INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
CABLE: CENTRATOM - TELEX 460392 - I

SMR/220-32

COLLEGE ON SOIL PHYSICS
2 - 20 November 1987

"Steady Infiltration into Crusted Soils"
"Transient Absorption Through a Crust"
"Steady Infiltation from Line Sources and Forrows"

Professor P. RAATS
Institute for Soil Fertility
Haren, The Netherlands

Transactions, 10th International
Congress of Soil Science, Vol. I
Physics and Technology of Soils.
"Nauka" Publishing House, Moscow. 1974.

STEADY INFILTRATION INTO CRUSTED SOILS

P.A.C. Raats
Agricultural Research Service, U.S. Department of Agriculture, USA

Consider steady infiltration into a homogeneous soil profile with a crust at its surface. In the crust, the flux q is given by:

$$q = C(h_o - h) \qquad (1)$$

where C is the conductance of the crust, and h_o and h are the pressure heads at the top and bottom of the crust. In writing (1), the difference between the gravitational potentials at the top and at the bottom of the crust has been neglected. For a profile without a crust $C = \infty$. For a profile with an impermeable crust $C = 0$. The steady flux q in the soil is given by:

$$q = k[h] \qquad (2)$$

where $k[h]$ is the hydraulic conductivity of the soil at the pressure head h. In this paper I present a simple, graphical analysis of flows described by equations (1) and (2). References to earlier work are given elsewhere (Raats, 1973).

Plots of equation (1) for a given crust with a given pressure head at its surface and of equation (2) for a given soil are shown in figure 1. Equation (1) describes a straight line with slope C and intercept h_o on the h-axis. Equation (2) corresponds to a curve which is simply a plot of the hydraulic conductivity of the soil versus the pressure head. Any intersection of such a straight line and such a curve describes a solution of equations (1) and (2). The rate of infiltration and the corresponding subcrust pressure head are, respectively, the ordinate and the abscissa of the point of intersection.

For a given soil and a given crust, the maximum, steady rate of infiltration in the absence of poded water on the surface, q_m, is reached when $h_o = 0$. With $h_o = 0$, equation (1) describes a straight line through the origin. Two such straight lines and plots of equation (2) for two soils are shown in figure 2. The maximum, steady rates of infiltration and the corresponding subcrust pressure heads are, respectively, the ordinates and the abscissas of the intersections between the straight lines and the curves. Note that in soils with a large reduction in the hydraulic conductivity at pressure heads slightly below zero, a crust whose conductance is large may still have a significant effect. In other words, just a slight crust may cause the large pores to remain emply and thus considerably reduce the maximum rate of infiltration.

Let R be the rate of rainfall. If $R < q_m$ the rain is classified as non-ponding and if $R > q_m$ as ponding. If $R > q_m$ one can distinguish further a pre-ponding phase and a post-ponding phase. This generalizes the classification of Rubin (1966) for soils without crusts. For non-ponding rains $h_o < 0$. Of course, if water is allowed to accumulate on top of the crust, then the rate of infiltration will be larger than q_m.

The dimensionless flux q_* pressure head h_* and hydraulic conductivity k_* are defined by:

$$q_* = q/K \quad , \quad h_* = -h/h_c \quad , \quad k_* = k/K \qquad (3)$$

where K is the hydraulic conductivity of the soil when $h = 0$, and h_c is the critical pressure head defined by (cf., Raats and Gardner, 1971):

$$h_c = \frac{\int_0^{-\infty} k[h]\,dh}{K} \qquad (4)$$

In term of dimensionless variables, (1) and (2) become:

$$q_* = C_*(h_{*o} - h_*) \qquad (5)$$

$$q_* = k_*[h_*] \qquad (6)$$

where the dimensionless conductance C_* is defined by:

$$C_* = -h_c C/K \tag{7}$$

Values of h_c may range from -10 cm for a coarse sand to about -100 cm for a clay. Typical values of C and K are such that C_* can be of the order 1 but also several orders of magnitude smaller. In the following it will be shown that, if for a given combination of soil and crust $C_* < 1$, then, the crust will significantly reduce the maximum rate of infiltration.

A versatile, empirical relationship between k and h is given by (cf., Raats and Gardner, 1971):

$$k = \frac{K}{(h/h_{\frac{1}{2}K})^n + 1} \tag{8}$$

where $h_{\frac{1}{2}K}$ is the pressure head at which $k = \frac{1}{2}K$. The critical pressure head h_c corresponding to (8) is given by (Raats and Gardner, 1971):

$$h_c = \frac{\pi}{n\sin(\pi/n)} h_{\frac{1}{2}K} \tag{9}$$

Introducing (9) into (7) gives

$$C_* = -\frac{\pi C/K}{n\sin(\pi/n)} h_{\frac{1}{2}K} \tag{10}$$

Plots of equation (5) with $h_{*_0} = 0$ for several values of C_* and of equation (6) with (8) for n ranging from 2 to 16 are shown in figure 3. For $C_* = 1$ the dimensionless flux ranges from about .45 for a soil with $n = 2$ to about .85 for a soil with $n = 16$. For $C_* = 10$ the range is .9 to nearly 1, while for $C_* = .01$ the range is .037 to 0.013. Note that for $C_* = 1$ and 10 the soils with low values of n are affected most by the crust. Between $C_* = 1$ and .01 this trend gradually reverses itself.

The same graphical procedure can be used with other empirical relationships between k and h (cf., Raats and Gardner, 1971), including the relationships used by Hillel and Gardner (1969). Of particular interest is the form

$$k = Ke^{\alpha h} \tag{11}$$

The critical pressure head h_c corresponding to (11) is given by (Raats and Gardner, 1971):

$$h_c = -1/\alpha \tag{12}$$

Introducing (12) into (7) gives:

$$C_* = C/(\alpha K) \tag{13}$$

Introducing (11) into (5) and (6) gives:

$$C_* = -\frac{e^{h_*}}{h_*} \quad , \quad C_* = -\frac{q_*}{\ln q_*} \tag{14}$$

Plots of h_* and q_* as functions of C_* according to (14) are shown in figure 4.

