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Basic ideas in applied functional analysis

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Basic idea in applied functional Analysis

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Summary

Introduction. 1

example 1. Homogeneous Dirichlet problem 1

I. The Lax Milgram theorem (on a real space vector) 4

1. Scope, terms and proof of the main theorem 4
2. Case of the inclusion of one Hilbert space V in an other one H 7
3. the cases where the bilinear form is symetric (variational equations) 8

II Introduction of the Useful functional spaces 9

1. Recapitulation 9
2. notions on the distributions 11
3. Definitions on the Sobolev spaces $W^{m,p}$ (m integer) 13
4. Properties of Sobolev spaces $H^m(\Omega)$ (m integer) 14
5. Introduction of the Sobolev spaces $H^s(\Omega)$ for s non necessarily integer 16
6. Green formula in Sobolev spaces 17
7. Application of the Lax Milgram theorem to the Homogeneous Dirichlet problem 19
8 Bibliography 21

III Galerkin's method 22

1. Introduction 22
2. Formulation of the approached problems: existence and uniqueness 23
3. Result of convergence 23

IV Applications of the Lax Milgram theorem to several problems 25

1. Homogeneous Dirichlet problem for $\lambda = 0$. 26
2. Non Homogeneous Dirichlet problem $\lambda \geqslant 0$. 28
3. Homogeneous Neuman problem 29
4. non Homogeneous Neuman problem 30
5. Mixed Homogeneous "Dirichlet-Neuman problem 31

V. Variational or no Variational inequalities 32

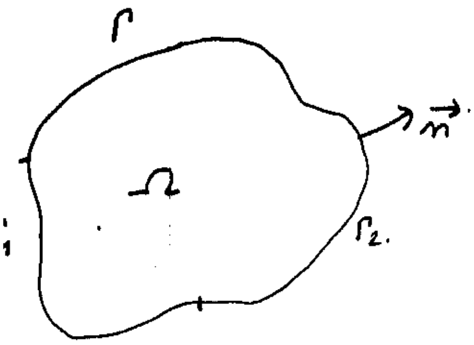
1. Motivations 32
2. minimization of a functional. 35
3. Variational Inequalities 41
4. Non variational Inequalities. 42
5. Bibliography (non exhaustive) 50.

V. Variational or no variational inequalities.

V.1. Motivation

Example 5: Semi permeable membrane.

Let us consider a fluid of pressure $p(\underline{x},t)$, occupying a part Ω of $\mathbb{R}^3$ bounded by a semi permeable membrane Γ (of negligible thickness); that is to say: Γ allows fluid to enter into Ω but not to leave.



Let us suppose that the outside pression is a constant one p_0, then the function $u(\underline{x},t) = p(x,t) - p_0$ satisfies a diffusion equation of the type

- (53) $\dfrac{\partial u}{\partial t} - \Delta u = g$ in Ω for $t>0$.

and also an initial solution.

- (54) $u(x,0) = u_0(x)$.

The boundary conditions are:

- (55) $\forall\, x \in \Gamma$ where $u(\underline{x},t)>0$ then $\dfrac{\partial u}{\partial n}(\underline{x},t)=0$.

(in fact: $u(x,t)>0$ means that the pression inside Ω in the neighbourhood of this point is bigger than p_0; but the fluid cannot leave Ω then the flux across Γ is 0 at this point: $q n = 0 = -k\dfrac{\partial u}{\partial n}$ from the Darcy law)

- (56) $\forall\, x \in \Gamma$ where $u(x,t)=0$ then $\dfrac{\partial u}{\partial n}(\underline{x},t) \geq 0$.

(in fact, as soon as the pressure p has a tendency to become smaller than p_0, the fluid enters in Ω and the flux $q.n = -k\dfrac{\partial u}{\partial n} \leq 0$ then $\dfrac{\partial u}{\partial n} \geq 0$ at this point and because the assumption of negligible thickness for Γ, instantaneously the difference u between p and p_0 is 0, we have for this reason to write $u=0$ in place of $u \leq 0$.)

