INTERNATIONAL ATOMIC ENERGY AGENCY
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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
MIRAMARE - P.O.B. 586 - 34100 TRIESTE (ITALY) - TELEPHONES: 224281/2/3/4/5/6 - CABLE: CENTRATOM

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MATHEMATICAL ASPECTS OF FLUID FLOW

Appendix

Outline of Leray-Schauder degree theory

T.B. Benjamin
University of Essex,
Colchester, U.K.

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This theory was created in a famous paper by J. Leray and J. Schauder which appeared in 1934 ('Topologie et équations fonctionnelles.' Ann. Sci. Ecole Norm. Sup., série 3, 51, pp. 45-78). Developments from first principles may be found in the books by

M. A. Krasnosel'skii 1964 Topological Methods in the Theory of Nonlinear Integral Equations. (In place of the term 'degree', K. uses the term 'rotation of a vector field'.),

J. T. Schwartz 1969 Non-linear Functional Analysis.,

and the book by Jane Cronin on Fixed Point Theorems which is less helpful to readers interested primarily in applications. Shorter accounts presenting the essentials needed for applications may be found in the books by

M. Berger & M. Berger 1968 Perspectives in Nonlinearity.,

D.H. Sattinger 1973 Topics in Stability and Bifurcation Theory (Springer Lecture Notes in Mathematics, No. 309), Chapter 7.

The theory is useful in applications to nonlinear equations in the form

$$u = Au,$$

where A is a nonlinear operator (mapping, transformation) which is compact (completely continuous) acting in a real Banach space. [Definition. Let $\bar{\Omega} \subset X$ be the closure of a bounded open set in a Banach space X. A mapping $A: \bar{\Omega} \to X$ is said to be compact if it is continuous and the range (image) $A(\bar{\Omega})$ is precompact, which means that its closure is compact. (Recall that a compact set is such that any infinite sequence in it has a strongly convergent subsequence – or is itself strongly convergent. The infinite sequence $\{\sin nx\}$ $(n = 1, 2, 3, \ldots)$ does not converge strongly in $L_2(0,\pi)$, for example, being only weakly convergent to the null function; but the sequence $\{n^{-1}\sin nx\}$ converges strongly, i.e. in the norm of L_2. An operator is said to be <u>completely continuous</u> if it transforms every weakly convergent sequence into a strongly convergent one, without passage to subsequence; but the two terms are sometimes used interchangeably in the literature.]

<u>Exercise</u>. Remembering that every L_2 function has a Fourier-series expansion, show that the linear operator defined by $Bu(x) = (I - d^2/dx^2)^{-1} u(x)$ is a compact mapping of a ball $\{u : \|u\| \le r\}$ in $L_2(0,\pi)$. Is this operator similarly compact in $C(0,\pi)$?

<u>Example of compact nonlinear operator.</u> Consider the boundary-value problem

$$\begin{cases} \Delta\phi + F(\underset{\sim}{x}, \phi) = 0 & \text{on} \quad D \quad (\subset \mathbb{R}^n), \\ \phi|_{\partial D} = 0. \end{cases}$$

Here the function $F(\underset{\sim}{x}, \phi)$, $\underset{\sim}{x} \in \bar{D}$ & $\phi \in \mathbb{R}$, is nonlinear in the second (scalar) variable. If $\bar{D}$ is fairly simple, e.g. a sphere or a rectangle, an explicit Green function may be found. Roughly speaking, this is a function $k(\underset{\sim}{x}, \underset{\sim}{\hat{x}})$ which is positive on $D \times D$, vanishes everywhere on $\partial D \times \bar{D}$ and $\bar{D} \times \partial D$, and satisfies

$$\Delta k = -\delta(\underset{\sim}{x} - \underset{\sim}{\hat{x}}) \qquad \text{(Dirac delta function)}$$

Accordingly, the problem can be recast in the form of a nonlinear integral equation of Hammerstein type

$$\phi(\underset{\sim}{x}) = \int_D k(\underset{\sim}{x}, \underset{\sim}{\hat{x}})\, F(\underset{\sim}{\hat{x}}, \phi(\underset{\sim}{\hat{x}}))\, d\hat{x} = BF\phi = A\phi, \quad \text{say.}$$

