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RENORMALIZED TURBULENCE THEORIES

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Renormalized Turbulence Theories

Two broad categories of nonlinear phenomena:

a) Coherent nonlinear states, Korteweg-de Vries, Nonlinear Schrodinger Eqn etc.

b) Turbulent states, Navier-Stokes, Magnetohydrodynamics, Vlasov-Maxwell...

Coherent solutions require very delicate balancing, very specific conditions. Time asymptotic state is expected to be turbulent for most systems.

A Typical system (which has been thoroughly investigated) is Vlasov-Poisson system. Both Perturbative as well as nonperturbative schemes have been used to study this system. A short summary of the highlights follows.

Perturbative

(i) Dupree's pioneering work,[1-2] named [6] 'Resonance Broadening'; replace the bare propagator $(\omega - \underset{\sim}{k}\cdot\underset{\sim}{u})^{-1} \rightarrow (\omega - \underset{\sim}{k}\cdot\underset{\sim}{u} + i\Gamma_k)^{-1}$ (renormalized), $i\Gamma_k$ stands for the stochastic diffusion from the turbulent fields.

(ii) Single renormalization theory replaced by fully renormalized theory, $i\Gamma_k$ is determined in terms of the spectrum rather than through the bare propagator.[3-4-5]

(iii) Dupree and Tetrault's β term, recovers energy conservation for e.s. drift waves.

Nonperturbative

Based on the Martin, Siggia, Rose (MSR) systematology.[12-13] $\Rightarrow$ Functional differential equation for Γ, the renormalized vertex.

Solutions remain elusive (Rigorous).

Ad-hoc Closure: Replace Γ by the bare vertex $\gamma \longrightarrow$ Direct Interaction Approximation (DIA).

Krommes introduces DIAC (Coherent) to obtain a solution in the diffusion approximation, Incorporation of the above features has led to the moder perturbation theories[6-10]. The theories are limited to second order in expansion, although the concept of 'order' is not unambiguous.

These theories fail to produce the commonly accepted weak turbulence limit, e.g, the Kadometsen spectrum equations[11].

energy conservation in e.s. drift waves is proved; equivalence with Dupree and Tetrault.

DIA does recover the weak turbulence equations in the appropriate limit, it also yields an expression for the dielectric function which agrees with the definition used in statistical mechanics[18-19].

The nonperturbative approach, however, has gone little beyond the perturbative results in physical applications to turbulence problems; e.g, the renormalized version of Kadometsev equations and the of the dielectric function have still not been derived.

The failure of Dupree's theory to yield the correct weak turbulence limit has been attributed to its lack of proper self-consistent treatment of the Vlasov Poisson system. The difference between

Dupree's treatment and the DIA can be seen by examining the propagators used in the two theories. Without going into details we state here that the propagator used by Dupree contains only the diffusion part (which is related to the self energy parts) and not the polarization part related say to the polarization cloud surrounding a test particle. It is possible, however, to construct a perturbation theory, in which all these effects, do appear in the appropriate order. One such theory is presented in these lectures. The theory is quite general, and can deal with the class of turbulence described by dissipative dispersive systems with a quadratic Nonlinearity. The turbulence is assumed to be stationary and homogeneous.

Correlation Expansion: A formal method called the 'correlation expansion' will be used to deal with the statistical ensemble of turbulence. The product of fluctuating quantities (the nonlinear terms) is decomposed into correlated and uncorrelated parts; this is done order by order. Either regular or diagrammatic techniques can be used. For formal purposes, the diagrammatic procedure will be adopted.

Renormalization: The existence of a formal reliable solution for the functionals of the fluctuating quantities requires renormalizability. In the present context it means that the compensating term added to renormalize the linear part of the operator must be exactly cancelled to each order by appropriate contributions from the nonlinearity. Schematically (N is the nonlinear term)

$$N \Rightarrow \underline{\text{Coherent}} + \underline{\text{Incoherent}}$$

$$\text{Coherent} + \text{Compensating terms} = 0 \quad \text{(Renormalization)}. \text{(a)}.$$

$$\text{Incoherent part} \equiv \text{Nonlinear source}$$

We shall now show that (a), the renormalization is indeed possible. The proof places the perturbation theory on a firm footing so that we can use with great confidence.

General theory will be presented first, and then Vlasov-Poisson system will be analysed in some detail.

General Theory

$$\mathcal{L}\, \phi(x) = N\, \phi(x) \qquad (5\text{-}a)$$

$\phi(x)$ is the field variable, $x = (\underset{\sim}{x}, t)$, L = linear operator, N = Nonlinear operator.