References

Hillel D. and W.R.Gardner. Soil Sci. 108: 137-142, 1969.

Raats P.A.C. The role of the unsaturated zone in hydrology. J.Hydrol. (in preparation), 1973.

Raats P.A.C. and W.R.Gardner. Water Resources Res. 7: 921-928, 1971.

Rubin J. Water Resources Res. 2: 739-749, 1969.

Summary

Crusts play a key role in determining the amounts of runoff and erosion and in the performance of septic tank drainage fields ~~elds.~~ In this paper a simple graphical method for analyzing steady infiltration into crusted soils has been developed. The results clearly show that the reduction in the rate of infiltration caused by a given crust depends on the hydraulic properties of the underlying soil. It is shown that any crust/soil combination may be characterized by a single parameter which is equal to minus the product of the crust resistance and the so-called critical pressure head of the soil, divided by the hydraulic conductivity of the soil when saturated.

2

Résumé

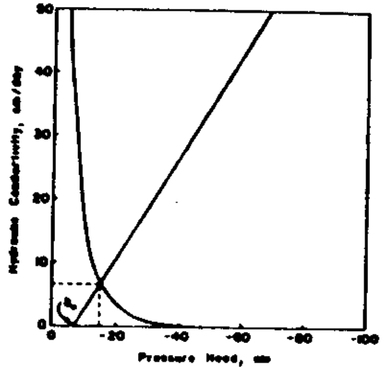
Dans un sol la croûte de surface joue un rôle déterminant dans l'importance du ruissellement et de l'érosion, et dans le rendement du drainage des fosses septiques. Dans cet article, on présente une méthode graphique simple qui permet d'analyser l'infiltration en régime permanent dans un sol avec une croûte. Les résultats montrent clairement que la réduction de vitesse d'infiltration due à une croûte dépend des propriétés hydrauliques du sol en dessous. On montre de plus que toute combinaison croûte/sol peut être caractérisée par un simple paramètre qui est égal à la valeur négative du rapport du produit de la résistance de la croûte par la charge critique du sol sur la conductivité hydraulique saturée du sol.

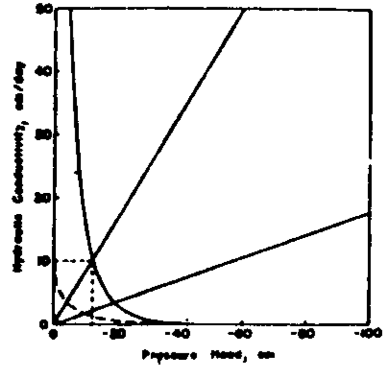
Zusammenfassung

Der Ausmass des Ablaufs und der Erosion sowie das Funktionieren überhaupt von Versickerungsanlagen für Abwässer werden hauptsächlich von Bodenkrusten bestimmt. In der vorliegenden Arbeit wurde eine einfache graphische Methode zur Analyse der Dauerabsickerung in verkrüstete Böden entwickelt. Die Ergebnisse zeigen eindeutig, dass die jeweilige durch eine bestimmte Kruste verursachte Verminderung der Absickerungsgeschwindigkeit von den hydraulischen Eigenschaften des darunterliegenden Bodens abhängig ist. Es wird dargelegt, wie jede Kruste-Boden-Kombination durch einen einzelnen Parameter charakterisiert werden kann. Dieser entspricht dem negativen Produkt des Krustenwiderstands mit dem sogenannten kritischen Drucküberschuss des Bodens, geteilt durch den hydraulischen Leitfähigkeit des wassergesättigten Bodens.

Резюме

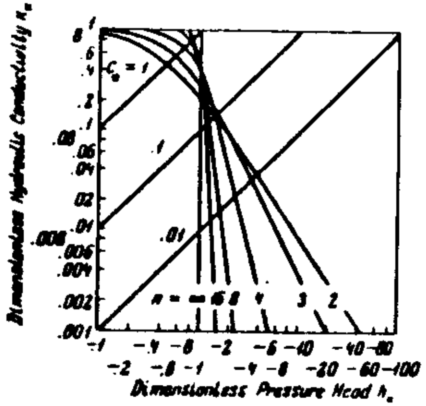
От поверхностных корковых образований значительно зависят объемы стока и эрозии, а также производительность полей орошения сточными водами. В данном докладе описан простой графический метод анализа установившейся инфильтрации в почвы, покрытые коркой. Полученные результаты четко показывают, что уменьшение скорости инфильтрации, вызванное данной коркой, зависит от гидравлических свойств подстилающей почвы. В докладе показано, что любое сочетание корка - почва может характеризоваться одним параметром, который равен отрицательной величине произведения сопротивления корки и так называемого критического напора почвы, деленного на коэффициент фильтрации насыщенной почвы.

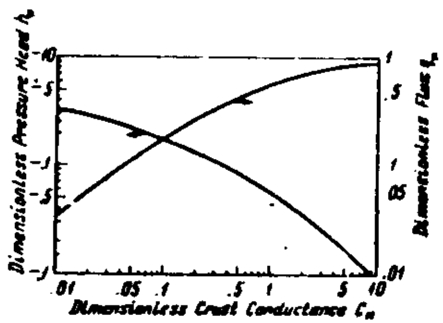




F i g. 1. Plots of equations (1) and (2) with arbitrary h_o.

F i g. 2. Plots of equations (1) and (2) for two crusts and two soils with $h_o = 0$.





F i g. 3. The effect of certain crusts upon the rate of infiltration into a class of soils.

F i g. 4. Dimensionless pressure head and dimensionless flux as a function of the dimensionless crust conductance for soils with an exponential dependence of the hydraulic conductivity upon the pressure head.

