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"Stochastic Approach of Soil Water Flow Through the Use of
Scaling Factors: Measurement Simulation "

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STOCHASTIC APPROACH OF SOIL WATER FLOW THROUGH THE USE OF SCALING FACTORS. MEASUREMENT AND SIMULATION.

1. Introduction.

One of the major problem met by hydrologists and soil physicists is to cope with spatial variability of soil hydraulic properties. They must be used to describe water flow in the vadose zone, but they can exhibit a very large degree of spatial variations as shown by many authors.

The scaling technique appears to be a promising simplified method for describing spatial variability of these properties (Warrick et al., 1977, Simmons et al., 1979, Russo and Bresler, 1981) and for modeling unsaturated flow processes at a large scale (Peck et al., 1977, Sharma and Luxmoore, 1979, Warrick and Amoozegar-Fard, 1979, Clapp et al., 1983, Ahuja et al., 1984).

The purpose of this paper is to provide more evidences of the interest of this technique.

2. Theory.

The method of scaling which is extensively used in hydraulics and fluid mechanics, has not been widely developed in soils physics, most probably due to the fact that vigorous assumptions are never met in nature. It will however be shown that heterogeneity from location to location within a field or at the scale of a watershed may be approximated with the use of a scaling coefficient for each site.

Scale factors derived from similitude analysis were introduced in Soil Science by Miller and Miller (1956), for soil water properties such as, hydraulic

conductivity $K(\theta)$, soil water pressure head $h(\theta)$ and diffusivity $D(\theta)$, assuming that water flows are governed by hydrodynamic laws of surface tension (σ) and viscosity (μ).

With the concept of similar media associated with the invariance of σ and μ, it can be shown from dimensional analysis that : :

$$h_i \lambda_i = C_1 \quad ; \quad K_i/\lambda_i^2 = C_2 \quad ; \quad D_i/\lambda_i = C_3 \qquad /1/$$

where λ_i characterizes the pore scale of the element i, C_1, C_2 and C_3 are three constants for isothermal conditions.

For similar media, the internal geometry differs only by λ_i. Such materials would have identical porosity and the same relative particle and pore size distributions. The consequences are that for a given volumetric water content $h_i(\theta)$, $K_i(\theta)$, $D_i(\theta)$ are related to reference values $h^*(\theta)$, $K^*(\theta)$ and $D^*(\theta)$ by :

$$h_i(\theta) = 1/\alpha_i \; h^*(\theta) \quad ; \quad K_i(\theta) = \alpha_i^2 \; K^*(\theta) \quad ; \quad D_i(\theta) = \alpha_i \; D^*(\theta) \qquad /2/$$

where $\alpha_i = \lambda_i/\lambda_m$ is the scale factor, λ_m being the characteristic length of the reference medium.

From dimensional argument, it is possible to develop a generic expression :

$$W_i = \alpha_i^p \; W^* \qquad /3/$$

with p = 2 for hydraulic conductivity or flux, p=-1 for water pressure head, p=1 for capillary diffusivity, p=1/2 for capillary sorptivity.

In fact, soils do not generally have identical values of porosity and therefore the theoretical concept of similar media does not hold. It is the reason why several authors use the degree of saturation $S=\theta/\theta_S$ where θ_S is the saturated water content. Furthermore one can always scale soil hydraulic properties through the functional normalization method (Tillotson and Nielsen, 1984) assuming that Eq. /3/ takes the form :

$$W_i = \alpha_{W,i}^p \; W^* \qquad /4/$$

where $\alpha_{W,i}$ is a scale factor associated with the location i and the property W. Considering that the physical model function W_i for each location is similar to the reference function W^* the model functions must have the form :

$$W_i = a_i \; f(S, \text{parameters}) \quad \text{and} \quad W^* = a^* f\,(S, \text{parameters}) \qquad /5/$$

where the curve shape function f(S, parameters) is independent of i.

