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SPRING COLLEGE ON PLASMA PHYSICS

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NEOCLASSICAL MHD EQUATIONS, INSTABILITIES AND TRANSPORT IN TOKAMAKS

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Neoclassical MHD Equations, Instabilities and Transport in Tokamaks

Objective: **Extend Resistive MHD Theory from Pfirsch-Schlüter Regime to Experimentally Relevant Banana-Plateau Collisionality Regime**

Resistive MHD	**Neoclassical MHD**
short mean free path ($\lambda \ll Rq$)	long mean free path (λ / Rq arbitrary)
collisional plasma ($\nu_* \gg \epsilon^{-3/2}$)	collisionless plasma (ν_* arbitrary)
Pfirsch–Schlüter regime	banana–plateau regime

Procedure: **Add Viscous Force Effects (Primarily Due to Parallel Viscous Stress) to Resistive MHD**

TOKAMAK PLASMA BEHAVIOR MOSTLY DESCRIBED THROUGH RESISTIVE MHD MODELS

- Macroscopic Behavior (via changes in magnetic topology):

Internal disruptions -- $m/n = 1/1$ tearing modes		resistive MHD models
Mirnov oscillations -- $m/n = 2/1, 3/1$ tearing modes		
Skin current penetration -- $m/n \geq 3/1$ double tearing modes		
Major disruptions -- $m/n = 2/1, 3/2$ tearing modes		

- Anomalous Transport (via Magnetic Flutter ?):

Drift wave type instabilities -- ∇p driven kinetic microinstabilities.

Resistive MHD instabilities ($m, n \gg 1$) -- ∇p driven resistive ballooning
$\nabla \sigma_{\parallel}$ driven rippling modes

- <u>Resistive MHD Model</u> $[\lambda_{mfp} \ll \ell_{\parallel} \equiv ((\underline{b}\cdot\underline{\nabla})\ln B)^{-1} \sim R_0 q]$:

Density conservation: $\frac{\partial \rho_m}{\partial t} + \underline{\nabla}\cdot\rho_m \underline{V} = 0$

Momentum conservation: $\rho_m \frac{d\underline{V}}{dt} = \frac{1}{c}\underline{J}\times\underline{B} - \underline{\nabla} p$

$$\Rightarrow \underline{J}_{\perp} = \frac{c}{B^2}\underline{B}\times\left(\underline{\nabla} p + \rho_m \frac{d\underline{V}}{dt}\right)$$

Ohm's law: $\underline{E} + \frac{1}{c}\underline{V}\times\underline{B} = \underline{J}/\sigma$

Energy conservation: $\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$

PARALLEL VISCOSITY CANNOT BE NEGLECTED IN BANANA-PLATEAU COLLISIONALITY REGIME

- Parallel Momentum Balance Equation ($\underline{b} \equiv \underline{B}/B$):

$$mn\frac{dV_{\parallel}}{dt} = nqE_{\parallel} - \nabla_{\parallel} p - \underbrace{\underline{b}\cdot\underline{\nabla}\cdot\underline{\underline{\pi}}}_{\sim \nu\lambda^2[(\underline{b}\cdot\underline{\nabla})\ln B]^2 V_{\parallel}} - \underbrace{nqJ_{\parallel}/\sigma_{\parallel}}_{\sim \nu V_{\parallel}}$$

(viscosity: $\underline{b}\cdot\underline{\nabla}\cdot\underline{\underline{\pi}}$; friction: $nqJ_{\parallel}/\sigma_{\parallel}$)

- Ratio of Electron Parallel Viscosity to Friction:

$$\frac{\underline{b}\cdot\underline{\nabla}\cdot\underline{\underline{\pi}}_{\parallel e}}{nq_e J_{\parallel}/\sigma_{\parallel}} \sim \begin{cases} \lambda^2(\underline{b}\cdot\underline{\nabla})^2 \sim \epsilon^2\frac{\lambda^2}{R_0^2 q^2} \ll 1 & \text{in Pfirsch-Schlüter collisionality regime} \\ \gg 1 \,? & \text{in banana-plateau collisionality regime} \end{cases}$$

$\lambda \equiv v_T/\nu$ -- collisional mean free path

$\epsilon \equiv r/R_0 \ll 1$ -- inverse aspect ratio

collisionality $\begin{cases} \lambda \ll R_0 q \text{ -- Pfirsch-Schlüter} \\ R_0 q/\epsilon^{3/2} \gg \lambda \gg R_0 q \text{ -- plateau} \\ R_0 q/\epsilon^{3/2} \ll \lambda \text{ -- banana} \end{cases}$