$\Rightarrow$ Fourier space $[\, k = (\underset{\sim}{k}, \omega)\,]$

$$G_k^{(0)^{-1}} \phi_k = \sum_{k_1+k_2=k} V_{k_1,k_2}\, \phi_{k_1} \phi_{k_2} \qquad (5\text{-}b)$$

Linear operator, bare linear propagator

strength of coupling

The field operator has a random(w) and an unrandom part (non necessarily d.c)

$$\phi_k = \bar{\phi}_k + \tilde{\phi}_k \qquad (5\text{-}c)$$

Ensemble averaging yields $[\langle\ \rangle]$ the set

(unrandom part) $$[G_k^{(0)}]^{-1} \bar{\phi}_k = \sum_{k_1+k_2=k} V_{k_1,k_2}\, \bar{\phi}_{k_1} \bar{\phi}_{k_2} + \sum_{k_1+k_2} V_{k_1,k_2} \langle \tilde{\phi}_{k_1} \tilde{\phi}_{k_2} \rangle \qquad (5\text{-}d)$$

(random part) $$[G_k^{(0)}]^{-1} \tilde{\phi}_k = 2 \sum_{k_1+k_2=k} V_{k_1,k_2} \left[\bar{\phi}_{k_2} \tilde{\phi}_{k_1} + \tfrac{1}{2} \tilde{\phi}_{k_1} \tilde{\phi}_{k_2} \right] \qquad (5\text{-}e)$$

To this equation for the fluctuating field, one adds the renormalizing term $i \Gamma_k \tilde{\phi}_k$ (add on both sides of 5-e) to obtain the basic equation

$$\tilde{\phi}_k = 2\, G_k \sum_{k_1+k_2=k} V_{k_1,k_2} \left[\bar{\phi}_{k_2} \tilde{\phi}_{k_1} + \tfrac{1}{2} \tilde{\phi}_{k_1} \tilde{\phi}_{k_2} \right] + G_k\, i \Gamma_k \tilde{\phi}_k \qquad (5\text{-}f)$$

where

$$G_k = \left[G_k^{(0)^{-1}} + i \Gamma_k \right]^{-1} \qquad (5\text{-}g)$$

is the renormalized propagator. (r.p).

Program :

(1) Successive iteration of (5-f) to an appropriate order

(2) Separation of each term into correlated and uncorelated parts.

(3) Cancellation of the correlated term by the compensating term $i\Gamma_K$ to each order thus obtaining an expression for $i\,\Gamma_K^{(n)}$.

(4) Showing that this cancellation can be done in every order to any desired order : thus proving the renormalizability.

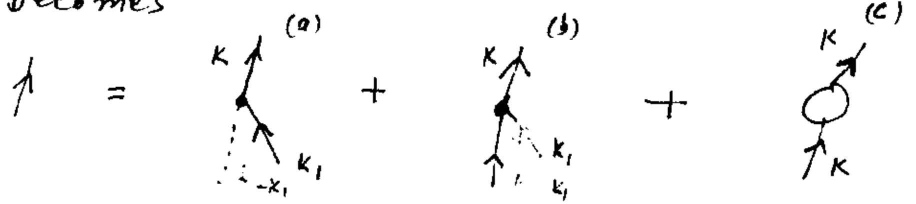
(5) obtaining expressions for physically interprettable quantities, e.g, the nonlinear dielectric response function etc

Technique : The program is best accompalished using diagrammatic techniques.

$$= \tilde{\varphi}_K \quad \text{or} \quad G_K \text{ when it is the top line.}$$

$$= 2\,\mathcal{C}_K \quad , \qquad O = i\,\Gamma_K \tag{6-a}$$

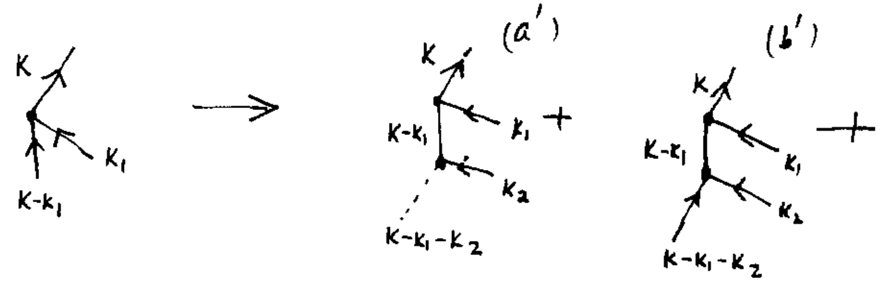
$\Rightarrow$ (5-f) becomes



(6-b)

