} - \sigma R^2 (B^{\phi})^2\right] \tag{8}$$

which can be readily demonstrated to correspond to the $\sqrt{g}R\nabla\phi \times \nabla Z \cdot F$ cylindrical MHD force component,

$$F_Z = \frac{\partial}{\partial\theta}\left[\sigma\sqrt{g}B^{\theta}\left(B^{\theta}\frac{\partial Z}{\partial\theta} + B^{\phi}\frac{\partial Z}{\partial\phi}\right) + R\frac{\partial R}{\partial\rho}\left(p_\perp + \frac{B^2}{2\mu_0}\right)\right] + \frac{\partial}{\partial\phi}\left[\sigma\sqrt{g}B^{\phi}\left(B^{\theta}\frac{\partial Z}{\partial\theta} + B^{\phi}\frac{\partial Z}{\partial\phi}\right)\right] - \frac{\partial}{\partial\rho}\left[R\frac{\partial R}{\partial\theta}\left(p_\perp + \frac{B^2}{2\mu_0}\right)\right] \tag{9}$$

which can be readily demonstrated to correspond to the $\sqrt{g}\,R\underline{\nabla}R \times \underline{\nabla}\phi \cdot \underline{F}$ cylindrical MHD force component, and

$$F_\lambda = \Phi'\left[\frac{\partial(\sigma B_\phi)}{\partial\theta} - \frac{\partial(\sigma B_\theta)}{\partial\phi}\right] \tag{10}$$

which corresponds to the $-\sqrt{g}\Phi'\ \underline{B} \times \underline{\nabla}\rho \cdot \underline{F}/B^2$ force component. The toroidal and poloidal magnetic fields in the covariant representation are B_ϕ and B_θ, respectively. In the 2D helically symmetric case, an expression identical to Eq. (7) is obtained, but with R replaced by X and Z replaced by Y. The MHD forces in this case are

$$F_X = \frac{\partial}{\partial\theta}\left[\sigma\sqrt{g}B^\theta\left(B^\theta\frac{\partial X}{\partial\theta} - B^\phi Y\right) + \frac{1}{h}\frac{\partial Y}{\partial\rho}\left(p_\perp + \frac{B^2}{2\mu_0}\right)\right]$$
$$- \sigma\sqrt{g}B^\phi\left(B^\phi\frac{\partial Y}{\partial\theta} + B^\phi X\right) - \frac{\partial}{\partial\rho}\left[\frac{1}{h}\frac{\partial Y}{\partial\theta}\left(p_\perp + \frac{B^2}{2\mu_0}\right)\right] \tag{11}$$

which corresponds to $\sqrt{g}\underline{\nabla}Y \times \underline{\nabla}\phi \cdot \underline{F}/h$ and

$$F_Y = \frac{\partial}{\partial\theta}\left[\sigma\sqrt{g}B^\theta\left(B^\theta\frac{\partial Y}{\partial\theta} + B^\phi X\right) - \frac{1}{h}\frac{\partial X}{\partial\rho}\left(p_\perp + \frac{B^2}{2\mu_0}\right)\right]$$
$$+ \sigma\sqrt{g}B^\phi\left(B^\theta\frac{\partial X}{\partial\theta} - B^\phi Y\right) + \frac{\partial}{\partial\rho}\left[\frac{1}{h}\frac{\partial X}{\partial\theta}\left(p_\perp + \frac{B^2}{2\mu_0}\right)\right] \tag{12}$$

which corresponds to $\sqrt{g}\underline{\nabla}\phi \times \underline{\nabla}X \cdot \underline{F}/h$.

After expanding each of the terms of Eq. (7) in a Fourier series, the Fourier amplitudes of R, Z and λ (of X, Y and λ in helical symmetry) are advanced iteratively following a path of steepest descent until the force amplitudes vanish within some tolerance level.[3] An equilibrium state that is achieved following the procedure outlined

can be diagnosed by evaluating the radial component of the force balance relation (1), namely $F_\rho = \sqrt{g}\underline{\nabla}\Theta \times \underline{\nabla}\phi \cdot \underline{F}$, or

$$F_\rho = -\frac{\Phi'}{\sqrt{g}}\left\{\frac{\sqrt{g}}{\Phi'}\frac{\partial p_\parallel}{\partial\rho} + \left(\iota - \frac{\partial\lambda}{\partial\phi}\right)\left[\frac{\partial}{\partial\rho}(\sigma B_\theta) - \frac{\partial}{\partial\theta}(\sigma B_\rho)\right]\right.$$
$$\left. + \left(1 + \frac{\partial\lambda}{\partial\theta}\right)\left[\frac{\partial}{\partial\rho}(\sigma B_\phi) - \frac{\partial}{\partial\phi}(\sigma B_\rho)\right]\right\} \tag{13}$$

where B_ρ is the radial magnetic field in the covariant representation.