Outline

- **Viscous Stresses and Forces**
- **Neoclassical MHD Equations**
- **Equilibrium**
- **Dielectric Constant**
- **Neoclassical MHD Instabilities**
- **Instability–Induced Transport**
- **Confinement Scaling**
- **Overall Transport Behavior**
- **Summary**

Viscous Stresses and Forces

- **Viscous Stress Tensor Has $\|, \times, \perp$ Components (Analogous to Heat Flux $\underline{q} = -\kappa_{\|}\nabla_{\|}T - \kappa_{\times}\underline{b} \times \underline{\nabla}_{\perp}T - \kappa_{\perp}\underline{\nabla}_{\perp}T$):**

$$\underline{\underline{\pi}} = \underset{(\text{parallel})}{\underline{\underline{\pi}}_{\|}} + \underset{(\text{cross})}{\underline{\underline{\pi}}_{\times}} + \underset{(\text{perpendicular})}{\underline{\underline{\pi}}_{\perp}}$$

	(parallel)	(cross)	(perpendicular)
Form: (Braginskii)	$\eta_0 \underline{\nabla V}$	$\eta_{3,4}\underline{b} \times \underline{\nabla V}$	$\eta_{1,2}(\underline{\nabla V})_{\perp}$
Magnitude	$\nu\left(\frac{\epsilon\lambda}{Rq}\right)^2$	ω_*	$\nu(k_{\perp}\rho)^2$

- **Dominant, Parallel Viscous Stress Has CGL Form for All Collisionality Regimes ($\underline{b} \equiv \underline{B}/B$):**

$$\underline{\underline{\pi}}_{\|} = (p_{\|} - p_{\perp})(\underline{b}\,\underline{b} - \underline{\underline{I}}/3)$$

- **Pressure–Anisotropy Caused by Flow Against $\underline{\nabla}B$:**

$$p_{\|} - p_{\perp} = -\frac{mn\mu\langle B^2\rangle}{\langle(\underline{b}\cdot\underline{\nabla}B)^2\rangle}\underline{V}\cdot\underline{\nabla}\ln B$$

Viscous Forces Due to Parallel Stress

- **Total Viscous Force:**

$$\underline{\nabla} \cdot \underline{\underline{\pi}}_{\|} = (p_{\|} - p_{\perp}) [(\underline{b} \cdot \underline{\nabla})\underline{b} - \underline{b}(\underline{b} \cdot \underline{\nabla}) \ln B] + \underline{b}(\underline{b} \cdot \underline{\nabla})(p_{\|} - p_{\perp}) - \underline{\nabla}(p_{\|} - p_{\perp})/3$$

- **Parallel Component of Viscous Force:**

$$\underline{B} \cdot \underline{\nabla} \cdot \underline{\underline{\pi}}_{\|} = -(p_{\|} - p_{\perp})(\underline{b} \cdot \underline{\nabla})B + (2/3)(\underline{B} \cdot \underline{\nabla})(p_{\|} - p_{\perp})$$

- **Flux Surface Average Parallel Viscous Force:**

$$\langle \underline{B} \cdot \underline{\nabla} \cdot \underline{\underline{\pi}}_{\|} \rangle = mn\mu U_\theta \langle B^2 \rangle$$

$$U_\theta \equiv \underline{V} \cdot \underline{\nabla}\theta / (\underline{B} \cdot \underline{\nabla}\theta) = V_\theta / B_\theta = V_{\|}/B + (cI/B^2)(\partial/\partial\psi)(\phi - \frac{T}{q}\ln n)$$

- **Parallel Momentum Balance Shows That Viscous Force Damps Poloidal Flows:**

$$mn\frac{d}{dt}\langle V_{\|} B \rangle = -\langle \underline{B} \cdot \underline{\nabla} \cdot \underline{\underline{\pi}}_{\|} \rangle + \cdots = -mn\mu U_\theta \langle B^2 \rangle + \cdots$$