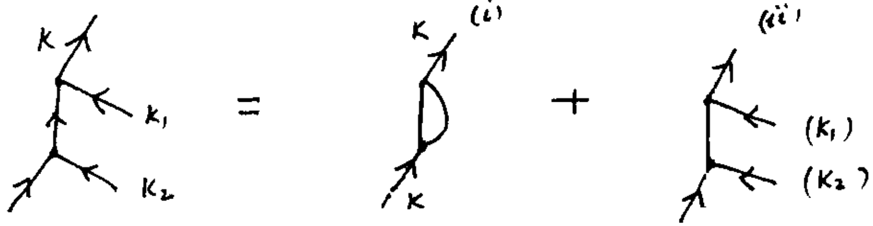
The Vertex in both (a) and (b) represents the strength V_{K_1,K_2}.
To make progress we must iterate Eq. 6(b) in some prescribed manner. Let us pick up term (b) (the basic nonlinear term). On

iteration it becomes [where energy momentum conservation at each vertex is displayed]



(7-a)

Clearly the strength of the processes is $|V|^2$. (b') in (7-a) has two parts (this is the guts of the procedure).



(7-b)

(i) corresponds to the case when $k_1+k_2=0$, (ii) where (k_1) and (k_2) are put in brackets ⇒ that $(k_1)+(k_2)\neq 0$. In other words (7-c)

$$\tilde{q}_{k_1}\,\tilde{q}_{k_2} = \langle\langle \tilde{q}_{k_1}\,\tilde{q}_{k_2}\rangle\rangle + (\tilde{q}_{k_1})(\tilde{q}_{k_2})$$

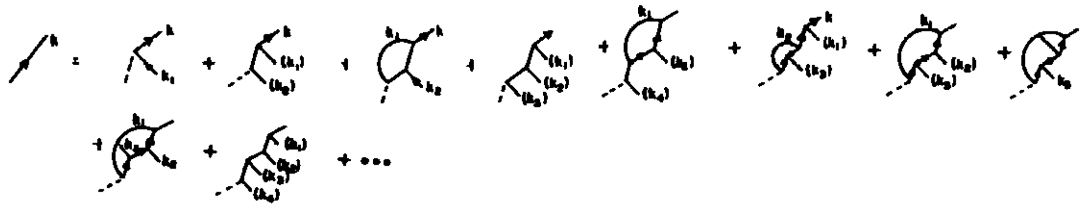
where the second part is uncorrelated, i.e $\langle(\tilde{q}_{k_1})(\tilde{q}_{k_2})\rangle = 0$.

(i) represent the self-energy terms. These self energy terms can be used to cancel all diagrams which contain $i\Gamma_k$ (the bubble) to yield

(7-d)

$$
\begin{aligned}
-i\Gamma_k = & \sum_{k_1} V_{k_1,k-k_1} G_{k-k_1} V_{k-k_1,k} \langle\langle \tilde{\varphi}_{k_1} \tilde{\varphi}^*_{k_1} \rangle\rangle \\
& + \sum_{k_1,k_2} V_{k_1,k-k_1} G_{k-k_1} V_{k_2,k-k_1-k_2} G_{k-k_1-k_2} V_{-k_1-k_2,k} \langle\langle \tilde{\varphi}_{k_1} \tilde{\varphi}_{k_2} \tilde{\varphi}^*_{k_1+k_2} \rangle\rangle \\
& + \sum_{k_1,k_2} V_{k_1,k-k_1} G_{k-k_1} V_{k_2,k-k_1-k_2} G_{k-k_1-k_2} V_{k_1,k-k_2} G_{k-k_2} V_{-k_2,k} \langle\langle \tilde{\varphi}_{k_1} \tilde{\varphi}^*_{k_1} \rangle\rangle \langle\langle \tilde{\varphi}_{k_2} \tilde{\varphi}^*_{k_2} \rangle\rangle \\
& + \sum_{k_1,k_2,k_3} V_{k_1,k-k_1} G_{k-k_1} V_{k_2,k-k_1-k_2} G_{k-k_1-k_2} V_{k_3,k-k_1-k_2-k_3} G_{k-k_1-k_2-k_3} \\
& \qquad \times V_{-k_1-k_2-k_3,k} \langle\langle \tilde{\varphi}_{k_1} \tilde{\varphi}_{k_2} \tilde{\varphi}_{k_3} \tilde{\varphi}^*_{k_1+k_2+k_3} \rangle\rangle + \cdots \; .
\end{aligned}
\tag{11b}
$$