An algorithm to determine the rotational transform that corresponds to a prescribed effective current can be included by noting that in the equilibrium state $\underline{F} = 0$ so that the vector $\underline{K}$ satisfies the same properties as the current density $\underline{j}$ in the isotropic pressure limit, namely that $\underline{K} \cdot \underline{\nabla}\rho = \underline{\nabla} \cdot \underline{K} = 0$. Thus, in analogy with the scalar pressure case, we define the effective plasma current as

$$I(\rho) \equiv \frac{1}{2\pi}\iiint d\theta d\phi d\rho\sqrt{g}(\underline{K}\cdot\underline{\nabla}\phi) = \frac{1}{2\pi}\iint d\theta d\phi(\sigma B_\theta) \tag{14}$$

Expanding B_θ, we can obtain an expression for $\iota(\rho)$ that corresponds to a prescribed $I(\rho)$.[5] A natural choice for stellarator configurations that have no induced currents is $I(\rho) = 0$.

APPLICATION TO STRAIGHT ELMO SNAKY TORUS[6]

As an application, we consider an ELMO Snaky Torus configuration in the helically symmetric limit with zero net plasma current within each flux surface. The diameter of the plasma is chosen as 0.46 m and the helical pitch is such that $1/h = 0.375$ m. To model the hot electron population that is required for gross plasma stability, the pressure moment $p(\rho,B)$ is expressed as

$$p(\rho,B) = p_h(\rho)\left[\frac{B_M(\rho)}{B}\right]^L \qquad (15)$$

where B_M is the minimum value of B on each surface, $p_h(\rho)$ is chosen as Gaussian with halfwidth Δ centred about a flux surface near the plasma edge, and the integer L is the anisotropy parameter. This is an appropriate model for a distribution of energetic trapped particles. We specifically concentrate on determining the conditions that are required for hot electrons to induce a magnetic well on the outermost flux surfaces in the absence of thermal plasma pressure. The flux surfaces, the mod-B contours, and the hot electron pressure contours are shown in Fig. 1 for a sequence of equilibria with variable Δ, L = 8 and zero thermal pressure. The radial extent of the energetic electron layer broadens and its peak moves slightly towards the magnetic axis with increasing Δ. The flux surfaces and the mod-B contours, on the other hand, are not drastically altered. To guarantee that the local magnetic well generated by the hot electrons extends to the plasma boundary, the hot electron beta (β_h) contribution must increase from β_h = 0.7% for Δ = 0.1 to β_h = 1.2% for Δ = 0.2 to β_h = 1.5% for Δ = 0.3. The differential volume profiles that correspond to each of these cases appear in Fig. 2. The central region has a magnetic hill which reverses to a well in the region of energetic electron pressure. To obtain equilibria with finite thermal pressure, we tailor the $M(\rho)$ profile so that the resulting pressure gradient is concentrated in the region of magnetic well of Fig. 2. The absence of a source of free energy in the region of magnetic hill should reduce the risk that MHD instabilities would destroy the configuration.

SUMMARY

A generalised energy principle has been developed for magnetic confinement systems with finite rotational transform, the minimisation of which yields MHD equilibria for 2D (axisymmetric and helically symmetric) and 3D configurations that have anisotropic plasma pressure in a unified manner. Confinement schemes that rely on auxiliary heating methods to generate populations of energetic particles can be as a

result self-consistently investigated and evaluated. To obtain MHD equilibria, a steepest descent procedure is implemented that iteratively advances the Fourier amplitudes of the inverse geometric coordinates and of a poloidal angle renormalisation parameter λ until the minimum energy state is reached.

This procedure has been applied to an ELMO Snaky Torus configuration in the limit of helical symmetry. The conditions that a trapped hot electron layer can develop a local magnetic well that extends to the plasma edge are investigated. A hot electron component with a relatively modest energy content (β_h < 1.5%) is demonstrated to produce such a well in this model. Equilibria with finite thermal pressure should have thermal pressure gradients localised to the region of magnetic well to avoid large scale MHD instabilities.