From Eqs. /4/ and /5/ :

$$a_i = \alpha_{W,i}^{p}\ a^{*} \qquad /6/$$

or, with the use of complementary constraint such as :

$$\frac{1}{N} \cdot \sum_{i=1}^{N} \alpha_{W,i} = 1 \qquad /7/$$

$$a^{*\ 1/p} = \frac{1}{N} \sum_{i=1}^{N} a_i^{1/p} \qquad /8/$$

where N is the number of measurement sites.

A proper choice for the function f(S, parameters) depends on the soil-water properties of interest. This function constitutes a physical model if it describes the measured data within limits of statistical error at each location.

Any soil-water property can be scaled with respect to a given physical model (Eq. /5/) if, and only if a common set of parameters can be estimated so that the function fits the measured data at each location. More generally, we will say that scaling applies to a soil if different soil water properties W_i can be scaled independently according to Eq. /4/ and if the scale factors $\alpha_{W,i}$ defined by Eqs. /6/ and /8/ for each property are identical for each location.

It should be pointed out that the scaling factors thus calculated depend on both the choice of the functional model and the normalizing constraint /Eq. 8/.

They have to be viewed as conversion factors which empirically relate properties of two or more soils or locations.

On the other hand, scale factors obtained from dimensional or inspectional analysis have physical meaning in terms of the system being studied.

3. Applications.

Four illustrative examples of the scaling theory are given : i) scaling of water retention and conductivity soil properties ; ii) prediction of hydraulic conductivity curve from knowledge of retention curve ; iii) scaling of an infiltration law obtained by double ring infiltrometer tests ; iv) stochastic modeling of infiltration and drainage.

3.1 Scaling of h(θ) and K(θ) curves

Soil water properties were determined on a one-hectare of bare sandy soil (Imbernon, 1981). The experimental procedure is described in detail by Vauclin et al, 1983. Briefly two sets of data are available for our analysis : i) – h(θ) and K(θ) curves obtained during internal drainage experiments, with use of neutron probe and tensiometers on 4 sites, 10 depths. ii) – h(θ) curves obtained during redistribution following infiltration at 19 sites, at 3 depths, with the use of neutron probes and tensiometers.

The experimental data are given in Figs. 1a and 2a. All the data were scaled by using the regression technique proposed by Simmons et al. (1979) and applied at the Brooks and Corey (1964) functions. The corresponding mean scale functions were found to be :

$$K^{*}(S) = K_o^{*} S^{b} \quad \text{with } K_o^{*} = 6{,}75\ 10^{-5}\ \text{m/s},\ b = 6{,}87 \qquad /10a/$$

$$h^{*}(S) = h_o^{*} S^{\beta} \quad \text{with } h_o^{*} = -0{,}166\ \text{m of water},\ \beta = -1{,}294 \qquad /10b/$$

The scaled values are reported Figs. 1b and 2b as well as the curves given by Eqs. /10a/ and /10b/.

The scaling factors of the water pressure head α_h, calculated at 23 locations were found to be lognormally distributed (e.g. the probability function of $X = Ln\alpha_h$ is normal with mean value $m_X = -0{,}1229$ and standard deviation $\sigma_X = 0{,}5274$).

Furthermore, the good linear correlation between α_h and α_K ($\alpha_K = 0{,}97\ \alpha_h$ with a coefficient of determination $r^2 = 0{,}85$) tends to show that the concept of similar porous media can be adequately accepted. Figures 1 and 2 show clearly that the result of the scaling procedure is to coalesce the original data into a narrow band around the scale mean curves.

3.2 Prediction of unsaturated hydraulic conductivity curve

An experimentation was conducted in Tunisia (Vachaud et al., 1985) in order to determine at the field scale, soil water balance for different surface covers : 1 ha of rainfed wheat, 0,4 ha of irrigated grass, 0,4 ha of bare soil. Nine sites of measurements were selected. At each site, textural components and bulk density (by gamma-densitomer) profile were determined. Time evolution of water content (by a neutron probe) and of water pressure head (by tensiometers) profiles were routinely measured. In addition, classical internal drainage experiments were performed at two sites, in order to determine both K(θ) –and h(θ)– curves at several depths.