After the full cancellation, the remaining part will be



(12a)

or

$$
\begin{aligned}
\tilde{\varphi}_k = & G_k \sum_{k_1} V_{k_1,k-k_1} \tilde{\varphi}_{k_1} 2\phi_{k-k_1} + G_k \sum_{k_1,k_2} V_{k_1,k-k_1} G_{k-k_1} V_{k_2,k-k_1-k_2} 2\phi_{k-k_1-k_2} (\tilde{\varphi}_{k_1})(\tilde{\varphi}_{k_2}) \\
& + G_k \sum_{k_1,k_2} V_{k_1,k-k_1} G_{k-k_1} V_{k_2,k-k_1-k_2} 2\phi_{k-k_2} \langle\langle \tilde{\varphi}_{k_1} \tilde{\varphi}^*_{k_1} \rangle\rangle \tilde{\varphi}_{k_2} \\
& + G_k \sum_{k_1,k_2,k_3} V_{k_1,k-k_1} G_{k-k_1} V_{k_2,k-k_1-k_2} G_{k-k_1-k_2} V_{k_3,k-k_1-k_2-k_3} 2\phi_{k-k_1-k_2-k_3} (\tilde{\varphi}_{k_1})(\tilde{\varphi}_{k_2})(\tilde{\varphi}_{k_3}) \\
& + G_k \sum_{k_1,k_2,k_4} V_{k_1,k-k_1} G_{k-k_1} V_{k_2,k-k_1-k_2} G_{k-k_1-k_2} V_{-k_1,k-k_2} G_{k-k_2} \\
& \qquad \times V_{-k_1-k_2-k_3,k} \langle\langle \tilde{\varphi}_{k_1} \tilde{\varphi}_{k_2} \tilde{\varphi}_{k_3} \tilde{\varphi}^*_{k_1+k_2+k_3} \rangle\rangle + \cdots \; .
\end{aligned}
\tag{12b}
$$

The proof of renormalization is a bit technical and involves several observations which are best illustrated using a particular example. Since the Vlasov-Poisson (V-P) system is a favorite of the theoreticians, we shall discuss it in detail. On our way to proving the general renormalizability of the system, we shall make contact with the known approaches in literature. It must be stated that although the example chosen is the V-P system, the methodology is true for all any dissipative dispersive system with quadratic nonlinearity. In the following pages there will be some repetition of the material covered earlier.

Vlasov - Poisson System

In the absence of external electric and magnetic fields, the perturbed Vlasov - Poisson system is the set of equations $[\ \hat{L}(k) = (-q/m)\, \underset{\sim}{k} \cdot \underset{\sim}{\partial},\ \underset{\sim}{\partial} \equiv \partial \underset{\sim}{v}]$

$$(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})\, f_k = \hat{L}(k)\, f_0\, \phi_k + \sum_{k_1} \hat{L}(k_1)\, f_{k-k_1}\, \phi_{k_1} \qquad (10\text{-}a)$$

and

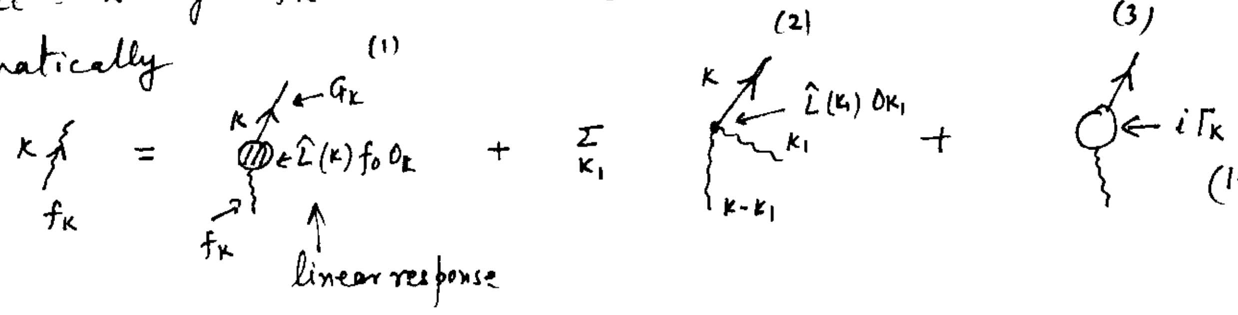
$$\phi_k = (4\pi q / k^2) \int d\underset{\sim}{v}\, f_k \equiv O_k\, f_k \qquad (10\text{-}b)$$

which are combined together to yield an equation of the type (5-f)

$$f_k = G_k\, \hat{L}(k)\, f_0\, O_k\, f_k + G_k \sum_{k_1 \neq k} \hat{L}(k_1)\, f_{k-k_1}\, O_{k_1}\, f_{k_1} + G_k\, i\Gamma_k\, f_k \quad (10\text{-}c)$$

with $\quad G_k = [\omega - \underset{\sim}{k} \cdot \underset{\sim}{v} + i\Gamma_k]^{-1} \qquad (10\text{-}d)$.

