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Some aspects of the theory of infiltration

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SOME ASPECTS OF THE THEORY OF INFILTRATION

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# PHYSICAL-MATHEMATICAL MODEL

## Balance of mass

$$\frac{\partial \theta}{\partial t} = -\frac{\partial F}{\partial x} \qquad (1)$$

where $t$ is the time, $x$ is the spatial coordinate (positive downward), $\theta$ is the volumetric water content and $F$ is the volumetric flux.

In (1), $x$ and $t$ are the independent variables, and $\theta[x,t]$ and $F[x,t]$ are the dependent variables. Often it is more convenient to take $\theta$ and $t$ as the independent variables, and $x[\theta,t]$ and $F[\theta,t]$ as the dependent variables. In that case (1) is transformed to

$$\left.\frac{\partial x}{\partial t}\right|_{\theta} = -\left.\frac{\partial F}{\partial \theta}\right|_{t} \qquad (2)$$

## Darcy's law

$$F = -k\frac{\partial h}{\partial x} + k\cos\alpha \qquad (3)$$

$$= -D\frac{\partial \theta}{\partial x} + k\cos\alpha \qquad (4)$$

where $h$ is the pressure head, $k$ is the hydraulic conductivity, $\alpha$ is the angle between the $x$-direction and the vertical direction, and $D$ is the diffusivity ($= k\,dh/d\theta$)

## Soil properties

1. retentivity curve: $h[\theta]$ (hysteretic!)



2. conductivity curve: $k[\theta]$ (not hysteretic, but $k[h]$ is hysteretic!)

## Initial conditions

1. uniform

$$h = h_\infty$$
$$\theta = \theta_\infty \qquad x \geq 0 \qquad t=0 \qquad (5)$$

2. equilibrium

$$h = h_0 + x = h_e$$
$$\theta = \theta_e[h_e] \qquad x \geq 0 \qquad t=0 \qquad (6)$$

The subscripts 0 and $\infty$ (respectively) denote values at the soil surface ($x=0$) and at large depth ($x \to \infty$)

## Boundary conditions

1. constant water content

$$\theta = \theta_0 \qquad x=0 \qquad t=0 \qquad (7)$$

2. constant flux

$$F = F_0 \qquad x=0 \qquad t=0 \qquad (8)$$

3. constant pressure head $h_s$ on top of a crust

$$F_0[t] = \gamma'(h_s - h_0) \qquad (9)$$

# CLASSICAL SOLUTIONS

Four solutions of the form $x = \hat{x}[\theta,t]$ or $x = \hat{x}[h,t]$

A. Steady downward flows: see Raats, P.A.C. 1973. Steady upward and downward flows in a class of unsaturated soils. Soil Sci. 115: 409-413 (Especially Eqns 5, 8, and 10)

B. Time-invariant profiles moving at speed $u$ → wetting front of constant shape

$$x = u t - \int \left(1 + \frac{\theta}{k} u + \frac{1}{k} f_2\right)^{-1} dk \qquad (11)$$

with

$$u = \frac{k_0 - k_\infty}{\theta_0 - \theta_\infty} \qquad \text{speed of wetting front}$$

$$f_2 = \frac{\theta_0 k_\infty - \theta_\infty k_0}{\theta_0 - \theta_\infty} \qquad \text{flux relative to wetting front}$$

For a general discussion of time-invariant moving profiles see P.A.C. Raats and W. R. Gardner, Movement of water in the unsaturated zone near a water table. Amer. Soc. Agron. Drainage Monograph (1973 or 1974)

C. One-dimensional (horizontal) absorption with constant water content boundary condition

$$x = \lambda[\theta, \theta_0, \theta_\infty]\, t^{1/2} \qquad (12)$$

where $\lambda = x / t^{1/2}$ is the Boltzmann similarity variable

D One-dimensional (vertical) infiltration with constant water content boundary condition

$$x = \sum_{n=1}^{\infty} \lambda_n[\theta, \theta_0, \theta_\infty]\, t^{n/2} \qquad (13)$$

where $\lambda_1 = \lambda$. This is Philip's series solution.

Later we will see that recent quasi-analytical solutions are also of the form $x = \hat{x}[\theta \text{ or } k, t]$.