Viscous Damping Frequency μ

- **Formula For μ Covers All Collisionality Regimes:**

$$\mu_e = \frac{2.3\epsilon^{1/2}\nu_e}{(1 + 1.07\nu_{*e}^{1/2} + 1.02\nu_{*e})(1 + 1.07\nu_{*e}\epsilon^{3/2})}$$

$$\nu_* \equiv \epsilon^{-3/2}\nu/\omega_b$$

- **μ Small in Pfirsch–Schlüter Regime ($\nu \gg \omega_b$):**

$$\mu \sim \epsilon^2\omega_b^2/\nu \sim \epsilon^2\nu(\lambda^2/R^2q^2)$$

- **μ Independent of ν in Plateau Regime ($\epsilon^{3/2}\omega_b \ll \nu \ll \omega_b$):**

$$\mu \sim \epsilon^2\omega_b$$

- **μ Reflects Viscous Drag on Parallel (Poloidal) Flow By Immobile Trapped Particles (Fraction $\epsilon^{1/2}$) in Banana Regime:**

$$\mu \sim \epsilon^{1/2}\nu$$

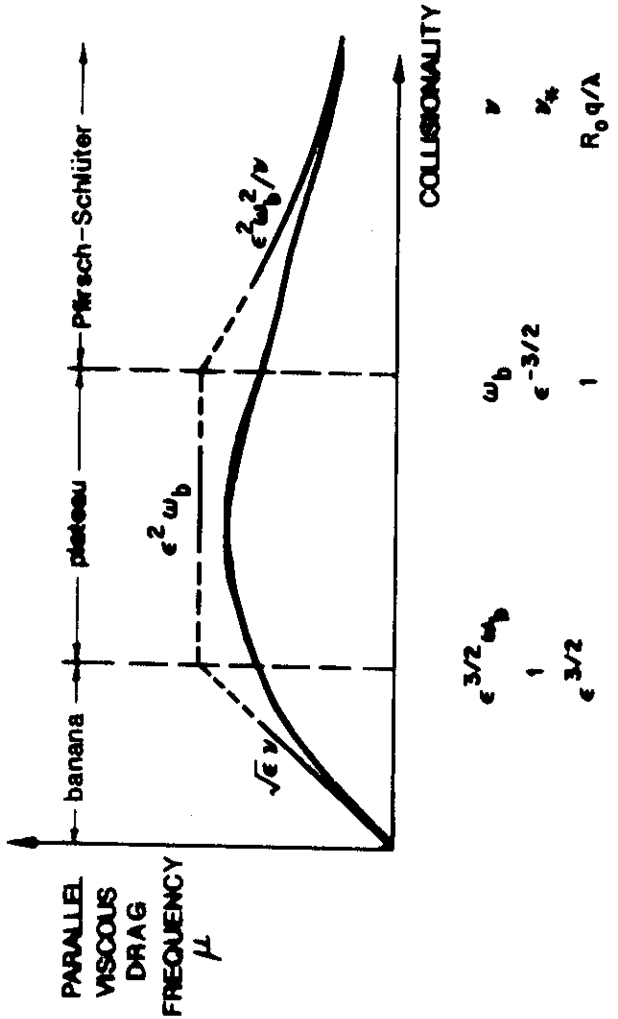


Fig. 1. Asymptotic scaling of the viscous drag frequency μ in the various collisionality regimes. The dashed portions of the curve indicate transition regions where the complete theory[15] provides smooth connection formulae.

Neoclassical MHD Equations

•Assumptions:

Small gyroradius, T=constant, $\partial B/\partial t = 0$ but $\partial \underline{B}/\partial t \neq 0$

•Starting Point is Conservation Equations:

$$\partial n/\partial t + \underline{\nabla}\cdot n\underline{V} = 0$$

$$mn\, d\underline{V}/dt = nq(\underline{E} + \frac{1}{c}\underline{V}\times\underline{B}) - \underline{\nabla}p - \underline{R} - \underline{\nabla}\cdot\underline{\pi}$$

where

$$\underline{R} \equiv nq(\underline{J}_\perp/\sigma_\perp + \underline{J}_\parallel/\sigma_\parallel)$$

•Perpendicular Flow ($\underline{E} = -\underline{\nabla}\phi + \frac{1}{c}\frac{\partial\psi}{\partial t}\underline{\nabla}\zeta$):

$$\underline{V}_\perp = \frac{c}{B^2}\times\underline{\nabla}(\phi + \frac{T}{q}\ln n) \qquad \sim \rho/\ell$$

$$+\frac{c}{nqB^2}\underline{B}\times(mn\frac{d\underline{V}_\perp}{dt} + \underline{R} + \underline{\nabla}\cdot\underline{\pi}) + \frac{\underline{\nabla}\psi}{R^2B^2}\frac{\partial\psi}{\partial t} \qquad \sim \rho^2/\ell^2$$