Notice that (10-c) has exactly the same structure as (5-f). However, for this system (which is a special case of (5-f)) we draw new diagrams to illustrate the basic thrust of the theory. We first notice that our equilibrium distribution function f_0 is now d.c ($k=0$), and hence k of f_k cannot be zero. We now write (10-c) diagramatically

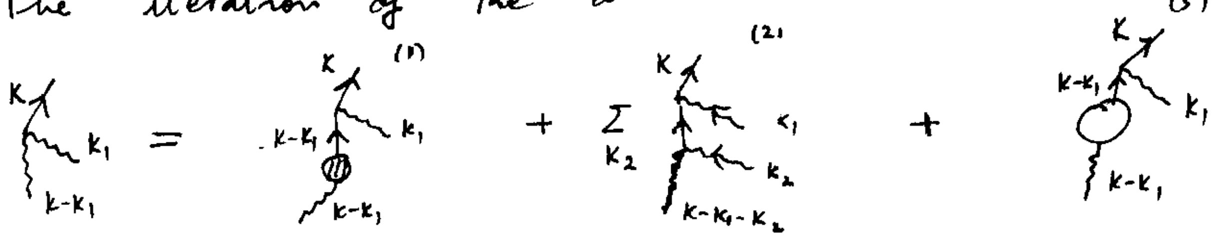


(10-e)

where all the symbols are self-explanatory, the solid lines are the propagators G_K, the wiggly lines are the fluctuating quantities f_K, the shaded bubble is indicative of the equilibrium distribution function f_0 (vertex of strength unity), the dot is the vertex of nonlinear coupling and the bubble denotes the renormalization term.

The first term is the lowest order coherent term, and the third term is the general frequency broadening (and shift) term. These two terms are not iterated.

The iteration of the 2nd term obviously yields



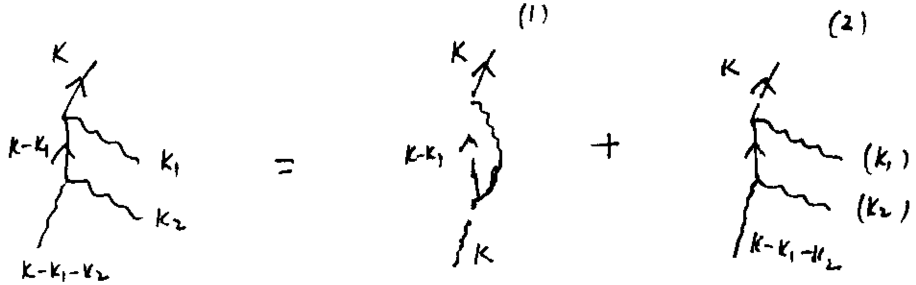
(11-a)

Points to Notice :-

(a) Since the K associated with either f_K or G_K cannot be zero, term (1) cannot give us a term proportional to the wave f_K, because that will require either $K_1 = 0$ or $K = K_1 = 0$. Thus (1) is an intrinsically incoherent (I.C) term, i.e, it cannot be phase coherent with f_K.

(b) Term (3) of (11-a) contains $i\Gamma_K$, so no more iteration.

(c) Before we iterate (2) further, we must separate the self energy part [which is just the correlation expansion mentioned before]

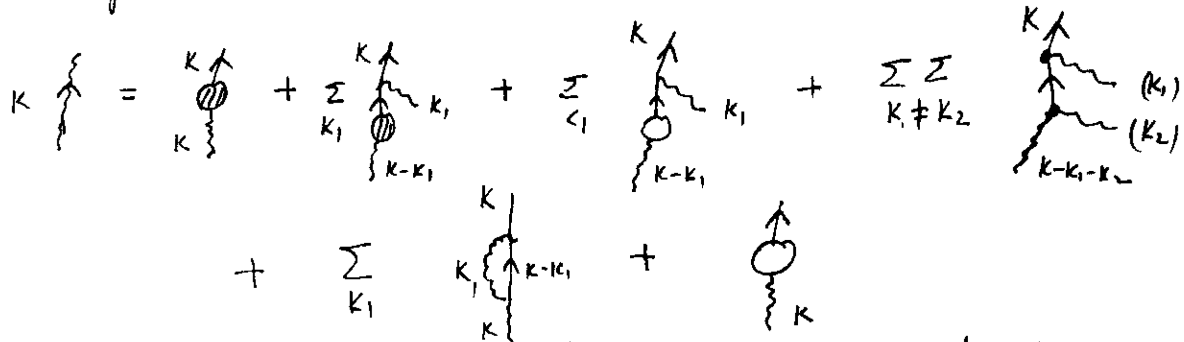


(12-a)

$$q_{k_1} q_{k_2} = \langle\langle q_{k_1} q_{k_2} \rangle\rangle + (q_{k_1})(q_{k_2})$$

Thus the correlation function has nonzero contribution only from the term $k_1+k_2=0$, all others are represented by (2) in 12-a. The waves with parenthesis mean that are uncorrelated with each other, $(k_1)+(k_2) \neq 0$.