## QUASI-ANALYTICAL SOLUTIONS

Integral conditions are used to constrain approximate solutions. Integration of (2) between $\theta$ and $\theta_\infty$ gives

$$f - f_\infty = \frac{\partial}{\partial t} \int_{\theta_\infty}^{\theta} x \, d\theta \qquad (14)$$

Time-integral condition (used by Philip and Knight)

$$\int_0^t \{f_0[t] - f_\infty\}\, dt = \int_{\theta_\infty}^{\theta_0[t]} x[\theta, t]\, d\theta \qquad (15)$$

Space-integral condition (used by Parlange)

$$\int_0^\infty \{f_x - f_\infty\}\, dx = \frac{\partial}{\partial t} \int_{\theta_0}^{\theta_\infty} \tfrac{1}{2} x^2 \, d\theta \qquad (16)$$

$$= \frac{\partial}{\partial t} \int_0^\infty x(\theta - \theta_\infty)\, dx$$

INTERMEZZO

Special cases of 15

1. One-dimensional absorption: introducing (12) into (15) gives

$$Q = S t^{1/2} \rightarrow dQ/dt = f_0 = \tfrac{1}{2} S t^{-1/2} \qquad (17)$$

where $Q = \int_0^t f_0 \, dt$ is the cumulative absorption and $S = \int_{\theta_\infty}^{\theta_0} \lambda \, d\theta$ is the sorptivity

2. One-dimensional infiltration: introducing (13) into (15) gives

$$I = f_\infty t + \sum_{n=1}^{\infty} S_n t^{n/2} \rightarrow dI/dt = f_0 = f_\infty + \sum_{n=1}^{\infty} \tfrac{n}{2} S_n t^{n/2-1}$$

where $I = \int_0^t f_0 \, dt$ and $S_n = \int_{\theta_\infty}^{\theta_0} \lambda_n \, d\theta$ (18)

($S_1 = S$)

Philip's 2-term infiltration eqn.

$$I = St^{1/2} + At \rightarrow dI/dt = f_0 = \tfrac{1}{2}St^{-1/2} + A \qquad (19)$$

Large time asymptote

$$I = I_{capillary} + k_0 t \rightarrow dI/dt = f_0 = k_0 \qquad (20)$$

Estimates of $A/k_s$

| | |
|---|---|
| Green and Ampt | 2/3 |
| linear equation | 1/2 |
| Burger's equation | 0.36 |
| Talsma and Parlange | 1/3 |

Integration of Darcy's law

at $x$ $\quad f_x = -D\,\partial\theta/\partial x + k\cos\alpha$

at $x \rightarrow \infty$ $\quad f_\infty = k_\infty \cos\alpha$

Subtracting and dividing by $\{f_0[t] - k_\infty\cos\alpha\}$ gives

$$F[\Theta,t] = -\frac{D[\theta]}{f_0[t]-k_\infty\cos\alpha}\frac{\partial\theta}{\partial x} + K[\theta,t]\cos\alpha \qquad (21)$$

where $\Theta[\theta,t] = \dfrac{\theta - \theta_\infty}{\theta_0[t]-\theta_\infty}$ (22)

$$F[\Theta,t] = \frac{f[\theta,t]-k_\infty\cos\alpha}{f_0[t]-k_\infty\cos\alpha} \qquad (23)$$

$$K[\theta,t] = \frac{k[\theta]-k_\infty}{f_0[t]-k_\infty\cos\alpha} \qquad (24)$$

Integration from $x=0$ to $x$ gives (25)

$$x[\theta,t] = \{f_0[t]-k_\infty\cos\alpha\}^{-1}\int_\theta^{\theta_0[t]}\frac{D[\theta]}{F[\Theta,t]-\frac{k[\theta]-k_\infty}{f_0[t]-k_\infty\cos\alpha}\cos\alpha}\,d\theta$$



Introducing $x[\theta,t]$ into (15) gives

$$\{f_0[t]-k_\infty\cos\alpha\}\int_0^t\{f_0[t]-k_\infty\cos\alpha\}\,dt$$

$$\int_{\theta_\infty}^{\theta_0[t]}\frac{(\theta-\theta_\infty)\,D[\theta]}{F[\Theta,t]-\frac{k[\theta]-k_\infty}{f_0[t]-k_\infty\cos\alpha}\cos\alpha}\,d\theta \qquad (26)$$

This integral mass balance relates the surface water content $\theta_0[t]$ and the flux $f_0[t]$. If $\theta_0[t]$ is given, then $f_0[t]$ can be calculated, and vice versa. With $\theta_0[t]$ and $f_0[t]$ known, the water content profile can be calculated from the expression for $x[\theta,t]$ given earlier. If $F[\Theta,t]$ is chosen judiciously then no iterations are necessary (See papers by Philip and Knight)