Remarks 17: a) This problem is not linear because the boundary conditions

(57) $\Gamma_1 = \{ x / x \in \Gamma$ where $u(x,t) > 0. \}$
(58) $\Gamma_2 = \{ x / x \in \Gamma$ where $u(x,t) = 0 \}$.
and because this partition is depending on the solution u, Γ_1 and Γ_2 are unknown à priori, they have also to be determined. This kind of problem are called "free boundary problems"

Remarks 18 : a) the set of possible solution is not a vector space but an affine space $V + u_0$, V being a vector space to define; for this reason, to obtain the weak formulation of the problem, we shall multiply (53) by $v - u \in V$ in place to multiply by v

b) We have in effect to look for the solution in the subset defined by.
(59) $K = \{ v / v \in V + u_0$ s.t $\gamma_0 v \geq 0$ on $\Gamma \}$.

If we want now to obtain the weak formulation we have to multiply by $v - u$ $\forall v \in K$ and with the usual green transformation, we have for u solution of (53) ..... (56) :

$$\int\!\!\int_\Omega \left\{ \frac{\partial u}{\partial t}(v-u) + \operatorname{grad} u \cdot \operatorname{grad}(v-u) - g(v-u) \right\} dx = \int_\Gamma \frac{\partial u}{\partial n}(v-u)\, d\Gamma \qquad \forall v \in K$$

but $\int_\Gamma \frac{\partial u}{\partial n} u \, d\Gamma = 0$ because the alternative (55) (56).

and $\int_\Gamma \frac{\partial u}{\partial n} v \, d\Gamma \geq 0$ because $\frac{\partial u}{\partial n} \geq 0$ and $v \in K \Rightarrow v \geq 0$ on Γ.

The conclusion is: if u is a solution sufficiently regular of (53).... (56), then u is solution of

$$(60) \quad \begin{cases} u(t) \in K & \forall t \geq 0 \\ \left(u'(t), v - u(t) \right) + a\left(u(t), v - u(t)\right) \geq \langle g(t), v - u(t) \rangle & \forall v \in K \end{cases}$$

(we denote here by $u(t)$ the mapping which at the instant t associates to $\forall x \in \Omega$ the element $u(x,t) \in R$; $\langle \, , \, \rangle$ represents the

. . . . u' and $u'(t)$ in the ... $\frac{\partial u}{}$

a is the bilinear continuous form defined on $V + u_0$ by

$$a(v, w) = \int_\Omega \text{grad}\, v \; \text{grad}\, w \; dx \qquad \forall\, v \text{ and } w \in V + u_0 .$$

This kind of problem (60) is called an evolution functional inequality of parabolique type.

In a steady case, we should obtain a problem of the type:

$$(61) \quad \begin{cases} u \in K. \\ a(u, v-u) \geq (f, v-u) \quad \forall v \in K. \end{cases}$$

Which looks like the problem resolved by the Lax Milgram Theorem except that we have here an inequation. and also that K is no longer a vector space but a convex set

This kind of problems is very frequent in Physics and mechanics. We can evoke some of them

- Air. condition in thermique (control by the boundary or by some source in Ω)
- Unilateral contact in solid mechanics.
- Problems in plasticity where the constitutive law implies a notion of barrer of the type $\mathcal{F}(\sigma) \leq 0$. in Ω. (σ: stress field tensor) with for instance:

elastic behavior. $\forall x \in \Omega$ where $\mathcal{F}(\sigma(x)) < 0$.

plastic behavior. $\forall x \in \Omega$ where $\mathcal{F}(\sigma(x)) = 0$.

<u>Remark 18</u>: The characteristic element of these problems evoked above is that the solution has to be found in a <u>convex set K:</u>

a convex set K is subset of a vector space defined by the following property.

(62) if u and $v \in K$ then.

$$\theta u + (1-\theta) v \in K \qquad \forall \quad \theta \in [0, 1] \subset R$$

for example

$$K = \{ v \,/\, v \in H_0^1(\Omega) \text{ such that } |\text{grad}\, v| \leq 1 \}$$

Remark 20: We can distinguish two kinds of such a functional inequalities:

The first ones can be associated to a problem of minimization of a functional. We already met this case for the equations (paragraph I.3). These equations or inequations are then called "variational equations or inequations"

The second kind is not related to any problem of minimization of a functional.

We shall study successively the two kinds of problems.

V. 2. Minimization of a functional.

The more general result is the following.

Theorem 12 (Existence): Let V be a reflexif, separable, Banach space and $K \subset V$ a closed convex subset of K
Let now J be a functional $J : v \longrightarrow J(v) \not\equiv \infty$ defined over K such that.