Combining. (10-e), (11-a) and (12a) lead to



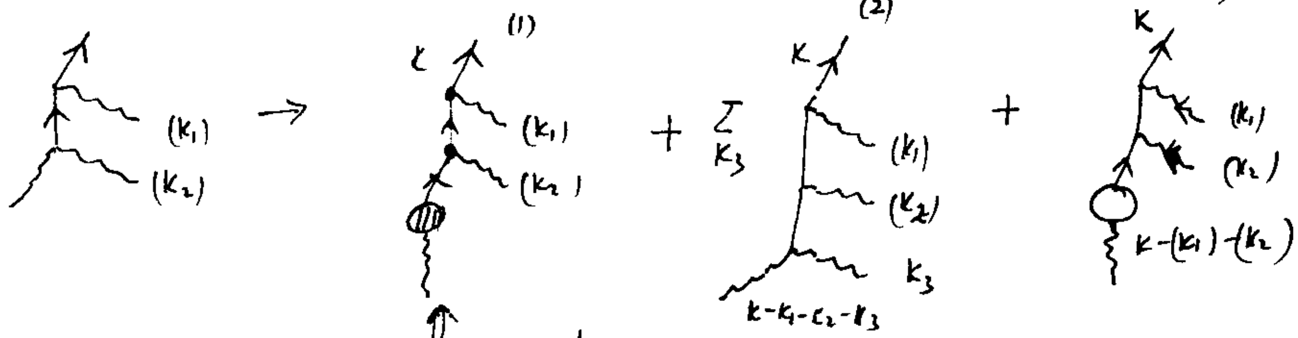
(12-b)

which is renormalized perturbation theory to the 2nd order, [where 'Order' is unambiguously defined as the maximum number of vertices appearing in the diagrams (excluding the shaded bubble which has unit strength)] provided we ask for the cancellation

(13-a)

of the compensating term . with the self energy term. Equation (13-a) is more or less the same theory as given by Rudakov - Tsytovich[5] and Choi - Horton.[20]

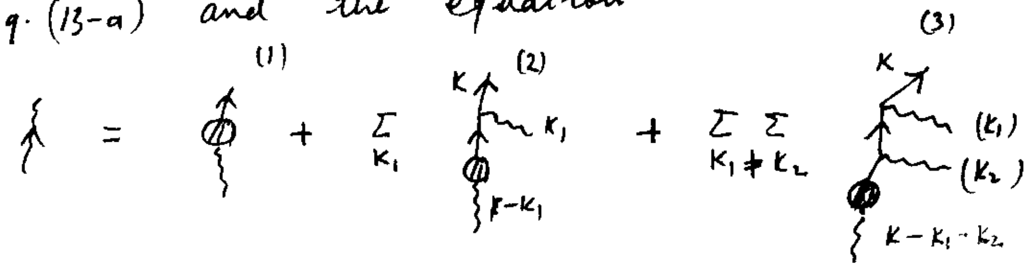
We must notice, however, that to this lowest order there is another contribution which comes from the term containing the double sum in (12-b). When iterated this term does indeed contain a second order term (because the shaded bubble vertex is of order unity). The second order contribution is the first term in the iteration



(13-b)

2nd order term

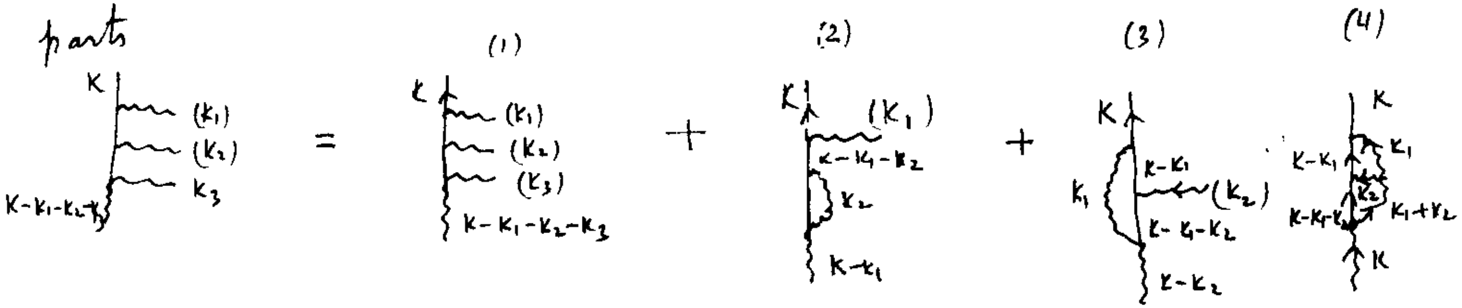
Thus the complete set of renormalized eqns. to 2nd order consists of Eq. (13-a) and the equation