•Perpendicular Current:

$$\underline{J}_\perp = \frac{c}{B^2}\underline{B}\times\left[\underline{\nabla}P + \rho_m\frac{d\underline{V}_\perp}{dt} + \sum_s \underline{\nabla}\cdot\underline{\pi}_s\right] \qquad \sim \rho/\ell + \rho^2/\ell^2$$

Neoclassical MHD Equations Add Viscosity Effects to Resistive MHD

•**Density Conservation Equation:**

$$\partial n/\partial t + \underline{\nabla}\cdot n\underline{V}_{\perp e} + (\underline{B}\cdot\underline{\nabla})(nV_{\|i}/B - J_{\|}/eB) = 0$$

•**Charge Continuity Equation ($\underline{\nabla}\cdot\underline{J} = 0$):**

$$-\underline{\nabla}\cdot\frac{c\rho_m}{B^2}\underline{B}\times\frac{d\underline{V}_{\perp i}}{dt} = (\underline{B}\cdot\underline{\nabla})(\frac{J_{\|}}{B}) + \underline{\nabla}\cdot\left[\frac{c}{B^2}\underline{B}\times(\underline{\nabla}P + \underline{\nabla}\cdot\sum_s \underline{\underline{\pi}}_{\|s})\right]$$

•**Total Parallel Momentum Balance Equation:**

$$\rho_m\frac{d}{dt}(V_{\|i}B) = -(\underline{B}\cdot\underline{\nabla})P - \underline{B}\cdot\underline{\nabla}\cdot\sum_s \underline{\underline{\pi}}_{\|s}$$

•**Parallel Electron Momentum Balance (Ohms's Law):**

$$-\frac{I}{cR^2}\frac{\partial\psi}{\partial t} = -\frac{J_{\|}B}{\sigma_{\|}} - (\underline{B}\cdot\underline{\nabla})(\phi - \frac{T_e}{e}\ln n) + \frac{1}{ne}\underline{B}\cdot\underline{\nabla}\cdot\underline{\underline{\pi}}_{\|e}$$

•**Convective Derivative:**

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{V}_E\cdot\underline{\nabla}, \quad \underline{V}_E = \frac{c}{B^2}\underline{B}\times\underline{\nabla}\phi$$

Properties of Neoclassical MHD Equations

•**Simplify to Reduced Equations of Resistive MHD In Limit of Small Parallel Viscous Stress ($\underline{\underline{\pi}}_{\|}, \mu \longrightarrow 0$)**

•**Add to Resistive MHD Equations a Parallel Flow Evolution Equation**

•**Satisfy an Energy Conservation Equation:**

$$\frac{\partial}{\partial t}\int d^3x\left[\frac{\rho_m}{2}(V_{\perp i}^2 + V_{\|i}^2) + \frac{B_\theta^2}{8\pi} + P\ln n\right]$$

$$= -\int d^3x\left[\frac{J_{\|}^2}{\sigma_{\|}} + \frac{J_{\perp}^2}{\sigma_{\perp}} + \sum_s \underline{V}_s\cdot\underline{\nabla}\cdot\underline{\underline{\pi}}_{\|s}\right]$$

•**Dissipation Mechanisms**

$J_{\|}^2/\sigma_{\|}$ – **Joule heating**

$J_{\perp}^2/\sigma_{\perp}$ – **classical diffusion** ($\sim \nu\rho^2/r_p^2$)

$\sum_s \underline{V}_s\cdot\underline{\nabla}\cdot\underline{\underline{\pi}}_{\|s}$ – **neoclassical diffusion**
($\sim mn\mu U_\theta^2\langle B\rangle \sim \mu\rho_\theta^2/r_p^2$)

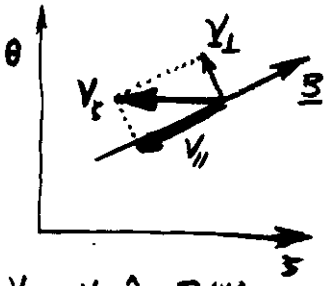
Equilibrium

- **After Transients ($t \gg 1/\mu_i \sim \tau_{ii}$), in Steady State:**

$$P = P_o(\psi), \qquad U_\theta = U_{\theta o}(\psi)$$

- **Viscous Force Damps Poloidal Ion Flow to Zero:**

$$0 = \langle \underline{B} \cdot \underline{\nabla} \cdot \sum_s \underline{\underline{\pi}}_{\parallel s} \rangle \simeq m_i n_i \mu_i U_{\theta i} \langle B^2 \rangle$$