(i) $J(v) \longrightarrow +\infty$ if $\|v\| \to +\infty$ (Hypothesis without object if K is bounded)
(ii) J is semi continuous from below for the weak topology

Then $\exists u \in K$ (non necessarily unique) such that

$$(63) \qquad J(u) \leq J(v) \quad \forall v \in K$$

Proof: Let $j = \inf_{v \in K} J(v)$ possibly $j = -\infty$.

and let $\{u_m\}$ be a minimizing sequence that is to say.

$$J(u_m) \longrightarrow j \quad \text{when } m \to \infty.$$

From (1) we know that $\|u_m\| \leq c$ because $j < +\infty$.

By the theorem 9 of weak compacity, we know that $\exists$ a subsequence $\{u_{m'}\} \subset \{u_m\}$ such that

$u_{m'} \xrightarrow{\text{weakly}} w$ in V; $w \in K$ because K is a closed convex (for a convex set the notion of closure are the same for strong or weak topology).

The assumption (ii) means that:

Here $J(u_{m'}) \longrightarrow j$ so

$$j = \liminf J(u_{m'}) \geq J(w)$$

Therefore, by definition of j, $J(w) = j = \inf J(v)$ $\forall v \in K$.

and $u = w$ is solution of (63)

<u>Theorem 13 (Uniqueness)</u>

<u>If J is a continuous convex functional, then (ii) is verified. and with the other assumptions, we have a fortiori existence of a solution for (63). If J is strictly convex, we have now the uniqueness for u. If J is convex but not strictly the set X of the solutions is convex</u>

The first part of this theorem is a classic result of analysis. Let us prove the uniqueness of u:

J is strictly convexe means:

(64) $$J[(1-\theta)v + \theta w] < (1-\theta) J(v) + \theta J(w) \quad \forall\ 0 < \theta < 1 \text{ and } \forall v \neq w$$

Suppose there exists two solutions $u_1 \neq u_2$ for (63)

then $J(u_1) \leq J\left(\frac{u_1+u_2}{2}\right)$ and $J(u_2) \leq J\left(\frac{u_1+u_2}{2}\right)$

$$\Rightarrow \quad \frac{J(u_1) + J(u_2)}{2} \leq J\left(\frac{u_1+u_2}{2}\right)$$

but the strict convexity of J implies.

$$J\left(\frac{u_1+u_2}{2}\right) < \frac{J(u_1) + J(u_2)}{2}$$

Whence the contradiction : Therefore. $u_1 = u_2$.

If J is convex but non strictly; let u_1 and $u_2 \in X$ set of the solutions. then

$$J(u_1) \leq J(v) \quad \text{and } J(u_2) \leq J(v) \quad \forall v \in V.$$

$$J[(1-\theta)u_1 + \theta u_2] \leq (1-\theta) J(u_1) + \theta J(u_2) \leq (1-\theta) J(v) + \theta J(v) = J(v)$$

$\forall \theta \in [0,1]$ and $v \in K$. then. $(1-\theta)u_1 + \theta u_2 \in K$. q.e.d.

Whence the convexity of X

Theorem 14 (characterization of the solution)

i) if J is a convex and differentiable functional, the solution u of (63) is characterized by

$$(65)\quad \begin{cases} u \in K \\ \langle J'(u), v-u \rangle \geq 0 \quad \forall\, v \in K \end{cases}$$

ii) if $J = J_1 + J_2$; J_1 and J_2 being convex functionals. J_1 being differentiable but not J_2; the solution u of (63) is characterized by

$$(66)\quad \begin{cases} u \in K \\ \langle J_1'(u), v-u \rangle + J_2(v) - J_2(u) \geq 0 \quad \forall\, v \in K \end{cases}$$

Proof.

J is differentiable in $u \in V \Leftrightarrow$

$\exists$ a neighborhood $N(u)$ of u and a continuous linear form $J'(u) \in V'$ such that

$$J(v) - J(u) = \langle J'(u), v-u \rangle + \varepsilon_u (v-u) \, \| v-u \| \qquad \forall\, v \in N(u)$$

With $\varepsilon_u : V \longrightarrow R$ such that

$$\lim_{w \to 0} | \varepsilon_u (w) | = 0 .$$

$J'(u)$ is called differential of J in u.