(14-a)

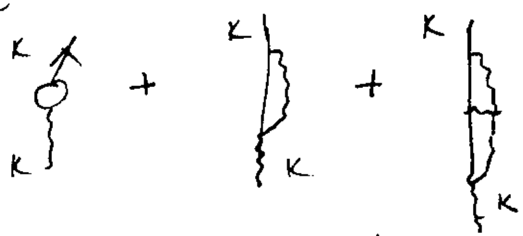
The inclusion of term (3) is the essential difference between the theory of Dupree - Tetrault[8] and previous theories. In this formalism, it [illegible] naturally in its order. Notice that the neglect of this term is a serious omission and had led to the non conservation of energy in the drift wave problem.

To continue with the procedure, we go back to Eq. (13-b), the first and the third term cease to iterate further, while the (2) term must decomposed into correlated and uncorrelated parts



(14-b)

Notice that the 2nd term of (14-b) and (3) term of (11-a) cancel because of (13-a). However, (13a) must be modified to yield ~~the~~ correct to the III order



$$= 0 \qquad (15a)$$

and similarly we can obtain a correct expression for f_k upto the third order and so on upto any desired order.

The general rules for the perturbation analysis are the following:

a) The terms containing the shaded bubble, and the 'hollow bubble' (frequency broadening and shift terms, $i\Gamma_k$ terms) are not iterated further. Neither are the self energy terms which are used to cancel $i\Gamma_k$ terms.

b) Any term not belonging to the above categories and containing an external wiggly line without parenthesis needs to be separated according to the correlation expansion prescription.

c) After separation, the terms without a self energy structure need iterating to produce the next order.

Proof of Renormalizability:

$i\Gamma_K$ can be expressed as a sum of self energy terms (all possible irreducible self energy structures), i.e, there is an exact cancellation between these two types of terms. Therefor f_K can be determined without explicit dependence on these terms.

The iteration procedure has several characteristic properties:

(1) To a given order all possible self energy structures must appear except those (and never those) that contain self-energy substructures. The latter part of the statement follows simply from the fact that we cease iterating on any term in which a simple self-energy diagram once appears. Therefore no higher order diagram can contain a self-energy structure of the lower order. Clearly the diagrams of the type

cannot appear. The allowable self energy graphs are called completely overlapping diagrams. Now, in order to prove that all types of completely overlapping patterns are possible we note that the prescription of correlation expansion allows all possible contractions which will give rise to all possible self-energy structures excepting the excluded ones already mentioned.

(2) A give diagram can appear only once. There is no repeated diagram.
All the diagrams can appear only in two ways; one is the iteraction, and the other is the separation [$K_2 \longrightarrow (K_1)$]. Since it is only the lower wiggly line which can contract with one or a combination of the passive lines [lines with (K) do not correlate with each other]: Thus the new diagrams result only from correlation with the lowest wiggly line (which was introduced in the last iteration), and could not possibly appear in the previous order.

(3) For non self-energy diagrams containing self-energy sub structures, all types of structures which appeared in the lower perturbative order are reproduced totally in higher perturbative orders.
All possible self-energy structures are produced by the correlation of the lowest wiggly line with the upper lines. The operation [$K_2 \rightarrow (K_1)$] in the higher order repeats all that happened in the lower order in addition to creating new terms. Thus regardless of the structure of the upper

[illegible] however, each higher order iteration will generate all diagrams of the described variety in the lower orders.

4) Because of (1), the self energy structures to the given order are the completely overlapping structures. Therefore it is not possible for any new type of self energy substructure (which was not present in the lower order) to appear in higher order order diagrams.

Observation (1) suggests that $-i\Gamma_K$ be chosen to be the sum of all types of possible self-energy structure which are completely overlapping. The combination (2)-(4) imply that as soon as the cancellation takes place for the lowest order diagrams containing $i\Gamma_K$, the same cancellation must take place for the higher order diagrams that have any structure plus $i\Gamma_K$ with the diagrams that have the same structure with all the self energy structures.

With the above proof, one can write down an expression for $i\Gamma_K$ (and hence G_K) as well as for the fluctuating distribution ~~fn.~~ function f_K to any desired order.