THREE APPLICATIONS (of 25 and 26)
(see literature for details)

A One-dimensional absorption

$$x[\theta,t] = f_0[t]^{-1}\int_\theta^{\theta_0}\frac{D[\theta]}{F[\Theta,t]}\,d\theta \qquad (27)$$

$$f_0[t]\int_0^t f_0[t]\,dt = \int_\theta^{\theta_0}\frac{(\theta-\theta_\infty)\,D(\theta)}{F[\Theta,t]}\,d\theta \qquad (28)$$

It follows that

$$S = \left\{2\int_{\theta_\infty}^{\theta_0}\frac{(\theta-\theta_\infty)\,D[\theta]}{F[\Theta,t]}\,d\theta\right\}^{1/2} \qquad (29)$$

$$\lambda = 2S^{-1}\int_\theta^{\theta_0}\frac{D[\theta]}{F[\Theta,t]}\,d\theta \qquad (30)$$

B Constant flux absorption

$$Z = f_0\, x[\theta, t] = \int_{\theta}^{\theta_0[t]} \frac{D[\theta]}{F[\Theta, t]}\, d\theta \qquad (31)$$

$$T = f_0^2\, t = \int_{\theta_\infty}^{\theta_0[t]} \frac{(\theta - \theta_\infty) D[\theta]}{F[\Theta, t]}\, d\theta \qquad (32)$$

Note that $f_0$ only enters via the reduced variables $Z$ and $T$: $Z[\theta, T]$ profiles do not explicitly depend on $f_0$

Ponding time: $\theta_0 \to \theta_s$

$$T_p = f_0^2\, t_p = \int_{\theta_\infty}^{\theta_0 = \theta_s} \frac{(\theta - \theta_\infty) D[\theta]}{F[\Theta, t]}\, d\theta \qquad (33)$$

$$= 1/2\; S^2 \qquad (34)$$

$$t_p = 1/2 \left( S / f_0 \right)^2 \qquad (35)$$

C. Constant flux infiltration

Same as B, except that the terms resulting from setting $\cos\alpha = 1$ should be added. In particular the ponding time:

$$T_p = (f_0 - k_\infty)^2\, t_p = \int_{\theta_\infty}^{\theta_0 = \theta_s} \frac{(\theta - \theta_\infty) D[\theta]}{F[\Theta, t] - \dfrac{k[\theta] - k_\infty}{f_0 - k_\infty \cos\alpha} \cos\alpha}\, d\theta \qquad (36)$$

At ponding there is a switch from $f_0$ = constant to $\theta = \theta_s$ = constant

Note that $f_0$ enters not only via $Z$ and $T$ but also via the integrand.



Literature on quasi-analytical solutions

. Parlange
Series of 11 papers published in Soil Science in the period 1971-1975
See also his review on "Water transport in soils" in Ann. Rev. Fluid Mechanics 12: 77-102

.. Philip and Knight
Series of 3 papers published in Soil Science in the period 1973-1974
See also paper by Knight in "Advances in infiltration." ASAE conference Chicago 1983

1. Following are the abstracts of 5 papers using the approach of Philip and Knight

a) White, I., D. E. Smiles, and K. M. Perroux. 1979. Absorption of water by soil: the constant flux boundary condition. Soil Sci. Soc. Am. J. 43:669-664.

Absorption of water into soil as the result of a constant flux condition at the soil surface is examined.

Experiments for a fine sand show that the surface water content, movement of the wetting front, and the water content profiles may be predicted from the soil water diffusivity function using the notion of the flux-concentration relation of Philip (1973).

Reduced space and time variables $X = V_0 x$ and $T = V_0^2 t$ are introduced. Use of these variables greatly simplifies treatment of the system and reveals that the surface water content and the reduced position of the wetting front are uniquely defined by $T$ while the water content profiles at any value of $T$ are unique in terms of $X$.

b) White, I. 1979. Measured and approximate flux concentration relations for absorption of water by soil. Soil Sci. Soc. Am. J. 43:1074-1080.