$$\Rightarrow U_{\theta io} = 0, \;\; \underline{V}_o = \Omega_o R^2 \underline{\nabla}\zeta \;\; \text{(toroidal)}$$
$$\text{where} \;\; \Omega_o(\psi) = -c\frac{\partial}{\partial\psi}(\phi_o + \frac{T_i}{e}\ln n)$$

$$V_\zeta = \underline{V}_o \cdot \hat{\zeta} = \frac{\Omega_o(\psi)}{R}$$

- **Poloidal Electron Flow Driven by $E_\parallel^A$ and $\underline{\nabla} P_o$:**

$$U_{\theta e} = \left[-\frac{cI}{ne}\frac{dP_o}{d\psi} + \frac{e}{m_e \nu_e}\langle E_\parallel^A B \rangle \right] / (1 + \mu_e/\nu_e)\langle B^2 \rangle$$

- **Average Ohm's Law Includes Trapped Particle Effects (via viscosity) on Conductivity and Bootstrap Current:**

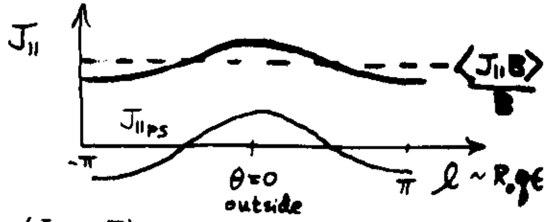
$$\langle B J_\parallel \rangle = \frac{\sigma_\parallel \langle E_\parallel^A B \rangle}{1 + \mu_e/\nu_e} - \underbrace{\frac{\mu_e/\nu_e}{1 + \mu_e/\nu_e} cI \frac{\partial P_o}{\partial \psi}}$$

conductivity trapped-particle (↑ $1 + \mu_e/\nu_e$)

$$\sim \frac{\mu_e}{\nu_e} c \frac{B}{B_\theta}\frac{\partial P_o}{\partial r} \text{ -- bootstrap current}$$

Neoclassical Parallel Currents and Transport

- **Parallel Current Includes Varying (Pfirsch–Schlüter, $J_{\parallel PS}$) and Average Currents ($\langle J_\parallel B \rangle$):**



$$J_\parallel = J_{\parallel PS} + \langle J_\parallel B \rangle / B$$

$$J_{\parallel PS} = -\frac{cI}{B}\frac{dP_o}{d\psi}(1 - \frac{B^2}{\langle B^2 \rangle}), \quad \langle J_{\parallel PS} B \rangle = 0$$

- **Viscosity–Induced Particle Flux Includes Varying (Pfirsch–Schlüter) and Average Parts:**

$$\langle n\underline{V}_\pi \cdot \underline{\nabla}\psi \rangle = \frac{cI}{e}\langle \left(\frac{1}{B^2} - \frac{1}{\langle B^2 \rangle}\right) \underline{B} \cdot \underline{\nabla} \cdot \underline{\underline{\pi}}_{\parallel e} \rangle + \frac{cI}{e\langle B^2 \rangle}\langle \underline{B} \cdot \underline{\nabla} \cdot \underline{\underline{\pi}}_{\parallel e} \rangle$$

varying -- Pfirsch-Schlüter   average -- banana-plateau

- **Net Viscosity–Driven Particle Transport Includes Pfirsch–Schlüter and Banana–Plateau Diffusion, and Neoclassical Pinch:**

$$D^{nc} \sim \nu_e \rho_e^2 q^2 + \mu_e \rho_{\theta e}^2$$

$$V_{\text{pinch}} \sim \left(\frac{\mu_e}{\nu_e}\right) \frac{cE_\parallel^A}{B_\theta}$$

Dielectric Constant

- **Perpendicular Dielectric Constant Can Be Identified Through Divergence of Ampere's Law:**

$$\underline{\nabla} \cdot \underline{J}_\perp = -\frac{1}{4\pi} \underline{\nabla} \cdot \underline{\underline{\epsilon}}_\perp \cdot \frac{\partial \underline{E}_\perp}{\partial t}$$

- **In Resistive MHD Polarization Drift Leads to Classical $\epsilon_\perp$:**

$$\underline{\nabla} \cdot \underline{J}_\perp = \underline{\nabla} \cdot \frac{c}{B^2} \underline{B} \times \rho_m \frac{d\underline{V}_\perp}{dt}$$

$$\Longrightarrow \epsilon_\perp^{cl} = 1 + c^2/V_A^2, \quad V_A \equiv B/(4\pi\rho_m)^{1/2}$$