It can be shown that if F is continuous in both variables, this A is a compact operator in the Banach space $\mathbb{C}(\bar{D})$. In fact, the set of functions $\{u = Av : v \in \bar{\Omega}\}$, where $\bar{\Omega}$ is the closure of any bounded open set in $C(\bar{D})$ is equicontinuous, and compactness of A is then assured by the Arzela-Ascoli theorem. A solution of the integral equation automatically satisfies the boundary condition $\phi|_{\partial D} = 0$; and there is a standard argument showing that if F is not only continuous but is Hölder continuous, then a solution of the integral equation (i.e. a fixed point of the nonlinear mapping A) is $C^{2+\alpha}(\bar{D})$ and therefore a classical solution of the original p.d.e. If $F \in C^\infty(\bar{D} \times \mathbb{R})$, then a solution $\phi \in C^\infty(\bar{D})$.

(Note that there are many other ways to obtain a compact operator equation from such a boundary-value problem — for example, by use of the Sobolev space $\overset{\circ}{W}_{1,2}(D)$, i.e. the completion of the set $C_0^\infty(D)$ under the norm $\|u\|_1 = \left(\int_D \nabla u\, dx\right)^{\frac{1}{2}}$, and by use of the Riesz representation theorem. This approach is effective even when an explicit Green function is not known.)

Exercise. Specifying appropriate assumptions about the function $F(\underset{\sim}{x}, u) : \bar{D} \times \mathbb{R} \to \mathbb{R}$, recast the above boundary-value problem as an operator equation in (i) $L_2(D)$, (ii) $\overset{\circ}{W}_{1,2}(D)$.

It should be noted incidentally that a degree theory is also available for compact mappings of Fréchet spaces, which are vector spaces with a 'distance function' or 'metric' but which do not have a norm (i.e. they are not Banach spaces). There are many collections of functions that cannot be represented as Banach spaces: for example, the collection of continuous but not necessarily bounded functions defined on $\mathbb{R}$, which includes x^{100} and e^{x^2}. There is no way to put a norm on such a set of functions.

A distance function (metric) for a vector space $\mathcal{F}$ is defined by just two axioms:

i) $d(u,v) \geq 0 \quad \forall\, u,v \in \mathcal{F}$, with $d(u,v) = 0$ if and only if $u = v$;

ii) TRIANGLE INEQUALITY, $\quad d(u,v) \leq d(u,w) + d(v,w) \quad \forall\, u,v,w \in \mathcal{F}$.

It follows from (i) and (ii) that d is symmetric, i.e. $d(u,v) = d(v,u) \quad \forall\, u,v \in \mathcal{F}$. (Exercise. Show this.)

Example of metric. For continuous functions on $\mathbb{R}$, define 'semi-norms' by

$$\|u\|_n = \max_{x \in [-nl,\, nl]} |u(x)| \qquad (n = 1,2,3,\ldots),$$

where l is a fixed positive number. [Or, in the case of 'locally' measurable and square-integrable functions ($L_{2(loc)}(\mathbb{R})$), define semi-norms by

$$\|u\|_n = \left(\int^{nl} u^2\, dx\right)^{\frac{1}{2}} \qquad (n = 1,2,3,\ldots).\ ]$$

Then a metric is defined by

$$d(u, v) = \sum_{n=1}^{\infty} \frac{\|u - v\|_n}{2^n (1 + \|u - v\|_n)} .$$

The open ball centred on the zero element θ in the Fréchet space $\mathcal{F}$ with this metric is the set $\{u \in \mathcal{F} : d(\theta, u) < r\}$, where evidently $r < 1$.

<u>Exercise</u>. Confirm that this d satisfies the axioms (i) and (ii).

With regard to most applications, it is adequate to consider Leray-Schauder degree as being defined by the following set of established properties, 1. to 6.

Let X be a real Banach space with zero element θ. Let Ω be a bounded open subset of X, with boundary $\partial\Omega$ and closure $\bar{\Omega} = \Omega \cup \partial D$. [For example, $\Omega = \{u \in X : \|u\| < r\}$ (open ball centred on θ), $\partial\Omega = \{u \in X : \|u\| = r\}$, and $\bar{\Omega} = \{u \in X : \|u\| \leq r\}$.]