All the raw data $K_i(\theta)$ and $h_i(\theta)$ were analyzed through the functional normalization procedure, previously described. The resulting mean scale functions are :

$$h^*(S) = -0{,}0713\ S^{-4{,}42}\ \text{(m of water)} \ ; \ K^*(S) = 2{,}46\ 10^{-6}\ S^{10{,}47}\ \text{(m/s)} \quad /11/$$

An example is given Fig. 3 for the water retention soil property. The scaling factors α_h and α_K were found to be lognormally distributed and well linearly correlated ($\alpha_h = 0{,}89\ \alpha_K$ with $r^2 = 0{,}89$). In addition, a multiple regression between α_h, the silt + clay contents (S+C) and the dry bulk density (ρ_d) led to :

$$\text{Ln}\alpha_h = -2{,}09\ \text{Ln}\ (S+C) - 4{,}22\ \rho_d + 13{,}53 \quad /12/$$

with $r^2 = 0{,}69$ for 29 points).

This regression was used predict the $K(\theta)$ relation at a site where ρ_d and (S+C) are known, using the relation :

$$K_r(S) = \alpha_r^2\ K^*(S) \quad /13/$$

In order to obtain an evaluation of this prediction, the site was choosen at a location where $K(\theta)$ was known, but not used in the original scaling treatment. For this site, S+C = 8%, $\rho_d = 1{,}62\ g/cm^3$ and, from eq. /12/, $\alpha_r = 10{,}2$.

The corresponding hydraulic conductivity curve calculated by Eq. /13/ with $\theta_S = 0{,}354\ cm^3/cm^3$ (estimated by 90 % of the porosity) is reported in Fig. 4 and fairly agrees with measured values. It should be noted that errors associated with the experimental and calculated values can be estimated (Balabanis, 1983).

3.3 Scaling of infiltration laws

An experience was conducted in the People's Republic of China (Henan Province), at the scale of 3000 m^2. Apart from the spatial analysis of various observations, such as grain size, bulk density, hydraulic conductivity, ... (Lei Zhi Dong et al, 1987), a series of 10 double ring constant head infiltration tests was also done on the site. The cumulative infiltration measurements were analyzed in light of the vertical heterogeneity of the soil profile and of the horizontal spatial variability of the field.

Infiltration parameters were identified from equation :

$$I = St^{\frac{1}{2}} + At \quad /14/$$

(Philip, 1957). A special procedure, developped by Lei Zhi Dong et al, 1987) was used to test the validity of the parameters S and A, whose values are given table 1.

The probability distribution function of both A and S corresponds to log-normal distributions. In order to cope with the large variability of these two parameters, and to obtain easily an estimation of the infiltration of water, under the same conditions as during the experiment, at any time, and in any point of the field, we have used the method of scaling published by Sharma et al (1980).

By application of eq. /3 /, we can define scaled infiltration I^* and scaled time t^* such as :

$$I_i = 1/\alpha_i\ I^* \quad \text{and} \quad t_i = \alpha_i^{-3}\ t^* \quad /15/$$

If the theory of similar media applies, α_i, at a given site, can be computed from measurement of S_i or A_i at every site. On table 1 are also given the values of α_{A_i} and α_{S_i}, relative respectively to those two parameters, and given by :

$$\alpha_{A_i} = \left(\frac{A_i}{\bar{A}}\right)^{\frac{1}{2}} \quad \text{and} \quad \alpha_{S_i} = \left(\frac{S_i}{\bar{S}}\right)^2 \quad /16/$$

where $\bar{A}$ and $\bar{S}$ are the arithmetic mean of measured values. Obviously no correlation can be obtained between the two sets (Sharma et al, 1980 ; Tillotson-Nielsen, 1984). This discrepancy with the theory will be discussed later on.

