INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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SMR/23 - 2

AUTUMN COURSE ON APPLICATIONS OF ANALYSIS TO MECHANICS

22 September - 26 November 1976

RANDOM EFFECTS IN FLUID FLOW

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Random Effects in Fluid Flow — H.K. Moffatt

1.1 Introduction

In these lectures, I propose to give a bird's eye view of some important problems that arise in turbulent flow. I shall not attempt to treat the central (unsolved) problem of the dynamics of an evolving turbulent system, but shall concentrate rather on the effects of a turbulent velocity field $\underset{\sim}{u}(\underset{\sim}{x}, t)$ on other fields, scalar or vector, which are influenced in some way by the turbulence, e.g. a field of temperature fluctuations $\theta(\underset{\sim}{x}, t)$, or density fluctuations $\rho(\underset{\sim}{x}, t)$ in a weakly compressible (low Mach number) situation, or magnetic field $\underset{\sim}{B}(\underset{\sim}{x}, t)$ in an electrically conducting fluid. It is not possible in 8 lectures to give anything like a complete or comprehensive treatment of these problems on which there is a vast literature; the best that I can do is to select certain topics in which I have been particularly involved myself over the last 10 years or so, and to treat these topics by way of providing an indication of techniques and reasoning that can be applied in some situations in which random non-linear effects play an essential part. There is of course no general theory of non-linear differential equations governing the evolution of random quantities, and mathematical rigour of necessity has to give way to "inspired physical reasoning." Of course if physical reasoning is really inspired, then it may be expected to be followed in due course by an appropriately rigorous mathematical treatment confirming the initial conclusions; a good example of this is provided by the "inspired physical reasoning" fluctuations on the solar magnetic field, and of the crucial effect that such fluctuations play in the problem of the Sun's dynamo cycle; the mathematical follow-up came in two quite independent and complementary treatments of the problem by Braginskii (1964) and Steenbeck, Krause & Rädler (1966); I shall discuss the latter treatment later in these lectures. By contrast, physical reasoning that is in any way unsound can never expect to be followed up by a rigorous treatment, and may expect in the course of time to die a natural death.

1.2 Basic equations of fluid mechanics

Now, partly because this is the first lecture in the series entitled 'Mechanics', and partly because the participants in this Course came from a very wide background of scientific disciplines, it may be useful if in this introductory lecture I first discuss some elementary properties of the equations governing fluid flow, and some well-known general features of fluid flow systems.

First if $\rho(\underset{\sim}{x}, t)$ is the density field and $\underset{\sim}{u}(\underset{\sim}{x}, t)$ is the velocity field, the equation of mass conservation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{u}) = 0 \qquad (1.1)$$

or equivalently

$$\frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + \underset{\sim}{u} \cdot \nabla \rho = -\rho \nabla \cdot \underset{\sim}{u}. \qquad (1.2)$$

Here $D/Dt \equiv \partial/\partial t + \underset{\sim}{u} \cdot \nabla$ is the 'Lagrangian' time derivative. In an 'incompressible' flow, the density of fluid elements remains constant, i.e. $D\rho/Dt = 0$, and so from (1.2),

$$\nabla \cdot \underset{\sim}{u} = 0. \qquad (1.3)$$

we shall assume it satisfied unless explicitly stated to the contrary. Of course, in imposing the condition (1.3) we effectively filter out sound waves from the equations of fluid flow. If we want to consider the problem of generation of sound by turbulence (e.g. the turbulence in a jet) then we have to return to the exact equation (1.2).

The momentum equation (or Cauchy equation) is

$$\rho\, Du_i/Dt = \partial\sigma_{ij}/\partial x_j + \rho f_i \tag{1.4}$$

and f_i represents any extraneous force distribution.

where $\sigma_{ij}(\underset{\sim}{x}, t)$ is the stress tensor. In a Newtonian fluid, and under the incompressibility condition (1.3), this is given by

$$\sigma_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.5}$$

where $p(\underset{\sim}{x}, t)$ is the pressure distribution, and μ the viscosity coefficient. Eqns. (1.4) and (1.5) give the well-known Navier-Stokes equation

$$\rho\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u}\right) = -\nabla p + \mu\nabla^2\underset{\sim}{u} + \rho\underset{\sim}{f}\,. \tag{1.6}$$

Frequently we are concerned with situations in which the density is uniform (as well as constant in time); in this situation, (1.6) takes the simpler form

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u} = -\nabla P + \nu\nabla^2\underset{\sim}{u} + \underset{\sim}{f} \tag{1.7}$$

where $P = p/\rho$, $\nu = \mu/\rho$ (the kinematic viscosity), and where any conservative (irrotational) ingredient of the body force $\underset{\sim}{f}$ may be supposed absorbed in the definition of P; in this way we may generally assume that $\nabla\cdot\underset{\sim}{f} = 0$.