Transient absorption through a crust

$$h = h_\infty, \quad \theta = \theta_\infty, \quad x > 0, \quad t = 0 \qquad (37)$$

$$f_0[t] = -k[h_0]\,\partial h/\partial x\big|_{x=0} = \gamma(h_s - h_0[t]) \qquad (38)$$

Reduced variables

$$Z = \gamma x \qquad T = \gamma^2 t \qquad F = f/\gamma \qquad (39)$$

Flow problem in terms of reduced variables

$$\frac{\partial \theta}{\partial T} = -\frac{\partial F}{\partial Z} \qquad (40)$$

$$F = -k\,\partial h/\partial Z = -D\,\partial\theta/\partial Z \qquad (41)$$

$$h = h_\infty, \quad \theta = \theta_\infty, \quad Z > 0 \quad T = 0 \qquad (42)$$

$$F_0[T] = -k[h_0]\,\partial h/\partial Z\big|_{Z=0} = (h_s - h_0[T]) \qquad (43)$$

The crust conductance γ appears no longer explicitly. Unique dependence on Z, T

$$\Theta = \frac{\theta - \theta_\infty}{\theta_0[T] - \theta_\infty} \qquad (45)$$

$$F[\Theta, T] = F/F_0 \qquad (46)$$

Water content profiles

$$F_0[T]\, Z[\theta, t] = \int_\theta^{\theta_0[T]} \frac{D[\theta]}{F[\Theta, t]}\, d\theta \qquad (47)$$

$$F_0[T] \int_0^T F_0[T]\, dT = \int_{\theta_\infty}^{\theta_0[T]} \frac{(\theta - \theta_\infty)\, D[\theta]}{F[\Theta, t]}\, d\theta \qquad (48)$$

This solution can be generalised to vertical flow and time-dependent γ

$$F_0[T] = dQ/dT \qquad (49)$$

$$F_0 = h_s - h_0 \qquad (50)$$

$$Q = \int_0^T F_0[T]\, dT \rightarrow \text{function of } h_0 \qquad (51)$$

since both θ_0 and F_0 in the integral mass balance are functions of h_0

Hence T and h_0 are uniquely related

$$T = \int_0^Q F_0^{-1}\, dQ$$

$$= \int_{h_s - h_\infty}^{h_s - h_0} (h_s - h_0)^{-1} \frac{dQ}{d(h_s - h_0)}\, d(h_s - h_0)$$

$$= \frac{Q[h_s - h_0]}{h_s - h_0} + \int_{h_s - h_\infty}^{h_s - h_0} \frac{Q[h_s - h_0]}{(h_s - h_0)^2}\, d(h_s - h_0) \qquad (52)$$

Application of Philip/Knight approach

Smiles, D. E., J. H. Knight, and K. M. Perroux. 1982. Absorption of water by soil: the effect of a surface crust. Soil Sci. Soc. Am. J. 46:476–481.

The one-dimensional absorption of water by a uniform soil through a relatively impermeable surface layer is analyzed using conventional soil physics theory. The analysis permits calculation of the evolution of the water potential on the interface between the soil and the crust, the water content profiles, and the cumulative volume of water absorbed.

In the first instance the approach is tested for situations where the conductance of the crust is constant.

Extensions of the analysis to cases where the conductance is time-dependent are foreshadowed.

Critique and improvement (using eqn (16))

Parlange, J.-Y., W.L. Hogarth, and M.B. Parlange. 1984. Optimal analysis of the effect of a surface crust. Soil Sci. Soc. Am. J. 48:494–497.

Recently, experiments have been presented for the horizontal flow of water by soil with a surface crust and compared to an analytical model. It was found that the water content at the surface was consistently less than predicted by about a 1% water content. Such a small discrepancy may seem to be more of theoretical rather than practical interest. In fact, due to the rapid variation of soil properties near saturation, a very small error in water content at the surface implies that the model is unreliable to predict the relationship between cumulative absorption and soil water content at the soil surface, especially in the early stages of infiltration. A new model is developed here, based on an optimal principle. The results from this model are closer to the experimental observations, but some discrepancy remains. To eliminate the possibility of experimental error a simple theoretical example is also considered, where an exact solution exists, which illustrates in detail the accuracy of the optimal model.

Application to swelling material

Smiles, D.E., P.A.C. Raats, and J. H. Knight. 1982. Constant pressure filtration: the effect of a filter membrane. Chem. Eng. Sci. 37

Abstract—Theory developed to describe water movement and volume change in soils may be applied to many industrially important particulate liquid suspensions. The theory is used here to predict the important aspects of constant-pressure filtration where the filter membrane significantly impedes the escape of the liquid.

The method requires measured liquid content–liquid potential and liquid content—liquid diffusivity relations of the suspensions, and the conductance of the filter membrane.

Illustrative calculations for saturated bentonite slurry are presented. These calculations predict the evolution of the liquid and solid profiles in both material and physical space, and the cumulative volume of liquid expelled, as a function of time during filtration.

Experiments using bentonite at two different pressures, and with a range of values of membrane conductance, confirm integral predictions of the model.

Reprinted from the *Soil Science Society of America Proceedings*
Volume 34, Number 5, September–October 1970
677 South Segoe Road, Madison, Wisconsin 53711 USA

Steady Infiltration from Line Sources and Furrows[1]

P. A. C. Raats[2]

ABSTRACT

Steady infiltration from an array of equally spaced line sources or furrows at the surface of a semi-infinite soil profile is analyzed. The discussion is based on the assumption that the hydraulic conductivity is an exponential function of the pressure head. It is shown that, under this assumption, the matric flux potential and the stream function for plane flows satisfy the same linear partial differential equation. Explicit expressions for the stream function, the flux, the matric flux potential, the pressure head, and the total head are obtained. Some implications with regard to furrow irrigation are discussed. The solution provides a rational basis for the discussio of leaching under furrow irrigation.