Following the suggestion of Sharma, the harmonic mean :

$$\alpha_{H,A_i,S_i} = \frac{2\alpha_{A_i}\ \alpha_{S_i}}{\alpha_{A_i} + \alpha_{S_i}} \quad /17/$$

was finally used to determined the scaled infiltration I^* and the scaled time t^* from eq./15/. This combination accounts for some weighting between the initial stage of infiltration (S) and the late stage period (A). Example of results obtained with this parameter is given fig. 5. All the measured values, after scaling, coalesce around a unique scaled infiltration curve which is given by :

$$I^* = \bar{S}\ t^{*\frac{1}{2}} + \bar{A}\ t^* \quad /18/$$

In the present experience, the fact that factors of similitude obtained for A and S are quite different would tend to reject the assumption of similar media, contrary to what has been obtain previously. A careful analysis of the experimental techniques which are involved here should however been done before generalizing this conclusion. The identification of S (sorptivity) is indeed extremely sensitive to measurement of cumulative infiltration at short time, a domain where in practice very few data are available. In the contrary A is affected by measurements at long time, where experimental values should be taken cautionly due to probable inference of lateral flow.

We would therefore tend to say that in this last case the scaling procedure should be viewed more as a technique of similitude applied to a given experimentation (Tillotson-Nielsen, 1984) than as a method of scaling soil properties in the sense of Miller and Miller.

3.4 Stochastic modeling of water flows

Let us now look at the consequences of the variability of soil water properties for water flow modeling. Isothermal soil water movement such as infiltration or drainage is classicaly described at the macroscopic level by Richards'equation which, in one spatial vertical dimension, has one of the two following forms :

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(D(\theta)\frac{\partial\theta}{\partial z}\right) - \frac{\partial K(\theta)}{\partial z} \qquad /19/$$

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(K(\theta)\left(\frac{\partial h}{\partial z} - 1\right)\right) \qquad /20/$$

For homogeneous soils characterized by $K(\theta)$, $h(\theta)$ and $D(\theta)=K(\theta)\, dh/d\theta$ curves, the solution of Eqs./19/ and /20/ for relevant initial and boundary conditions, provides time and space evolutions of soil water content (Haverkamp et al., 1981). In non homogeneous soils we can consider a field on its horizontal extension as a collection of vertical columns, each of them being described by its proper $K_i(\theta)$, $h_i(\theta)$ and $D_i(\theta)$ curves. If all the soil columns can be approximated as similar media, we can easily show that the following set of transformations :

$$S = \theta/\theta_S \quad ; \quad t^* = (\alpha^3/\theta_S)t \quad ; \quad z^* = \alpha z \qquad /21/$$

applied at Eqs./17/ or /18/ leads to :

$$\frac{\partial S}{\partial t^*} = \frac{\partial}{\partial z^*}\left(D^*(S)\frac{\partial S}{\partial z^*}\right) - \frac{\partial K^*(S)}{\partial z^*} \qquad /22/$$

$$\frac{\partial S}{\partial t^*} = \frac{\partial}{\partial z^*}\left(K^*(S)\left(\frac{\partial h^*}{\partial z^*} - 1\right)\right) \qquad /23/$$

The solutions of Eqs./22/ or /23/ are space invariant and unique for a fictitious soil characterized by the "mean" hydrodynamic soil properties : $K^*(S)$, $h^*(S)$ and $D^*(S)$. Applying the inverse transformations /21/ we obtain stochastic solutions of Eqs./19/ or /20/ since α_i can be viewed as a random variable defined by its probability density function.

Two short examples of application can be given on the measurements done by Imbernon in Senegal :

1 - Infiltration using Philip's quasianalytical solution.