Equations (1.3) and (1.7) constitute a system of equations that in principle determines the evolution of $\underset{\sim}{u}(\underset{\sim}{x}, t)$ and $P(\underset{\sim}{x}, t)$ in [illegible] suitable initial eliminated via the equation

$$\nabla^2 p = -\nabla.(\underset{\sim}{u}.\nabla\underset{\sim}{u}) \tag{1.8}$$

but inversion of the Laplacian operator in general involves explicit consideration of the boundary conditions. Nevertheless it is clear from (1.8) that the value of $p(\underset{\sim}{x},t)$ is determined by values of $\underset{\sim}{u}(\underset{\sim}{\xi},t)$ for a wide range of the separation $|\underset{\sim}{x}-\underset{\sim}{\xi}|$; it is this <u>non-local</u> property of p (and hence of the stress tensor σ_{ij}) that leads to some particular difficulties in the context of turbulence dynamics.

1.3 <u>Character of solutions at low and high Reynolds number</u>

It is usual to characterise fluid flows by the value of the Reynolds number

$$R = u_0 l_0 / \nu \tag{1.9}$$

where u_0 is a typical velocity (e.g. the average value of $|\underset{\sim}{u}|$ over the fluid volume) and l_0 a typical length scale (e.g. a length characteristic of the geometry of the fluid boundaries). There may of course be other dimensionless parameters that are of importance in particular situations; the Reynolds number is however the most important and most fundamental, providing a measure of the relative importance of inertial and viscous effects in eqn. (1.7):

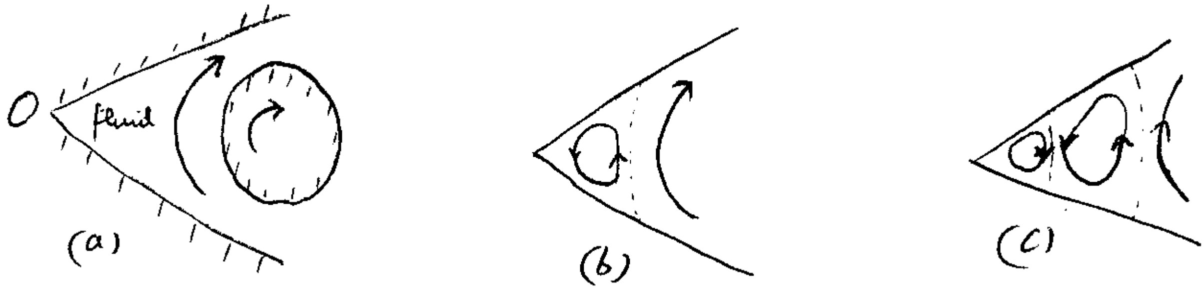
$$\frac{|\underset{\sim}{u}.\nabla\underset{\sim}{u}|}{|\nu\nabla^2\underset{\sim}{u}|} = O(R) \tag{1.10}$$

at all points of a flow field where $|\underset{\sim}{u}| = O(u_0)$, $|\partial u_i/\partial x_j| = O(u_0/l_0)$ and $|\partial^2 u_i/\partial x_j^2| = O(u_0/l_0^2)$. These latter conditions are of course violated on surfaces of near-discontinuity of u_i and particular care is needed in the treatment of eqn. (1.7) in the vicinity of

When $R \ll 1$, viscous forces dominate in eqn. (1.7) and inertia forces are (to leading order) negligible. Provided transient or high frequency forcing effects are excluded, eqn. (1.7) reduces to the quasi-static form

$$\mu \nabla^2 \underset{\sim}{u} = \nabla p \qquad (1.11)$$

(when $\underset{\sim}{f} = 0$). It might be thought that an equation as 'simple' as this could yield few surprises. Let me give just one example, which some of you may already be familiar with, to show what strange behaviour fluids can exhibit even in this 'creeping flow' diffusion dominated situation. Consider the problem of two-dimensional flow near a sharp corner (fig. 1a); imagine that a flow is



generated by some means, e.g. by the rotation of a solid cylinder as llustrated. How does the velocity field behave as the corner O is pproached? It might be expected that the fluid would have lifficulty' in penetrating right to the corner against strong scous retardation, and that a corner eddy might form (figure 1b); t the problem of penetrating to the corner still remains and one eddy forms, then a second will form (fig. 1c), then a ird, then a fourth, and so on! In fact the solution f (1.11), for the stream-function $\psi(r,\theta)$ of the flow, defined by

$$\underset{\sim}{u} = \nabla \wedge (\underset{\sim}{k}\,\psi) = -\underset{\sim}{k} \wedge \nabla\psi, \quad \underset{\sim}{k} = (0,0,1), \qquad (1.12)$$

$$\psi(r,\theta) \sim A r^{p} \cos(q \log r + \epsilon) \qquad (1.13)$$

where $p\ (>2)$ and q are real numbers depending only on the angle α between the planes, and A and ϵ are constants determined by conditions far from the corner; this shows that the stream-function (and so in fact both velocity components) oscillates infinitely rapidly as the corner ($r = 0$) is approached! This result relates to a steady state situation. Just how such a flow is established from a zero-velocity situation (e.g. by starting the rotation of the cylinder at some initial instant $t = 0$) presents an intriguing problem that does not yet appear to have been studied.