Additional Key Words for Indexing: **partially saturated soil, plane flows, matric flux potential, stream function, furrow irrigation.**

Interest in two- and three-dimensional movement of water in partially saturated soils is increasing rapidly. Analytical approaches to such problems have been reviewed by Philip (1969). Resistance network analog and digital computer solutions for two-dimensional steady flows have been reviewed briefly by Bouwer (1969). Whisler (1969) analyzed steady flow in an inclined soil slab with an electric analog. Digital computer solutions for some two-dimensional transient flows have been developed recently by Amerman (1969)[3], Hornberger et al. (1969), Rubin (1968), and Taylor and Luthin (1969). One may expect that large computers and improved numerical methods will enable future development of numerical solutions for transient flow problems of practical sizes. Advantages of analog and numerical solutions are that actual soil properties, and complex geometries and boundary conditions can be dealt with. However, it appears desirable that analytical and analog or numerical studies be pursued side by side. Analytical solutions, even if they are based upon simplifying assumptions, usually have the advantage of showing more clearly the structure of the flow and its dependence upon parameters characterizing the soil and the flow geometry.

The main purpose of this paper is to present an analytical solution for flow from an array of line sources at the surface for a soil whose hydraulic conductivity is an exponential function of the pressure head. The results show the main features of steady furrow irrigation.

PLANE, STEADY FLOWS

Let x, y, z be a rectangular Cartesian coordinate system, the z-direction being the vertical direction. Consider a steady flow, which is plane with respect to the y-direction. The balance of mass may then be written as:

$$\frac{\partial\theta u}{\partial x} + \frac{\partial\theta w}{\partial z} = 0 \qquad [1]$$

where θ is the volume of water per unit bulk volume, and u and w are the components of the velocity in the x- and z-directions, respectively. Expressions for the components of the volumetric flux are given by Darcy's law:

$$\theta u = -k\frac{\partial H}{\partial x} \qquad \theta w = -k\frac{\partial H}{\partial z} \qquad [2]$$

$$= -k[h]\frac{\partial h}{\partial x} \qquad = -k[h]\frac{\partial h}{\partial z} + k[h] \qquad [3]$$

$$= -D[\theta]\frac{\partial \theta}{\partial x} \qquad = -D[\theta]\frac{\partial \theta}{\partial z} + k[\theta] \qquad [4]$$

where k is the hydraulic conductivity, H is the total head, which is the sum of the pressure head h and the gravitational head $-z$ (i.e., $H = h - z$), and $D = k\,dh/d\theta$ is the diffusivity. It is assumed that the steady flow is established through monotonic changes in pressure head, so that hysteresis need not be considered.

The forms of [3] and [4] suggest the introduction of a potential ϕ defined by:

$$\phi = \int_{h_o}^{h} k[h]\,dh = \int_{\theta_o}^{\theta} D[\theta]\,d\theta \qquad [5]$$

where h_o and θ_o are reference values and $\theta_o = \theta[h_o]$. In terms of ϕ, Darcy's law may be written as:

$$\theta u = -\frac{\partial\phi}{\partial x} \qquad \theta w = -\frac{\partial\phi}{\partial z} + k[\phi]. \qquad [6]$$

At any point the flux may be regarded as the sum of a matric component and of a gravitational component. The matric component is given by the gradient of ϕ and, therefore, it is appropriate to call ϕ the *matric flux potential*. Equation [5] implies that surfaces of equal pressure head, water content, and matric flux potential coincide. Some remarks concerning the history of the use of ϕ will be found at the end of this section.

Equation [1] is satisfied by a stream function ψ, defined as:

$$\theta u = -\frac{\partial\psi}{\partial z} \qquad \theta w = +\frac{\partial\psi}{\partial x}. \qquad [7]$$

By combining the pairs of expressions for the velocity components given in equations [6] and [7] one gets an analog of the Cauchy-Riemann conditions:

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial z} \qquad \frac{\partial\phi}{\partial z} - k[\phi] = -\frac{\partial\psi}{\partial z}. \qquad [8]$$

Note the presence of the term $-k[\phi]$. Substitution of [6] into [1] gives the partial differential equation for ϕ

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{\partial k[\phi]}{\partial z}. \qquad [9]$$

Elimination of ϕ from [8] gives:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{\partial k[\phi]}{\partial x}. \qquad [10]$$

At this point it is convenient to assume that k and h are related by an expression of the form:

$$k = k_o \exp(\alpha h). \qquad [11]$$

The parameters k_o and α characterize the soil. The parameter k_o represents the hydraulic conductivity of the saturated soil. Introducing [11] into [5] and solving for k gives (taking $\theta_o = 0$ and $h_o = -\infty$ as reference values):

$$k = \alpha\phi. \qquad [12]$$

From [11] and [12] it follows that the pressure head h is given by:

$$h = \frac{1}{\alpha}\ln\phi - \frac{1}{\alpha}\ln k_o/\alpha. \qquad [13]$$

The hydraulic head H is given by:

$$H = \frac{1}{\alpha}\ln\phi - \frac{1}{\alpha}\ln k_o/\alpha - z. \qquad [14]$$

Substitution of [12] into [9] gives:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = \alpha\frac{\partial\phi}{\partial z}. \qquad [15]$$

Substitution of [12] into [10] and using the first of equations [8] gives:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} = \alpha\frac{\partial\psi}{\partial z}. \qquad [16]$$

Surprisingly, the matric flux potential ϕ and the stream function ψ satisfy the same partial differential equation. Therefore, *corresponding to any plane flow problem there exists a conjugate problem which is obtained by interchanging the matric flux potential ϕ and the stream function ψ.* This generalizes a well-known property of plane flows satisfying Laplace's equation.

A transformation of the type introduced in equation [5] was given around 1880 by Kirchhoff in his lectures on heat transfer (Kirchhoff, 1894). For this reason, the transformation is often referred to as the Kirchhoff transformation and the potential ϕ is sometimes called the Kirchhoff potential. The matric flux potential was introduced by Klute (1952) in an analysis of horizontal movement of water in a partially saturated soil. Gardner (1958) presented a partial differential equation for movement of water in a partially saturated soil, using the matric flux potential as the dependent variable and including the effect of gravity. To solve the equation for steady flows, he suggested the technique of separating the variables. He also noted that the introduction of the exponential relationship between k and h, equation [11] above, leads to a further simplification, equation [15] above, making analytical solutions for certain steady flows possible. The first detailed example of such an analytical solution was given by Philip (1968) in his discussion of steady infiltration from buried point sources and spherical cavities. Wooding (1968) analyzed steady infiltration from a circular pond. Philip (1969) briefly discussed the solution for a single, horizontal line source in an infinite region.