With the proper conditions (Philip, 1957) Eq./20/ has the following solution :

$$z^*(S,t^*) = \sum_{q=1}^{n} f_q^*(S)\, t^{*q/2} \qquad /24/$$

for the water content profiles, and :

$$I^*(t^*) = \sum_{q=1}^{n} A_q^*(S)\, t^{*q/2} \qquad /25/$$

for the cumulative infiltration. f_q^* are solutions of ordinary differential equations and A^*_q are estimated by $\int_{Sn}^{1} f_q^*(S)\, dS$.

The distribution of water infiltration at the field scale can be described by the "inverse" stochastic solutions :

$$z(S,t^*,\alpha) = 1/\alpha\; z^*(S,t^*) \qquad /26/$$

$$I(t^*,\alpha) = \theta_S/\alpha\; I^*(t^*) \qquad /27/$$

where t^* stands for $(\alpha^3/\theta_S)t$.

As an example, Fig. /6/ represents the expected value $E\{I\}$ calculated by Eqs./22/ (with n=4) and /27/ as a function of time. Soil hydraulic properties correspond to Eqs./10/.

As a validation of the method experimental measured cumulative infiltration values are given Fig. /6/ together with their domain of uncertainty.

2 - Gravity drainage

Similarly, one can describe the drainage of soil under gravity with the use of the simplified approach of Libardi et al., 1980, the solution of Richard's equation can be written as :

$$\theta(z,t) = \theta_S\left(1 + \alpha^2\, \frac{b-1}{\theta_S}\, K_o^*\, \frac{t}{z}\right)^{1/1-b} \qquad /28/$$

where b is already introduced in Eqs./10/.

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- the Centre de Recherches du Génie Rural, at Mornag, TUNIS (Tunisia) and the Office de Recherche Scientifique d'Outre-Mer, ORSTOM, PARIS
- the General Station of Environmental Hydrogeology, Bureau of Geology and Mineral Resources, ZHENGZHOU, Henan Province and the Dept. of Hydraulic Engineering, TSINGHUA UNIV. BEIJING, People's Republic of China.

We thanks our colleagues for making the data available.

TABLE 1

Site	A_i	$\alpha_{A_i}=(\frac{A_i}{\bar{A}})^{\frac{1}{2}}$	S_i	$\alpha_{S_i}=(\frac{S_i}{\bar{S}})^2$	$\alpha_{H,A,S}$
F0	0.4	1.31	1.75	0.73	0.94
F1	0.15	0.80	4.4	4.7	1.37
F2	0.18	0.88	0.9	0.2	0.325
F3	0.5	1.46	0.9	0.2	0.35
F4	0.06	0.51	3.1	2.3	0.82
F5	0.08	0.58	2.8	1.87	0.81
F6	0.15	0.80	1.8	0.77	0.79
F7	0.10	0.65	2.5	1.5	0.91
F8	0.36	1.24	2.2	1.15	1.19
F9	0.34	1.21	0.1	0.02	4.10^{-3}
$\bar{A} = 0.232$			$\bar{S} = 2.04$		
$\bar{A} = \frac{1}{n}\sum_{i=1}^{n} A_i$			$\bar{S} = \frac{1}{n}\sum_{i=1}^{n} S_i$		$\alpha_{H,A,S} = \frac{2\alpha_A\alpha_S}{\alpha_S+\alpha_A}$

Table 1 - Shanqiu, Henan Province
Values of parameter S and A determined for every infiltration test, and corresponding scaling factors.

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Fig. 2 - Bambey-Senegal. Comparison between measured (a) and scaled (b) Hydraulic conductivity curve K(S).