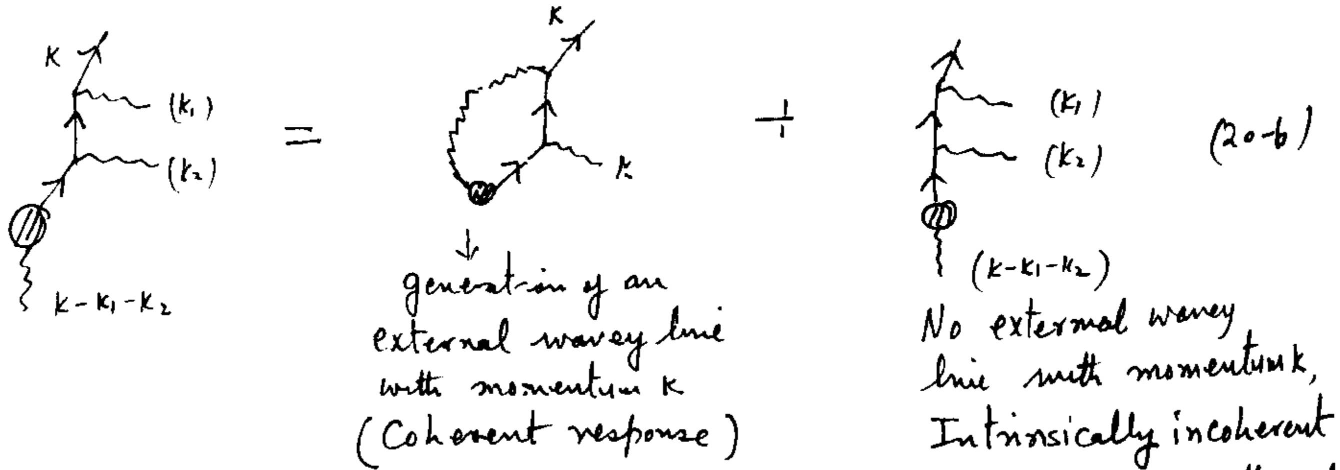
It is clear that both G_K and $i\Gamma_K$ must be complex indicating that $i\Gamma_K$ represents both frequency broadening and shift (nonlinear).

Coherent and Incoherent Distribution functions

With the concept of 'order' uniquely defined in our perturbation scheme, and with having demonstrated the renormalizability to all orders in the perturbation theory, we are now ready to sum up the results of the theory. It is important and also convenient to separate all the left over nonlinear terms into coherent and incoherent contributions. Even at the cost of repeating, we state that the coherent terms form that class of terms which are proportional to f_k, i.e, where the nonlinear terms have contracted in such a fashion that a wavey line with momentum k remains after contraction; the rest of the terms where no external wavey line can have a momentum k are grouped together to be termed the incoherent source. The point of this distinction is that all the coherent terms can be bracketed with the linear terms to define the renormalized propagator, while the incoherent terms act as a source for the fluctuating quantities. Generically, the final expression for f_k could be written as

$$\epsilon_k f_k = S_k \quad \text{or} \quad f_k = g_k S_k \tag{20-a}$$

where the renormalized propagator is $\epsilon_k^{-1} = g_k$ and S_k is the sum of all the intrinsically incoherent terms. We present the calculation to 2nd order. We notice that the third term in Eq.(14-a) still has an active wavey line which can contract with with the lines denote by (k_1) and (k_2). Clearly

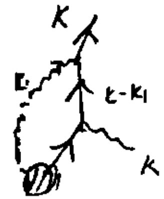


Notice that no other contraction is possible, because an attempt to contract (k_2) with $k-k_1-k_2$ will lead to a propagator (solid line) with momentum 0, a possibility which is not allowed. Within the framework of Eqs. (14-a) and (20-b), it is straight forward to break

$$f_k = \underset{\text{coherent}}{f_k^{(c)}} + \underset{\text{incoherent}}{f_k^{(i)}} \qquad (21\text{-}a)$$

where

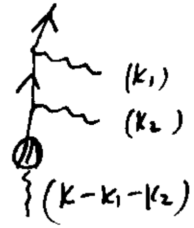
$f_k^{(c)} =$ 

$+ \sum_{k_1}$ 

(21-b)

and

$f_k^{(i)} = \sum_{k_1}$ 

$+ \sum_{k_1 \neq k_2 \neq k} \sum$ 

. (21-c)

Notice that $f_k^{(c)} \equiv \mathcal{C} f_k$, where $\mathcal{C}$ is some operator (where all wavey lines have been contracted). Thus (21-a) becomes

$$(1-\mathcal{C}) f_k = f_k^{(i)} \quad \Rightarrow \quad \hat{E}_k f_k = S_k \quad . \qquad (21\text{-}d).$$

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