STEADY INFILTRATION FROM A LINE SOURCE

Consider an array of equally spaced line sources at the soil surface (Fig. 1). Taking the line sources to be infinitely

[1] Contribution from the Corn Belt Branch, Soil & Water Conserv. Res. Div., ARS, USDA, Madison, Wis., in cooperation with the Wisconsin Agr. Exp. Sta. Approved for publication by the Director of the Research Division of the College of Agricultural & Life Sciences, Univ. of Wisconsin. Received Mar. 5, 1970. Approved June 19, 1970.

[2] Research Soil Scientist (Physics), USDA, Madison, Wis. and Assistant Professor, The University of Wisconsin.

[3] Amerman, C. R. 1969. Finite difference solutions of unsteady, two-dimensional, partially saturated porous media flow. Ph.D. Thesis, Purdue Univ., Lafayette, Ind. 221 p.

long, it is sufficient to describe the motion in the x,z-plane. Let $2L$ be the distance between the line sources and let $2S$ be the strength of the line sources per unit length in the y-direction. Before specifying the boundary conditions, it is convenient to introduce the following dimensionless variables:

$$X = (1/L)x \qquad Z = (1/L)z \qquad [17]$$

$$\Theta U = (L/S)\theta u \qquad \Theta W = (L/S)\theta w \qquad [18]$$

$$\Phi = (\alpha L/S)\phi \qquad \Psi = (1/S)\psi \qquad [19]$$

$$\gamma = (1/L)h \qquad \Gamma = (1/L)H. \qquad [20]$$

Note that:

$$S/L = \theta_\infty w_\infty \qquad [21]$$

where $\theta_\infty w_\infty$ is the uniform volumetric flux for $z \to \infty$. Introducing the dimensionless variables in [16] gives:

$$\frac{\partial^2\Psi}{\partial X^2} + \frac{\partial^2\Psi}{\partial Z^2} = \alpha L \frac{\partial\Psi}{\partial Z}. \qquad [22]$$

From the geometry of the problem it is evident that there are two bounding stream lines, (i) a stream line vertical downward from the line source, and (ii) a stream line running along the surface and vertical downward at the center between any two of the line sources. In terms of the dimensionless variables these conditions may be stated as:

$$X = 0 \qquad Z > 0 \qquad \Psi = 0 \qquad [23]$$

$$0 < X < 1 \qquad Z = 0 \qquad \Psi = 1 \qquad [24]$$

$$X = 1 \qquad Z > 0 \qquad \Psi = 1. \qquad [25]$$

An additional condition follows from the fact that for large Z the flow will be uniform (cf. equation [21]):

$$0 < X < 1 \qquad Z \to \infty \qquad \Psi = X. \qquad [26]$$

THE FLOW PATTERN

The solution of [22] subject to [23]-[26] may be written as:

$$\Psi = X + \Psi_p. \qquad [27]$$

The first term on the right-hand side of [27] may be regarded as a steady flow due to a source S distributed uniformly over $0 < X < 1$ at $Z = 0$. The second term represents a perturbation of the uniform flow arising from the source being actually at $X = 0$, $Z = 0$. The form of Ψ_p can be developed by separating the variables:

$$\Psi = X + \Psi_{pX} \cdot \Psi_{pZ} \qquad [28]$$

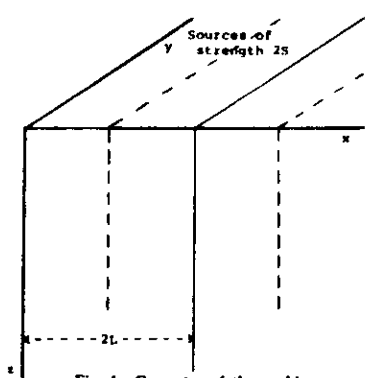


Fig. 1—Geometry of the problem.

where Ψ_{pX} depends upon X only, and Ψ_{pZ} depends upon Z only. Substitution of [28] into [22] and rearranging gives:

$$\Psi''_{pX}/\Psi_{pX} = -\Psi''_{pZ}/\Psi_{pZ} + \alpha L\,\Psi'_{pZ}/\Psi_{pZ} = -\lambda \qquad [29]$$

where λ is a constant. It follows that

$$\Psi''_{pX} + \lambda\Psi_{pX} = 0 \qquad [30]$$

$$\Psi''_{pZ} - \alpha L\Psi'_{pZ} - \lambda\Psi_{pZ} = 0. \qquad [31]$$

For $\lambda > 0$, general solutions of [30] and [31] are, respectively:

$$\Psi_{pX} = C_1 \cos\sqrt{\lambda}X + C_2 \sin\sqrt{\lambda}X \qquad [32]$$

$$\Psi_{pZ} = C_3 \exp p_n z \qquad [33]$$

where C_1, C_2, and C_3 are constants and p_n is given by:

$$p_n = -\alpha L/2 \pm \{\lambda + (\alpha L/2)^2\}^{1/2}. \qquad [34]$$

The expression for p_n is found by substituting [33] into [31]. After substituting [32] and [33] into [28], it is a simple matter to show that the solution for the dimensionless stream function Ψ which satisfies the conditions [23]-[26] is given by:

$$\Psi = X + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\exp - p_n z}{n}\sin n\pi X \qquad [35]$$

with

$$p_n = -\alpha L/2 + \{(\alpha L/2)^2 + (n\pi)^2\}^{1/2}. \qquad [36]$$

Note that according to [35] and [36] the flow pattern is a unique function of αL. In particular, the flow pattern is independent of the source strength S. For $\alpha L = 0$ equation [22] reduces to Laplace's equation and [36] reduces to:

$$p_n = n\pi \qquad [37]$$

In Fig. 2 the flow patterns for αL = 1, and 5 have been plotted. (All calculations related to this study were performed by the Univac 1108 digital computer at the University of Wisconsin. The number of terms used in the various infinite series ranged from one to several hundred, depending on the form of the series and the X, Z coordinates at which the series is to be evaluated). A comparison of Fig. 2a and 2b, indicates that with large αL gravity has a stronger effect resulting in a flow that is more concentrated around the Z-axis.