- **In Neoclassical MHD Viscosity Adds Parallel Inertia Term That Leads (for $t \gg \mu_i^{-1}$) to Enhanced $\epsilon_\perp$ Which Depends on B_θ:**

$$\underline{\nabla} \cdot \underline{J}_\perp = \underline{\nabla} \cdot \frac{c}{B^2} \underline{B} \times \underline{\nabla} \cdot \sum_s \underline{\underline{\pi}}_{\| s} \sim IB \frac{\partial}{\partial \psi} \frac{c\rho_m}{B} \frac{\partial}{\partial t} (V_{\| i} B)$$

$$\Longrightarrow \epsilon_\perp^{nc} = 1 + c^2/V_{A\theta}^2, \quad V_{A\theta} \equiv B_\theta/(4\pi\rho_m)^{1/2}$$

$$\simeq \frac{B^2}{B_\theta^2} \epsilon_\perp^{cl} \sim 10^2 \epsilon_\perp^{cl}$$

Neoclassical MHD Instabilities

- **Analogous to Resistive MHD Instabilities:**

  Resistive-g $\Longrightarrow$ Pressure–gradient–driven via viscosity instead of magnetic field curvature

  Tearing $\Longrightarrow$ Neoclassical tearing driven primarily by $\underline{\nabla} p$ through bootstrap current

  Rippling $\Longrightarrow$ New $\tilde{n}$ type modes possible because $\sigma_{\| eff} = \sigma_\| / (1 + \mu_e/\nu_e) = f(n)$

- **Modes Purely Growing in Toroidal $\underline{E}_o \times \underline{B}$ Rest Frame:**

$$\omega_{\underline{E}} = -nc \frac{d\phi_o}{d\psi} \simeq -\frac{nq}{r} \frac{c}{B} \frac{d\phi_o}{dr}$$

$$\sim \omega_{*e} \quad \text{for} \quad \frac{e}{T_e} \frac{d\phi_o}{dr} \sim -\frac{1}{n_e} \frac{dn_e}{dr}$$

Pressure–Gradient–Driven Neoclassical MHD Instabilities

• **Growth Rate:**

$$\gamma_\mu = \frac{n^{2/3}}{\tau_{A\theta} S_\theta^{1/3}} \left[\left(\frac{\mu_e}{\nu_e}\right)^{1/2} \left(\frac{\beta_\theta}{2 r_p}\right) \right]^{2/3}$$

$$\beta_\theta \equiv 8\pi P_o / B_\theta^2, \quad \frac{1}{r_p} \equiv |r\, d \ln P_o/dr|, \quad S_\theta \equiv \tau_R/\tau_{A\theta}$$

$$\tau_{A\theta} = (4\pi\rho_m)^{1/2} R_o q / B_\theta, \quad \tau_R = 4\pi\sigma_{||} r^2/c^2$$

• **Resistive Layer Width:**

$$\delta_\mu = \frac{r}{2^{1/2} n^{1/3} S_\theta^{1/3}} \left[\left(\frac{\mu_e}{\nu_e}\right)^2 \left(\frac{\beta_\theta}{2 r_p}\right) \right]^{1/6} \frac{1}{(r\, dq/dr)^{1/3}}$$

• **Differences from Resistive–g Modes:**

1) ∇p free energy accessible via parallel viscosity instead of magnetic field curvature – mode is "universally" unstable in a tokamak plasma, leads to turbulent fluctuations.

2) Larger growth rate and resistive layer width:

$$\frac{\gamma_\mu}{\gamma_g} \sim \frac{\delta_\mu}{\delta_g} \sim \left(\frac{\mu_e}{\nu_e}\right)^{1/3} \left(\frac{B}{B_\theta}\right)^{2/3} \sim 5$$

Tearing–Type Neoclassical MHD Instabilities

• **Instability Driven Primarily by ∇p Via Bootstrap Current:**

Growth Rate and Resistive Layer Width Similar to Pressure–Gradient–Driven Mode for $\mu_e \beta_\theta / \nu_e \gg S_\theta^{-2/5}$

• **Nonlinear Evolution for $\Delta' > 0$ Low Mode Number (e.g., $m/n = 2/1$) Neoclassical MHD Tearing Modes:**