J is a convex functional $\Leftrightarrow J(v) - J(w) \geq \frac{1}{\theta} \left[J[(1-\theta) w + \theta v] - J(w) \right]$

$\forall\, v$ and $w \in V$ and $\forall\ \theta \in [0,1]$.

if $\theta \to 0$ we obtain

$$(67)\quad J(v) - J(w) \geq \langle J'(w), v-w \rangle \qquad \forall\, v \text{ and } w \in V$$

Whence

a) if u is solution of (63) then

hence $\frac{1}{\theta}\left[J[u+\theta(v-u)] - J(u)\right] \geqslant 0 \quad \forall\, v \in K$ and $\theta \in R^+$.

and when $\theta \to 0$, $\langle J'(u), v-u \rangle \geqslant 0 \quad \forall\, v \in K$.

b) Reciprocally: if (65) is satisfied from (67) we obtain immediatly:

$$u \in K, \quad J(u) \leq J(v) \quad \forall\, v \in K.$$

ii) a) $J(u) \leq J(v) \quad \forall\, v \in K \Rightarrow J(u) \leq J[(1-\theta)u + \theta v] \quad \forall\, v \in K.$

$\Rightarrow \quad J_1(u) + J_2(u) \leq J_1[(1-\theta)u + \theta v] + (1-\theta)J_2(u) + \theta J_2(v) \quad \forall\, \theta \in [0,1].$

thanks to the convexity of J_2

$$\Rightarrow \quad J_1[(1-\theta)u + \theta v] - J_1(u) + \theta[J_2(v) - J_2(u)] \geqslant 0 \quad \forall\, v \in K.$$

if we divide now this inequality by θ and make θ go to 0 we obtain:

$$\langle J_1(u), v-u \rangle + J_2(u) - J_2(v) \geqslant 0 \quad \forall\, v \in K \qquad \text{q.e.d.}$$

b) Reciprocally: if (66) is satisfied.

$J(v) - J(u) = J_1(v) - J_1(u) + J_2(v) - J_2(u)$ and then thanks to (67),

$$J(v) - J(u) \geqslant \langle J_1'(u), v-u \rangle + J_2(v) - J_1(u) \geqslant 0 \quad \forall\, v \in K$$

<u>Remark 21:</u> (65) and (66) are two variational inequations but if J is no differentiable or no convexe we have a priori no charactization for the solution u. There exist therefore some problems of minimization of a functional which are not related to any variational inequation.

<u>Remark 22:</u> <u>If K is a convex cone of vertex 0</u>, then (61) is equivalent to

$$(65\text{ bis}) \quad \begin{cases} \langle J'(u), v \rangle \geqslant 0 \quad \forall\, v \in K. \\ \langle J'(u), u \rangle = 0. \end{cases}$$

in fact, by definition, C is a cone of vertex s iff

$$v \in C \Rightarrow \lambda(v-s) \in C \qquad \forall \lambda \geq 0$$

(a cone, is not necessarily a convex set)

Suppose then here that $s=0$ and K is a convex cone

$(65) \Rightarrow (65\text{ bis})$ because (65) implies in particular:

$$\langle J'(u), \lambda v - u\rangle \geq \quad \forall \lambda \geq 0 \text{ and } \forall v \in K.$$

that is to say:

$$\lambda \langle J'(u), v\rangle - (J'(u), u) \geq 0 \qquad \forall \lambda > 0 \text{ and } v \in K$$

for $\lambda \to \infty$ we obtain:

$$\left.\begin{array}{l} \langle J'(u), v\rangle \geq 0 \quad \forall v \in K \\ \text{for } \lambda \to 0 \text{ we obtain:} \\ \langle J'(u), u\rangle \leq 0 \end{array}\right\} \Rightarrow \langle J'(u), u\rangle = 0.$$

$(65\text{ bis}) \Rightarrow (65)$ is obvious

<u>Example 6</u>: V is an Hilbert space, K closed convex set $\subset V$.

$$J(v) = \| f - v \| \qquad f \text{ given in } V.$$

J satisfies obviously the assumptions of the theorem 13 and the solution u of $u \in K$, $J(u) \leq J(v)$ $\forall v \in K$. is characterized by:

$$\langle J'(u), v-u\rangle = \frac{-(f-u, v-u)}{\| v-u \|} \geq 0 \qquad \forall v \in K.$$

That is to say $(f-u, v-u) \leq 0 \quad \forall v \in K$.

Where $(\ ,\ )$ is the scalar product in V.

<u>interpretation</u>: $J(v)$ represents the distance of f to the convex. J is strictly convex, therefore u is unique. So we find again by this way a classic result of analysis about the existence and uniqueness of an element $u \in K$ which realizes the minimum distance from an element f of an Hilbert space to a closed convex set of this space.