6

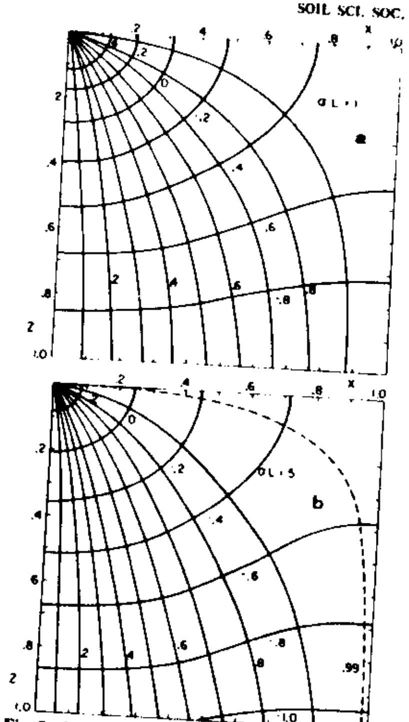


Fig. 2—Distribution of Ψ and Γ for αL = 1 and 5.

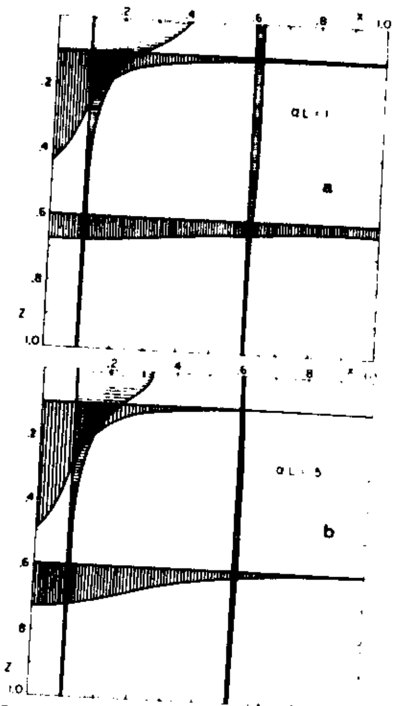


Fig. 3—Distribution of ΘU at X = 0.1 and 0.6, and of ΘW at Z = 0.1 and 0.6 for αL = 1 and 5.

THE DISTRIBUTION OF THE FLUX

Introduction of the dimensionless variables [17]-[19] into [7] gives:

$$\Theta U = -\frac{\partial\Psi}{\partial Z} \qquad \Theta W = \frac{\partial\Psi}{\partial X} \qquad [38]$$

Substituting [35] into [38] gives:

$$\Theta U = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{p_n \exp(-p_n Z)}{n}\sin n\pi X \qquad [39]$$

$$\Theta W = 1 + 2\sum_{n=1}^{\infty}\exp(-p_n Z)\cos n\pi X. \qquad [40]$$

Some examples of calculated flux profiles for αL = 1, and 5 are given in Fig. 3. For small depths the distribution of the flux is far from uniform, particularly if αL is large. For large depths ΘU approaches zero and ΘW approaches unity.

THE DISTRIBUTION OF THE MATRIX FLUX POTENTIAL

Substituting [12] into [6] and introducing the dimensionless variables [17]-[19] gives:

$$\Theta U = -(1/\alpha L)\frac{\partial\Phi}{\partial X}$$

$$\Theta W = -(1/\alpha L)\frac{\partial\Phi}{\partial Z} + \Phi \qquad [41]$$

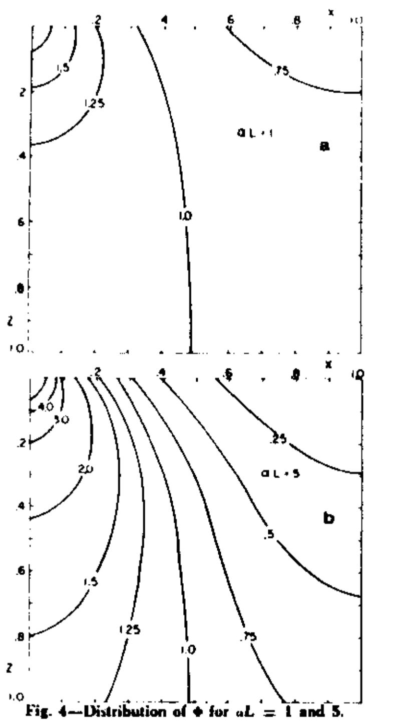


Fig. 4—Distribution of Φ for $\alpha L = 1$ and 5.

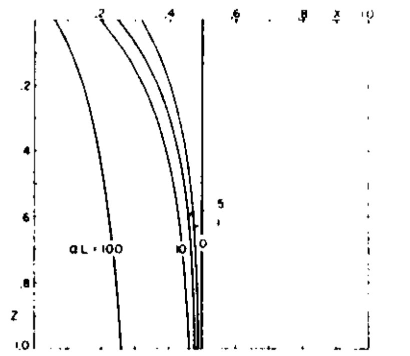


Fig. 5—Distribution of $\Phi = 1$ for $\alpha L = 0$, 1, 5, 10, and 100.