If a fluid can behave in such a curious manner at low Reynolds number, it may be readily anticipated that behaviour at high Reynolds number when non-linear inertia effects are of dominant importance may well present equally complex, and indeed startling, properties. When $R \to \infty$, viscous effects become negligible except in singular layers within the fluid or on its boundaries within which $\partial u_i/\partial x_j \to \infty$. The classical problem of flow in a converging or diverging channel (Jeffery-Hamel flow) provides a most instructive example, illustrating how a naive approach may lead to totally wrong conclusions. This naive approach would perhaps indicate that when $R \to \infty$, viscous effects should become negligible for both situations illustrated in figures 2a, b: in (a),

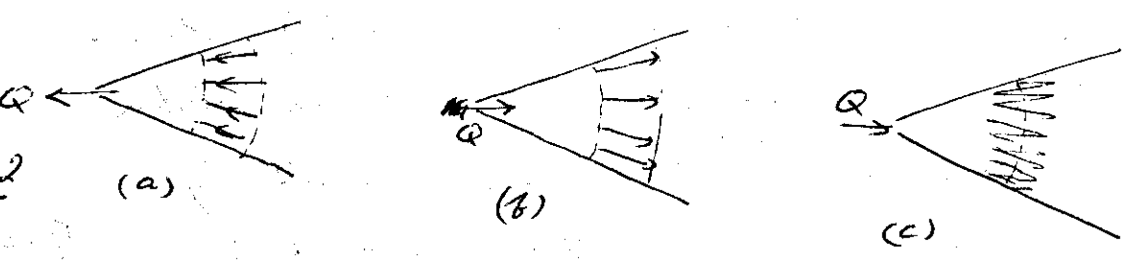


Figure 2

plane rigid surfaces; in (b) fluid is injected. In both cases the 'ideal' fluid situation is illustrated in which the radial velocity is $u_r = \pm Q^*/r\alpha$ ~~with~~ ~~Qexesde~~ where again α is the angle between the planes. In fact, solution of the exact Navier-Stokes equation for this problem, and subsequent procedure to the limit $R \to \infty$ (or equivalently $\nu \to 0$), indicates that while this picture is correct for case (a), it is totally incorrect for case (b); in this latter case the velocity field as given by the exact solution of the Navier-Stokes equations becomes <u>singular everywhere</u> in the limit $\nu \to 0$, so that in fact viscous forces remain comparable with inertia forces at every point no matter how small ν may be. It is a characteristic feature of turbulent flows also that, no matter how small ν may be, the smallest scale of variation of the flow simply adjusts itself in such a way that certain gross parameters, in particular the mean rate of dissipation of energy per unit mass ϵ, remain at most weakly dependent on ν in the limit $\nu \to 0$.

1.4 <u>Instability and transition to turbulence</u>

A flow $\underset{\sim}{U}(\underset{\sim}{x})$ satisfying the exact (steady) Navier-Stokes equations will be realisable in the laboratory only if it is stable to small perturbations. Linear stability theory consists in considering small perturbations $\underset{\sim}{v}(\underset{\sim}{x},t) = \underset{\sim}{u}(\underset{\sim}{x},t) - \underset{\sim}{U}(\underset{\sim}{x})$, linearising the equations for $\underset{\sim}{v}$ (and the associated pressure perturbation) and seeking solutions of the form

$$\underset{\sim}{v}(\underset{\sim}{x},t) = \mathrm{Re}\left(\hat{\underset{\sim}{v}}(\underset{\sim}{x})\, e^{pt}\right). \qquad (1.14)$$

Together with the relevant boundary conditions on $\underset{\sim}{v}$, this procedure generally leads to an eigenvalue problem for the complex parameter p. If $\mathrm{Re}\,p > 0$ for any eigenvalue, then the flow is unstable of finite amplitude perturbations also. I do not want to treat the difficult question of classification of different types of instability, which will be treated from different points of view by other lecturers on this course. It is sufficient to say here that turbulence invariably results from a catastrophic instability of a laminar (or quiescent) flow situation in which some ~~prs~~ parameter of the system changes by a large amount. Just which parameter makes this jump depends on the precise specification of the problem. For example, in the problem of Poiseuille flow along a tube of circular cross-section (a flow that is stable to infinitesimal disturbances at all Reynolds numbers, but catastrophically unstable to finite amplitude perturbations at sufficiently large Reynolds numbers) we may specify either the pressure drop Δp or the volume flux Q (Figures 3a, b):