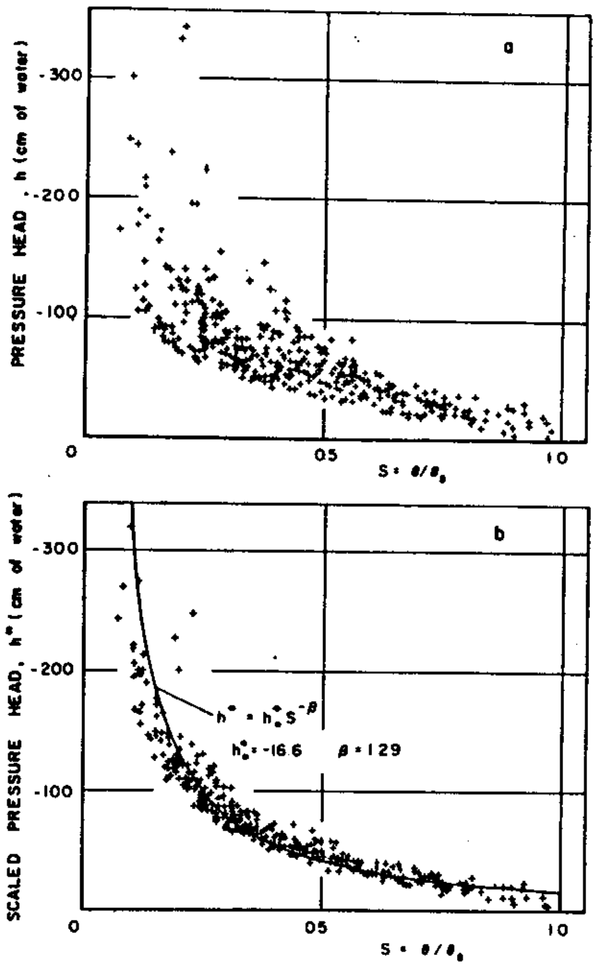
Fig. 3 - Mornag-Tunisia. Comparison between measured (a) and scaled (b) Soil water retention curve (h(S)).

Fig. 4 - Mornag-Tunisia. Prediction of K(S) from scaling factors of Fig. 3 and comparison with measured values.

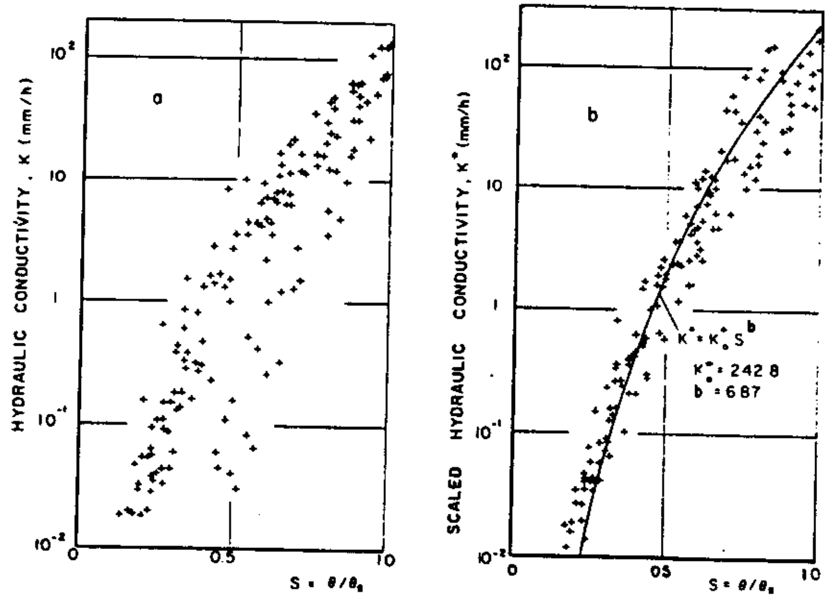
Fig. 5 - Shanqiu, Hean Province-China. Scaled cumulative infiltration curve using from α_i the harmonic mean between α_A and α_S (Table 1).

Fig. 6 - Bambey-Senegal. Comparison between measured and stochastically simulated values of cumulative infiltration for 19 sites.