Substituting [39] into the first of these two equations and integrating gives the following explicit expression for the dimensionless matric flux potential:

$$\Phi = 1 + (2L\alpha/\pi^2) \sum_{n=1}^{\infty} \frac{p_n \exp(-p_n Z)}{n^2} \cos n\pi X \, . \qquad [42]$$

In deriving [42], use was made of the condition that as $Z \to \infty$ the potential $\Phi \to 1$. The distributions of Φ for $\alpha L = 1$ and 5 are given in Fig. 4. Equation [5] implies that surfaces of equal matric flux potential ϕ are also surfaces of equal pressure head h and water content θ. From [13] and [19] it follows that the dimensionless pressure head γ is given by:

$$\gamma = (1/\alpha L) \{\ln\Phi - \ln k_0 L/S\} \, . \qquad [43]$$

The dimensionless pressure head does not depend only upon αL, but also upon k_0, L, and S. The matric flux potential Φ varies over a wider range when αL is larger, and, in view of the correspondence noted above, the same applies to the pressure head and the water content. In the region with $\Phi > 1$ the soil is wetter than it is for $Z \to \infty$. In the region with $\Phi < 1$ the soil is drier than it is for $Z \to \infty$. In Fig. 5 the loci of $\Phi = 1$ are given for several values of αL.

THE DISTRIBUTION OF THE HYDRAULIC HEAD

From [14], [17], [19], and [20] it follows that the dimensionless hydraulic head Γ is given by:

$$\Gamma = (1/\alpha L) \{\ln\Phi - \ln k_0 L/S\} - Z \, . \qquad [44]$$

The function $\Gamma + (1/\alpha L) \ln k_0 L/S = (1/\alpha L) \ln\Phi - Z$, which depends only on X, Z, and αL, and differs from Γ merely by the constant $(1/\alpha L) \ln k_0 L/S$, is plotted in Fig. 2. The stream lines and the lines of equal hydraulic head are orthogonal (cf., Raats, 1967, Sewell and van Schilfgaarde, 1963). The stream lines and the lines of equal matric flux potential are not orthogonal, except in the limits $R = (X^2 + Z^2) \to 0$ and $Z \to \infty$ (see also Philip, 1968; Wooding, 1968).

DISCUSSION

It was pointed out already that the flow pattern is a unique function of αL. The distributions of the dimensionless flux and the dimensionless matric flux potential are also unique functions of αL. The parameter L is a characteristic length of the flow region. The parameter α is roughly a measure of the coarseness or fineness of the soil. Taking [length] as the dimension of h, the dimension of α will be $[\text{length}]^{-1}$. According to Philip (1968), a typical value of α is 0.01 and the range 0.05 cm^{-1} to 0.002 cm^{-1} seems likely to cover most applications. Most of the data presented by Rijtema (1965) fall in this range. In this paper calculations for $\alpha L = 1$, and 5 have been presented. For a very fine-textured soil, with $\alpha = 0.002$ cm^{-1}, $\alpha L = 1$ and 5 correspond to line source spacings $2L = 10$ and 50 m, respectively. For a coarse-textured soil, with $\alpha = 0.05$ cm^{-1}, the values $\alpha L = 1$ and 5 correspond to line source spacings $2L = 0.4$ and 2 m, respectively.

A comparison of Fig. 2a and 2b, of 3a and 3b, and of 4a and 4b indicates that with large αL gravity has a stronger effect, resulting in a flow that is more concentrated around the Z-axis. This illustrates once again that with a finer textured soil the dimensions of the flow system must be larger for gravity to have a noticeable effect (cf., Corey et al., 1965; Kraijenhoff van de Leur, 1962; Miller and Miller, 1956). This basic principle is sometimes overlooked in discussions of numerical studies of two-dimensional flow problems.

The solution for flow from a line source is also exact for infiltration of water at a certain pressure head from furrows whose contours coincide with lines of equal hydraulic head H, or, in other words, from furrows whose contours are orthogonal to the stream lines. Examples of such contours have been drawn in Fig. 2. If the furrow is filled with water and if $h = 0$ at points in the furrow for which $Z = 0$, then everywhere along the furrow surface the pressure head will be positive, with a maximum at the point of intersection between the contour and the Z-axis. This implies that close to the furrow the pressure head in the soil will be positive and that the exponential relationship between k and h, equation [11], will not be valid. Moreover, some of the data presented by Rijtema (1965) suggest that equation [11] gives a poor approximation in the range −10 to 0 cm pressure head. However, two factors offset this inadequacy of equation [11]. First, for small $R = (X^2 + Z^2)^{1/2}$ the flow pattern is relatively independent of the parameter αL. Second, in practice the furrow surface will often be clogged or consist of a thin, relatively impermeable crust, resulting in a significant pressure drop over a small distance (Bouwer, 1969; Hillel and Gardner, 1969).

The flux profiles in Fig. 3 show that for small depths the distribution of the flux is far from uniform. The rapid downward movement under furrows in sandy soils was already noted by Loughridge (1908). The nonuniformity of the flux has important implications with regard to leaching of salts, as was pointed out by Bernstein and Fireman (1957), Haise (1948), and King (1902). Figure 4 shows that for small depths the distribution of the matric flux potential, and thus of the pressure head h or the water content θ, is also far from uniform.

It was pointed out before that corresponding to any plane problem there exists a conjugate problem which is obtained by interchanging the matric flux potential ϕ and the stream function ψ. Consider the flow resulting from keeping the points along the Z-axis at one value of Φ, and the points along the X-axis and along $(X,Z) = (1,Z)$ at another value of Φ. For such a flow the stream function is given by the right-hand side of [42] (Fig. 4), while the matric flux potential is given by the right-hand side of [35] (Fig. 2). The practical significance of this type of a flow is not apparent.

For large Z the flow becomes uniform and thus the same as a steady flow due to a source S distributed uniformly over $0 < X < 1$ at $Z = 0$. In particular the one-dimensional largetime solution may be used, giving the shape of the wetting front moving steadily downward. The reader is referred to Philip (1969) for a review of the relevant literature.