(1) Initial growth — $\tilde{\psi} \sim \exp(\gamma_\mu t)$

(2) Nonlinear growth driven by bootstrap current — $w \sim \tilde{\psi}^{1/2} \sim t^{1/2}$

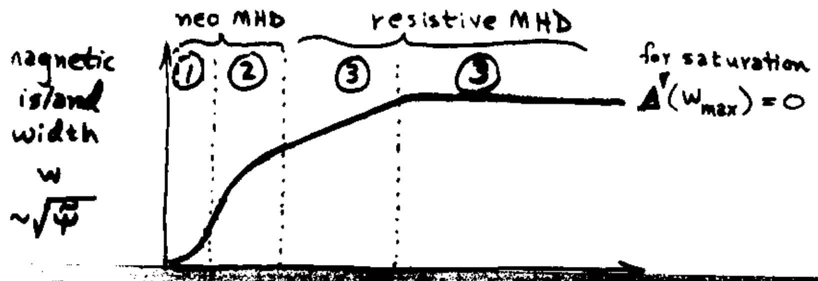
(3) Ultimate saturation $[\Delta'(w_{max}) \simeq 0]$ or disruption (overlapping islands) — $w \sim t$, Rutherford growth regime of resistive MHD

$$\frac{dw}{dt} = \frac{\eta}{\mu_0} [\Delta'(w) - \alpha w]$$

• **Nonlinear Evolution for Medium Mode Number (e.g., $n \sim 10$) Neoclassical MHD Tearing Modes ($\Delta' < 0$):**

Initial growth — $\tilde{\psi} \sim \exp(\gamma_\mu t)$

Growth into turbulent regime because many modes unstable and overlapping — turbulent saturation



Instability–Induced Transport

(Due to Either Type of Neoclassical MHD Instability)

•Mixing–Length Estimate of Turbulence Level:

$$\tilde{V}_{1x} \sim \gamma_\mu \delta_\mu, \quad \tilde{n} \sim \delta_\mu \frac{dn_o}{dr}, \quad \frac{\tilde{B}_x}{B} \sim \left(\frac{\gamma_\mu \delta_\mu^2}{4\pi\sigma_\parallel/c^2}\right)\left(\frac{k_\theta \delta_\mu^2}{L_s}\right)$$

•Strong Turbulence Estimate of Particle Diffusivity:

$$D \sim \sum \frac{\tilde{V}_x^2}{\gamma_\mu + k_x^2 D} \sim \sum \frac{\tilde{V}_x^2}{k_x^2 D}$$

$$\Longrightarrow D \sim \left[\sum \tilde{V}_x^2/k_x^2\right]^{1/2} \sim C_o \tilde{V}_x \delta_\mu \sim C_o \gamma_\mu \delta_\mu^2$$

$C_o \equiv$ spectrum sum ($\sim$ 10??)

•Strong Turbulence Estimate of Magnetic Field Diffusivity:

$$D_m \sim \frac{\langle \Delta x^2 \rangle}{\Delta \ell} \sim \sum \frac{(\tilde{B}_x/B)^2}{k_x^2 D_m}$$

$$\Longrightarrow D_m \sim \left[\sum (\tilde{B}_x/B)^2/k_x^2\right]^{1/2} \sim C_1 \delta_\mu \tilde{B}_x/B$$

•Electron Heat Diffusivity Estimated from $\chi_e \sim v_{T_e} D_m$:

$$\chi_e \sim v_{T_e} C_1 \delta_\mu \tilde{B}_x/B$$

Transport Coefficient Estimates

•Particle Diffusivity:

$$D \sim C_o D_\eta \left(\frac{\mu_e \beta_\theta}{\nu_e}\right)\left(\frac{r_q}{r_p}\right) \sim C_o D^{nc}$$

$$D_\eta \equiv \frac{4\pi\sigma_\parallel}{c^2} = \frac{\eta}{\mu_o} \text{ (in MKS)}, \quad \frac{r_q}{r_p} \equiv \left|\frac{d\ln P}{d\ln q}\right|$$

•Electron Heat Diffusivity (Valid for Arbitrary β_θ):

$$\chi_e \sim C_1 D_\eta \left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{\mu_e \beta_\theta}{\nu_e}\right)^2 \sim (5-10)D \gg D$$