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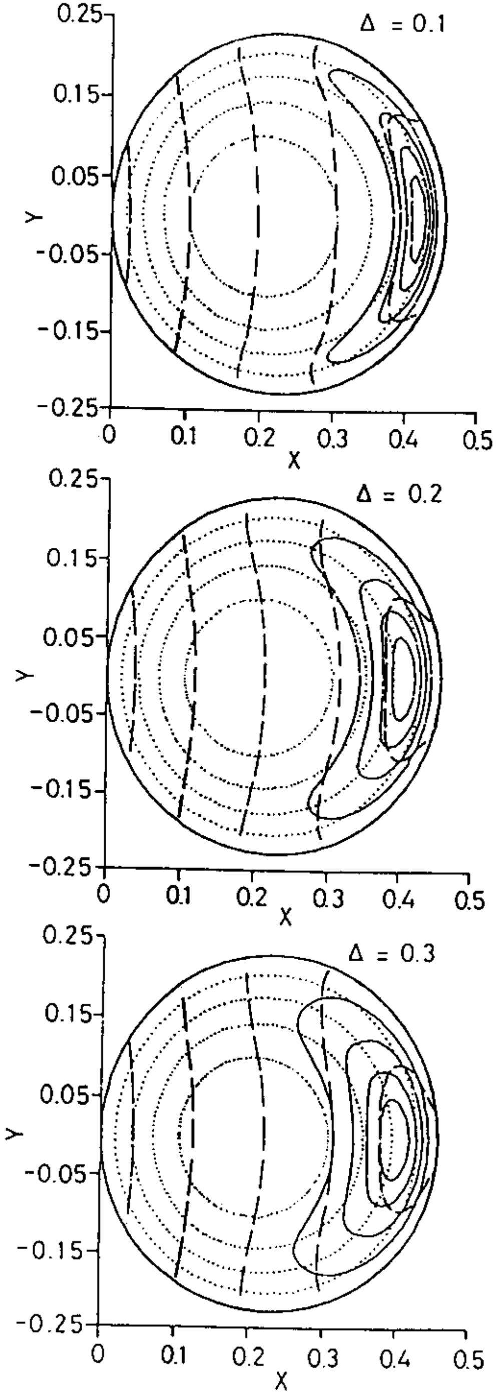
FIGURE CAPTIONS

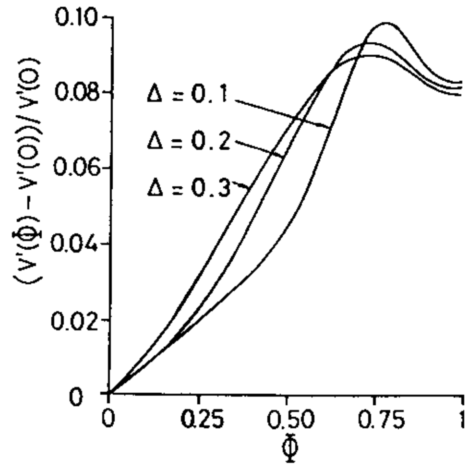
Fig. 1: The magnetic flux function Φ surfaces (the circular contours), the mod-B contours (the nearly vertical dashed lines), and the hot electron pressure surface (the crescent shaped contours) for helical ELMO Snaky Torus equilibria with zero thermal pressure and net current within each flux surface with $\Delta = 0.1$ and $\beta_h = 0.7\%$ (top figure), with $\Delta = 0.2$ and $\beta_h = 1.2\%$ (middle figure), and with $\Delta = 0.3$ and $\beta_h = 1.5\%$ (bottom figure).

Fig. 2: The differential volume profiles for the equilibria shown in Fig. 1 as a function of magnetic flux function Φ.



Δ = 0.1
Δ = 0.2
Δ = 0.3
Y
X
0.25
0.15
0.05
-0.05
-0.15
-0.25
0
0.1
0.2
0.3
0.4
0.5

FIG. 1



(V'(Φ) − V'(0))/V'(0)
Φ
Δ = 0.1
Δ = 0.2
Δ = 0.3
0.10
0.08
0.06
0.04
0.02
0
0.25
0.50
0.75
1

FIG. 2