Let $A : \bar{\Omega} \to X$ be a compact mapping, and let A have no fixed point on $\partial\Omega$ (i.e. no point $u \in \partial\Omega$ such that $u = Au$). Then there exists an integer

$$\deg(I - A, \Omega),$$

defined uniquely as a property of the mapping $I - A$ on $\partial\Omega$, and the following facts hold:

1. <u>Fixed-point principle</u>. If $\deg(I - A, \Omega) \neq 0$, then A has a fixed point in Ω. (Note that this fixed point is not necessarily unique.)

2. In the case of $I - u_0$, where I is the identity map and u_0 is a given element of X, $\deg(I - u_0, \Omega) =$ 0 or 1 accordingly as $u_0 \in \Omega$ or $u_0 \notin \bar{\Omega}$.

3. <u>Homotopy invariance</u>. Let A_s be a transformation depending on the real parameter s. Let A_s be compact in the sense $\bar{\Omega} \to X$ for every $s \in [0,1]$, and also continuous in sense $[0,1] \times \bar{\Omega} \to X$ (these conditions ensure compactness $[0,1] \times \bar{\Omega} \to X$). Let A_s have no fixed point on $[0,1] \times \partial\Omega$ (we say in this case that there is an <u>admissible homotopy</u> $I - A_s$ between $I - A_0$ and $I - A_1$). Then

$$\deg(I - A_0, \Omega) = \deg(I - A_1, \Omega).$$

[Note: This property is often useful in calculating Leray-Schauder degree. Given a complicated mapping $I - A$ on $\partial\Omega$, one looks for a simpler mapping (e.g. I) with which it has an admissible homotopy.]

4. Let A have fixed points u_m $(m = 1, 2, 3, \dots)$ in Ω which are all isolated. Then the total number k of fixed points is finite. If the index i_m of a fixed point is defined as

$$\deg(I - A, \Omega_m),$$

where $\Omega_m \subset \Omega$ is a neighbourhood of u_m small enough to contain no other fixed point, then -

$$\deg(I - A, \Omega) = i_1 + i_2 + \dots + i_k.$$

5. At the fixed point u_m in Ω, let A have a strong Fréchet derivative $A'(u_m)$. Let 1 not be an eigenvalue of the linear operator $A'(u_m)$ [i.e. there exists no non-zero $\xi \in X$ such that $\xi = A'(u_m)\xi$]. Then u_m is an isolated fixed point, and $i_m = (-1)^\beta$, where β is the sum of the algebraic multiplicities of those (real) eigenvalues of $A'(u_m)$ that satisfy $\lambda > 1$. (Note that in most applications, eigenvalues are simple – i.e. of multiplicity 1 – so that β is simply the number of eigenvalues exceeding unity. In general, if $S_N = \{\xi \in X : (\lambda I - A')^N \xi = \theta\}$ $(N = 0, 1, 2, \dots)$, then $S_0 = \{\theta\} \subset S_1 \subset S_2 \subset \dots \subset S_M = S_{M+1} = \dots$, and the integer M is called the algebraic multiplicity of the eigenvalue λ. For A compact, the Fréchet derivative A' is compact, and its spectrum $\{\lambda\}$ is discrete, with finite multiplicities M.)

6. If A has no fixed point in Ω, then $\deg(I-A, \Omega) = 0$ (This fact has already been exemplified in 2.) Coupled with 3, it implies that $\deg(I-A,\Omega) = 0$ also if the directions of the vector field $(I-A)(\partial\Omega)$ are all different from the direction of a fixed non-zero vector u_0, i.e. if

$$u - Au \neq a u_0 \qquad \forall\ u \in \partial\Omega,\ a \geq 0.$$

<u>Exercise</u>. Prove the last statement. (<u>Hint</u>. Use the fact that if $u \in \partial\Omega$, then u, Au and therefore $u - Au$ are all norm bounded.)

<u>Exercise</u>. Work out interpretations of properties 1 – 6 in the