LITERATURE CITED

1. Bernstein, L., and M. Fireman. 1957. Laboratory studies on salt distribution in furrow irrigated soil with special reference to the pre-emergence period. Soil Sci. 83: 249-263.
2. Bouwer, H. 1969. Theory of seepage from open channels. *In* Ven Te Chow [ed.]. Advances in hydroscience 5: 121-172. Academic Press, New York. 305 p.
3. Corey, G. L., A. T. Corey, and R. H. Brooks. 1965. Similitude for nonsteady drainage of partially saturated soils. Hydrology papers, Colorado State Univ., Fort Collins, Colorado. No. 9. 38 p.
4. Gardner, W. R. 1958. Some steady-state solutions of the unsaturated moisture flow equation with application to evaporation from a water table. Soil Sci. 85:228-232.
5. Haise, H. R. 1948. Flow pattern studies in irrigated course-textured soils. Soil Sci. Soc. Amer. Proc. 13:83-89.
6. Hillel, D., and W. R. Gardner. 1969. Steady infiltration into crust-topped profiles. Soil Sci. 108:137-142.
7. Hornberger, G. M., I. Remson, and A. A. Fungaroli. 1969. Numeric studies of a composite soil moisture groundwater system. Water Resources Res. 5:797-802.
8. King, F. H. 1902. Irrigation experiments in Wisconsin. USDA, Office of Experiment Stations. 119:313-352.
9. Kirchhoff, G. 1894. Vorlesungen über die Theorie der Wärme. Herausgegeben von M. Planck. Teubner Verlag, Leipzig. p. 13.
10. Klute, A. 1952. A numerical method for solving the flow equation for water in unsaturated materials. Soil Sci. 73: 105-116.
11. Kraijenhoff van de Leur, D. A. 1962. Some effects of the unsaturated zone on nonsteady free-surface groundwater flow as studied in a scaled granular model. J. Geophys. Res. 67:4347-4362.
12. Loughridge, R. H. 1908. Distribution of water in the soil in furrow irrigation. USDA, Office of Experiment Stations. 203:1-63.
13. Miller, E. E., and R. E. Miller. 1956. Physical theory for capillary flow phenomena. J. Appl. Phys. 27:324-332.
14. Philip, J. R. 1968. Steady infiltration from buried point sources and spherical cavities. Water Resources Res. 4: 1039-1047.
15. Philip, J. R. 1969. Theory of infiltration. *In* Ven Te Chow [ed.]. Advances in Hydroscience. 5:215-296. Academic Press. New York. 305 p.
16. Raats, P. A. C. 1967. The kinematics of soil water. p. 191-201. *In* Isotope and radiation techniques in soil physics and irrigation studies. Int. Atomic Energy Agency, Vienna.
17. Rijtema, P. E. 1965. An analysis of actual evapotranspiration. Agr. Res. Reports. no. 659, 107 p. Centre for Agricultural Publications and Documentation, Wageningen, The Netherlands.
18. Rubin, J. 1968. Theoretical analysis of two-dimensional transient flow of water in unsaturated and partially unsaturated soils. Soil Sci. Soc. Amer. Proc. 32:607-615.
19. Sewell, J. I., and J. van Schilfgaarde. 1963. Digital computer solutions of partially unsaturated steady-state drainage and subirrigation problems. Trans. Amer. Soc. Agr. Eng. 292-296.
20. Taylor, G. S., and J. N. Luthin. 1969. Computer methods for transient analysis of water table aquifers. Water Resources Res. 5:144-152.
21. Whisler, F. D. 1969. Analyzing steady-state flow in an inclined soil slab with an electric analog. Soil Sci. Soc. Amer. Proc. 33:19-25.
22. Wooding, R. A. 1968. Steady infiltration from a shallow circular pond. Water Resources Res. 4:1259-1273.

MULTIDIMENSIONAL (PLANE AND AXISYMMETRIC), QUASILINEAR STEADY FLOWS

A. The role of the matric flux potential $\varphi = \int_{-\infty}^{h} k\, dh = \int_{0}^{\Theta} D\, d\Theta$, and an assumed exponential relationship $k = k_0 \exp \alpha h$ were illustrated for plane flows using

- Raats, P.A.C. 1970. Steady infiltration from line sources and furrows. Soil Sci. Soc. Amer. Proc. 34: 709-714

and for axisymmetric flows using

- Raats, P.A.C. 1971. Steady infiltration from point sources cavities and basins. Soil Sci. Soc. Amer. Proc. 35: 689-694

Copies of these papers were distributed.

B. Flows to line and point sinks were discussed on the basis of figures from:

Raats, P.A.C. 1977. Laterally confined, steady flows of water from sources and to sinks in unsaturated soils. Soil Sci. Soc. Amer. J. 41: 294-303.

and

Warrick, A.W. and A. Amoozegar-Fard. 1977. Soil water Regimes near porous cup water samplers. Water Resources Research 13: 203-207.

C. Problems related to the so-called borehole permeameter were sketched briefly.
For further information:

- Philip, J. R. 1985. Approximate analysis of the borehole permeameter in unsaturated soil. Water Resources Research 21: 1025-1033
- Reynolds, W. D., D. E. Elrick, and B. E. Clothier. 1985. The constant head well permeameter: effect of unsaturated flow. Soil Sci. 139: 172-180
- Reynolds, W. D. and D. E. Elrick. 1985. In situ measurement of field-saturated hydraulic conductivity, sorptivity, and the α-parameter using the Guelph permeameter. Soil Science 140: 292-302

D It was pointed out that some of the problems with the borehole permeameter are related to the fact that the pressure of the water in the borehole is larger than the atmospheric pressure. A disk permeameter applying water at a pressure slightly below atmospheric pressure is being developed in Australia and Canada. Related theorie was discussed on the basis of

Wooding, R. A. 1968. Steady infiltration from a shallow circular pond. Water Resources

Research 4: 1259-1273.

- Weir, G. J. 1987. Steady infiltration from small shallow circular ponds. Water Resources Research. 23: 733-736.

E Downward flow around impermeable regions was discussed using illustrations from

- Maâledj, M. and L. Malabard. 197. Resolutions analogiques et numérique de problèmes d'irrigation des sols par canaux equidistant. C. R. Acad. Sc. Paris 276
- Babu, D. K. 1979. Steady state moisture flows in unsaturated soil containing a plane, isolated, impermeable barrier. Adv. Water Resour. 1: 35-46.
- Philip, J. R. 1987. An analog for infiltration and unsaturated seepage. Eos 68: 153-154.