•Other Diffusivity Coefficients:

$$D_{imp} \sim D, \quad \chi_i \sim (5/2)D$$

Confinement Scaling

•Ohmic Heating Power Balance Yields:

$$\beta_\theta \sim \left(\frac{m_e}{m_i}\right)^{1/6} \left(\frac{\mu_e}{\nu_e}\right)^{-2/3} \lesssim 0.3$$

$$\tau_E \sim n_e^{13/14} a^{4/7} R_o^{15/7} q^{5/7} B_\theta^{1/7}$$

(for $\mu_e/\nu_e \sim 1/\nu_{*e}, Z_{\text{eff}} \sim 1/n_e$)

($\tau_E \sim n_e a R_o^2 q^{3/4}$ for neo-Alcator scaling)

•Scaling with Input Power, Poloidal Field Strength, Density:

$$\tau_E \sim P_{in}^{-9/11} B_\theta^{8/11} n^{9/11} Z_{\text{eff}}^{-2/11} (\text{ohmic}, \ \mu_e/\nu_e \sim \epsilon^{1/2}/\nu_{*e})$$

$$\tau_E \sim P_{in}^{-5/7} B_\theta^{8/7} n^{3/7} \ (\text{transition}, \ \mu_e/\nu_e \sim \epsilon^{1/2}/\nu_{*e}^{1/3})$$

$$\tau_E \sim P_{in}^{-1/3} B_\theta^{8/3} n^{-1} Z_{\text{eff}}^{-2/3} \ (\text{ultimate}, \ \mu_e/\nu_e \sim \epsilon^{1/2})$$

(τ_E scales with B_θ because $\epsilon_\perp \sim c^2/V_{A\theta}^2$)

Overall Transport Behavior

•Radial Particle Flux ($\sim \delta^2$) Has Three Components:

$$\Gamma_\psi \equiv \langle n\underline{V} \cdot \underline{\nabla}\psi \rangle = \langle n_o \underline{V}_2 \cdot \underline{\nabla}\psi \rangle + \langle \tilde{n}_1 \underline{\tilde{V}}_1 \cdot \underline{\nabla}\psi \rangle + \langle (\underline{\tilde{B}}_1 \cdot \underline{\nabla}\psi) n_o \tilde{V}_{1\|}/B \rangle$$

•Second Order Flow Mostly Collisional Effects:

$\langle n_o \underline{V}_2 \cdot \underline{\nabla}\psi \rangle$ = classical + neoclassical (equil. and transient)

(transient neoclassical flow with $V_2 \sim (\mu_e/\Omega_e) V_{\theta i}$ for $d/dt > \mu_i$)

•Convection Driven Transport ($\tilde{n}_1, \tilde{V}_{1x}$ in phase):

$$\langle \tilde{n}_1 \underline{\tilde{V}}_1 \cdot \underline{\nabla}\psi \rangle \sim -D\,|\underline{\nabla}\psi|\,\frac{\partial n_o}{\partial r}, \quad D \sim C_o \gamma_\mu \delta_\mu^2 \text{ above}$$

•Magnetic Flutter Driven Transport (~Ambipolar, But Dominates Electron Heat Flux):

$$\sum q \langle \left(\underline{\tilde{B}}_1 \cdot \underline{\nabla}\psi\right) n_o \tilde{V}_{1\|}/B \rangle \sim \langle \left(\underline{\tilde{B}}_1 \cdot \underline{\nabla}\psi\right) \tilde{J}_\|/B \rangle$$

$$\sim \langle \left(\underline{\tilde{B}}_1 \cdot \underline{\nabla}\psi\right) \frac{c}{4\pi} (\underline{b} \cdot \underline{\nabla} \times \underline{\tilde{B}}_1) \rangle \sim 0$$

Summary

•Neoclassical MHD Equations Encompass:

Reduced equations of resistive MHD (for $\mu \longrightarrow 0$)

Arbitrary collisionality (banana, plateau, Pfirsch–Schlüter)

Neoclassical effects (equilibrium and transient transport)

Enhanced dielectric constant ($\epsilon_\perp \sim 1 + c^2/V_{A\theta}^2$)

Summary (continued)

•Neoclassical MHD Instabilities:

Are more virulent (γ, δ larger) than resistive MHD analogs

Evolve into regular nonlinear resistive MHD for low mode numbers ($n \sim 1, 2$) ⟹ preserves good correlation of theory with tokamak discharge phenomenology

Lead to small scale turbulence for medium mode numbers ($n \sim 3 - 30$)

•Neoclassical MHD Turbulent Transport

Has $\chi_e \gg D \sim D_{imp} \sim \chi_i$

Leads to τ_E scalings similar to experiment that depend on B_θ instead of B

NEOCLASSICAL MHD REFERENCES

Summary:

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