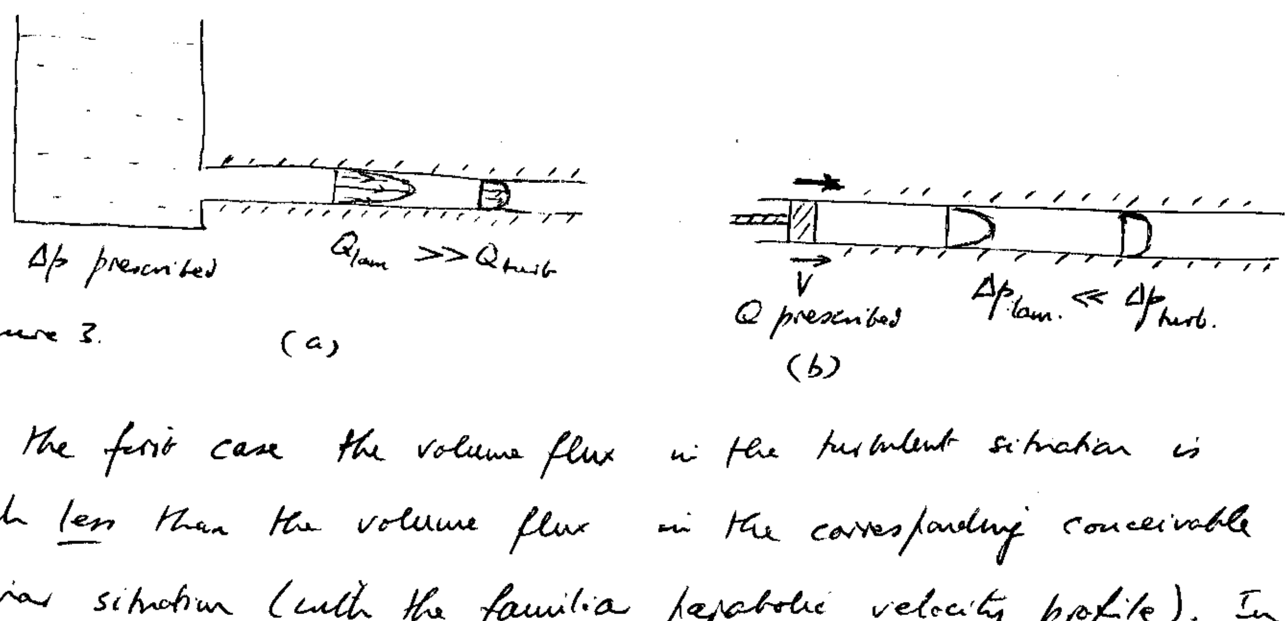


Figure 3. (a) (b)

In the first case the volume flux in the turbulent situation is much <u>less</u> than the volume flux in the corresponding conceivable laminar situation (with the familiar parabolic velocity profile). In the second case, the pressure gradient (or equivalently the wall stress) is much <u>greater</u> for the turbulent flow than for the laminar flow with the same value of Q — the force needed to drive the piston with a prescribed velocity takes a large jump when transition

Turbulent flows are of the greatest importance in a very wide range of circumstances in aeronautics, hydronautics, meteorology, oceanography, astrophysics, and chemical engineering. In the latter context the problem of turbulent mixing of different fluid constituents with and without chemical reactions is of central importance. In the terrestrial and astrophysical contexts, we have to contend also with complicating factors such as density stratification, Coriolis forces due to rotation of the systems under study, or magnetohydrodynamic effects when the fluid (as in the Sun's outer convective layers) is ionised and therefore a good conductor of electricity. In all cases however, the features that characterise a turbulent situation are (i) the existence of velocity fluctuations whose amplitude is so large that non-linear effects (involving the term $\underset{\sim}{u}\cdot\nabla\underset{\sim}{u}$ in eqn. (1.6) are certainly non-negligible; (ii) the existence of a large range of length-scales characterising these velocity fluctuations; (iii) the extreme complexity of the detailed evolution of $\underset{\sim}{u}(\underset{\sim}{x},t)$ making a deterministic theory totally impracticable, and a statistical approach essential; and (iv) the essentially dissipative (thermodynamically irreversible) character of the flow: in no sense is a turbulent fluid in a state of thermodynamic equilibrium; ~~and~~ ideas imported from classical statistical mechanics have little to offer in such a situation and may even be quite misleading.

With these superficial and inadequate remarks I must close this Introductory lecture; in the next lecture I will develop the notations and terminology of turbulence in the idealised situation in which all statistical properties are invariant under translations of the frame of reference — 'homogeneous' turbulence.