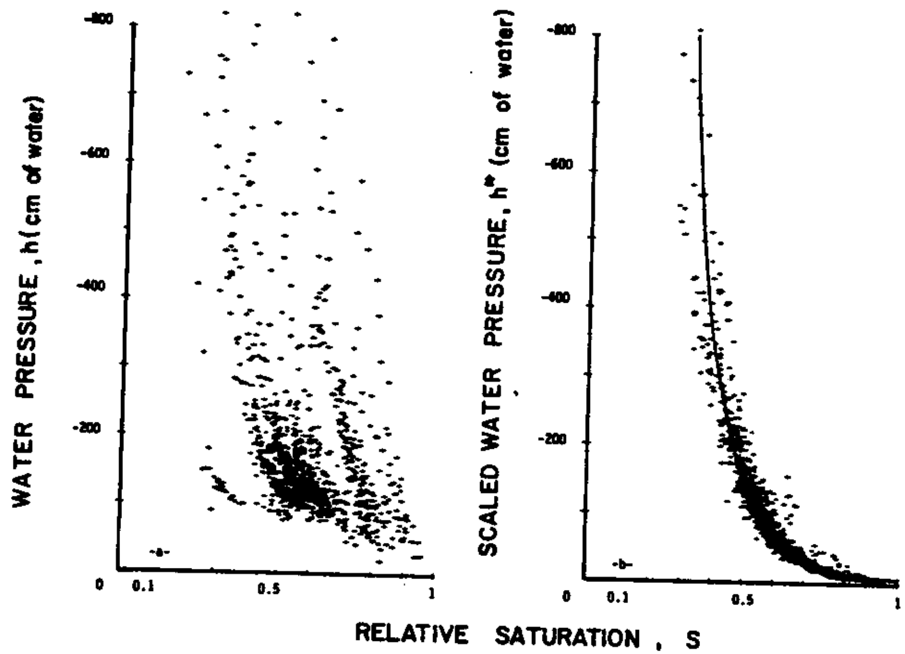
Fig. 7 - Bambey-Senegal. Comparison between measured and stochastically simulated values of change of water content a 1.1 m at 19 sites, following infiltration.



a
PRESSURE HEAD , h (cm of water)
-300
-200
-100
0
0.5
1.0
S = θ/θs
b
SCALED PRESSURE HEAD, h* (cm of water)
h* = h*o S^-β
h*o = -16.6 β = 1.29
S = θ/θs



a
HYDRAULIC CONDUCTIVITY, K (mm/h)
10^2
10
1
10^{-1}
10^{-2}
0
0.5
1.0
$S = \theta/\theta_s$
b
SCALED HYDRAULIC CONDUCTIVITY, K^* (mm/h)
$K^* = K_o^* S^b$
$K_o^* = 242.8$
$b^* = 6.87$


WATER PRESSURE , h (cm of water)
SCALED WATER PRESSURE, h* (cm of water)
-800
-600
-400
-200
0
0.1
0.5
1
-a-
-b-
RELATIVE SATURATION , S