The flux-concentration relation of Philip (1973) is calculated using moisture profiles measured during absorption of water by a fine sand when water was supplied at both constant hydraulic potential and constant rate. The general behavior of the measured flux-concentration relations was found to be consistent with those of the model soils calculated by Philip. Within the accuracy of the measurements, no significant time-dependence of the relation was found for the constant flux boundary condition over a large time span. Measurements of the time-averaged flux-concentration relation, however, suggest some significant time-dependence for short times. Simple approximations for the flux-concentration relation, based on these observations, are suggested. These approximations enable easily evaluated predictions to be made of the salient features of constant rate application of water to soil, which are sufficiently accurate for most practical purposes.

c) Perroux, K. M., D. E. Smiles, and I. White. 1981. Water movement in uniform soils during constant-flux infiltration. Soil Sci. Soc. Am. J. 45:237-240.

An analysis is presented for constant-flux infiltration of water in soil based on the flux-concentration relation. The analysis is compared with laboratory experiments on constant-flux infiltration into columns of fine sand and silty clay loam. It is shown that the effect of gravity is small for the early stage and that during this stage sufficiently accurate predictions of moisture profile development can be made by using the simpler absorption analysis of I. White, D. E. Smiles, and K. M. Perroux.

*Additional Index Words:* flux concentration, ponding.

d) Smiles, D. E., K. M. Perroux, S. J. Zegelin, and P. A. C. Raats. 1981. Hydrodynamic dispersion during constant rate absorption of water by soil. Soil Sci. Soc. Am. J. 45:453-458.

An analysis of hydrodynamic dispersion accompanying constant flux absorption of KCl solution by an initially relatively dry soil, is developed for the case when the hydrodynamic dispersion coefficient is pore water velocity-independent. It is shown that in this process both the water content and the soil water salt concentration are uniquely defined by $\theta(X,T)$ and $C(X,T)$, where $X = v_0 x$ and $T = v_0^2 t$ are space- and time-like coordinates, and $v_0$ is the constant surface flux of water.

Quasi-analytical methods based on the flux-concentration relation predict $\theta(X,T)$ while an error-function solution, based on a material coordinate $Q$ labeling parcels of water, predicts the salt profile.

The analysis is demonstrated using a chemically inert sandy soil. The results show that during transient, unsaturated flow a simple piston-flow model described the process over a range of water contents. The method may be extended to explore dispersion in structured and chemically reactive soils.

e) Boulier, J.F., J. Touma, and M. Vauclin. 1984. Flux-concentration relation-based solution of constant-flux infiltration equation: I. Infiltration into nonuniform initial moisture profiles. Soil Sci. Soc. Am. J. 48:245-251.

The quasi-analytical solution of the infiltration equation based on the flux-concentration relation for constant flux condition and initially uniform water content profile is extended here to nonuniform moisture profiles for fluxes either smaller or greater than the saturated hydraulic conductivity. In the latter case, the solution is developed for the postponding stage. It is shown that the measured flux-concentration relation is well-approximated by $F(\theta^*) = \theta^*$ and that no significant time dependence can be observed. The quasi-analytical solution is then successfully compared with both laboratory experiments performed on a sandy soil column and numerical solution of the Richards equation for the nonponding, preponding, and postponding stages of infiltration.

# INFILTRATION THROUGH CRUSTS

The infiltrability of soils is often strongly influenced by the resistance experienced by the water in crusts. The pressure drop in the crust implies that even under ponded conditions the largest pores in the subcrust soil remain air-filled.

Crust play a role in:

1. Judging the effects of various methods of tillage;
2. Judging the effects of relatively high ESP (= exchangeable sodium percentage);
3. Evaluation of the potential of "runoff farming";
4. The so-called "crust-method" for determining the hydraulic conductivity.

## Structure of crusts

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### SCANNING ELECTRON MICROSCOPE OBSERVATIONS ON SOIL CRUSTS AND THEIR FORMATION

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ABSTRACT

Scanning electron micrographs (SEM) of crusts of loessial soils are presented. SEM observations were performed on crusts formed by raindrop impact at various stages of their formation. The crust structure was compared to the natural undisturbed soil. During the crust formation, a middle-term stage developed at which coarse particles, stripped of the fine ones, composed the surface layer of the soil. At the final stage of the crust formation, the coarse particles were washed away, and a thin seal skin, about 0.1 millimeter thick, formed the uppermost layer of the soil. A depositional crust, which was formed mainly by the translocation of fine particles, was marked by the presence of a thin skin also about 0.1 millimeter thick, suggesting involvement of similar secondary mechanisms of formation. This work illustrates the use of SEM for the study of soil crust formation and structure.

## Steady flow through crusts

See the following paper