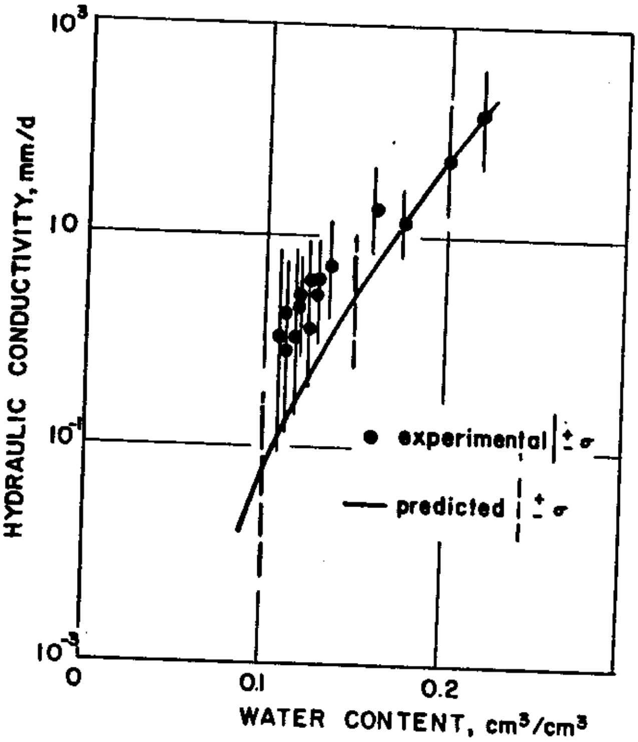
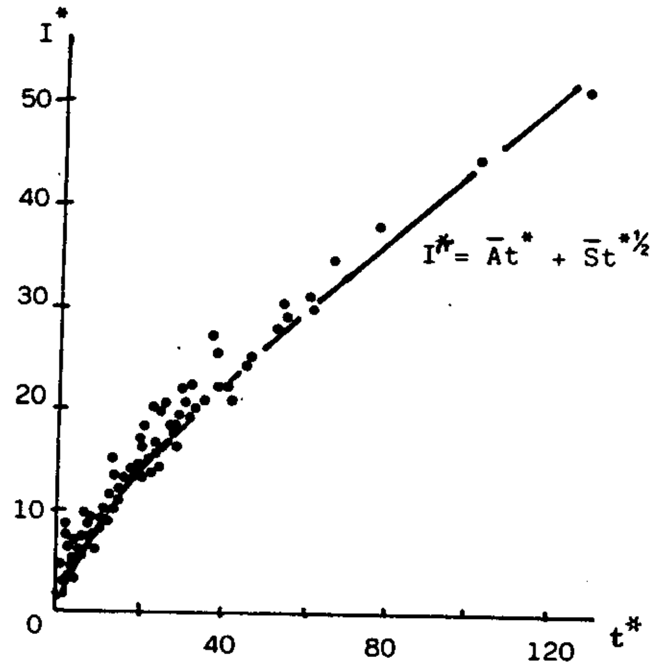

HYDRAULIC CONDUCTIVITY, mm/d
10³
10
10⁻¹
10⁻³
0
0.1
0.2
WATER CONTENT, cm³/cm³
experimental ± σ
predicted ± σ

Fig. 4



I*
50
40
30
20
10
0
40
80
120
t*
I* = Āt* + S̄t*½

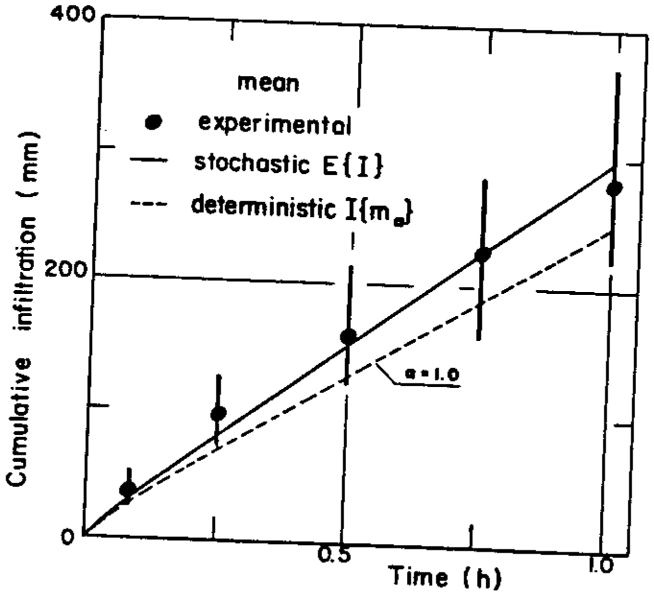

400
200
0
Cumulative infiltration (mm)
mean
experimental
stochastic E{I}
deterministic I{m_a}
a = 1.0
0.5
1.0
Time (h)

Fig 6

Fig 7