<u>Example 7</u>: $J(v) = \int_\Omega \left(|\operatorname{grad} v|^2 + |v|^2\right) dx - 2\int_\Omega f.v \, dx.$

$$K = \{ v \mid v \in H^1(\Omega) \quad v \geq 0 \text{ on } \Gamma \}$$

K is a convex cone of vertex 0; it is closed in $H^1(\Omega)$ (Proof p 45)
J is strictly convex then $\exists\, u \in K$ unique such that

$$J(u) \leq J(v) \quad \forall\, v \in K.$$

u is characterized by

$$(69) \quad \begin{cases} u \in K \\ a(u,v) \geq \int_\Omega f.v\, dx & \forall\, v \in K \\ a(u,u) = \int_\Omega f.u\, dx \end{cases}$$

with a defined by $a(v,w) = \int_\Omega (vw + \operatorname{grad} v \operatorname{grad} w)\, dx$

in fact $\quad (J'(u), v) = 2\left[a(u,v) - (f,v)\right]$

Let us interpret now:

$\mathcal{D}(\Omega) \subset K$, we can then apply (69) for $v = \pm\varphi \in \mathcal{D}(\Omega)$; we obtain:

$$\int_\Omega (-\Delta u + u)\,\varphi\, dx = \int_\Omega f.\varphi\, dx \quad \forall\, \varphi \in \mathcal{D}(\Omega)$$

$$\Leftrightarrow \quad (70) \quad -\Delta u + u = f \quad \text{in } \mathcal{D}'(\Omega).$$

and now, from (70) and (69) we obtain.

$$\int_\Omega f.v\, dx = \int_\Omega (-\Delta u + u)\, v\, dx \leq a(u,v) \quad \forall\, v \in K.$$

$$\Leftrightarrow \quad -\int_\Gamma \frac{\partial u}{\partial n}.\gamma_0 v\, d\Gamma + a(u,v) \leq a(u,v) \quad \forall\, v \in K$$
$$= \quad \text{if } v = u.$$

Whence

$$(71) \quad \begin{cases} \int_\Gamma \frac{\partial u}{\partial n}\gamma_0 v\, d\Gamma \geq 0 & \forall\, v \in K, \\ \qquad = 0 & \text{if } v = u. \end{cases}$$

reciprocally $(70) + (71) \Rightarrow (69)$
u and $v \in K \Rightarrow \gamma_0 u$ and $\gamma_0 v \geq 0$ on Γ
hence $(71) \Leftrightarrow \frac{\partial u}{\partial n} \geq 0$ and $\frac{\partial u}{\partial n}.u = 0$ on Γ

Therefore either u or $\frac{\partial u}{\partial n} = 0$.

and finally, u is the unique solution of

$$(72)\quad \begin{cases} -\Delta u + u = f \\ u \geq 0 \text{ on } \Gamma \\ \dfrac{\partial u}{\partial n} \geq 0 \text{ on } \Gamma \\ u \cdot \dfrac{\partial u}{\partial n} = 0 \text{ on } \Gamma \end{cases}$$

because the uniqueness of u, the two parts of the boundary

$\Gamma_0 = \{ x \,/\, x \in \Gamma$ such that $u = 0$ almost every where$\}$.

$\Gamma_1 = \{ x \,/\, x \in \Gamma$ such that $\dfrac{\partial u}{\partial n} = 0$ almost every where$\}$.

are well determined by the knowledge of u.

V.3. Variational inequalities

There is an immediate application of the theorems 12, 13 and 14

Theorem 15: Let V be a Hilbert space.
K a closed convex subset of V
a a continuous, symetric, coercive bilinear form on V
$f \in V'$ dual of V
Then $\exists\, u \in K$ unique such that

$$(78) \qquad a(u, v-u) \geq (f, v-u) \quad \forall v \in K.$$

Proof: Let us consider the functional

$$J(v) = a(v,v) - 2(f,v)$$

• J is strictly convex, in fact: for $\forall\ \theta \in]0,1[$

$$J[(1-\theta)u + \theta v] = a((1-\theta)u + \theta v, (1-\theta)u + \theta v) - 2\langle f, (1-\theta)u + \theta v\rangle$$
$$= A - 2\theta\langle f, v\rangle - 2(1-\theta)\langle f, u\rangle$$
$$A = (1-\theta)^2 a(u,u) + \theta^2 a(v,v) + 2\theta(1-\theta)\, a(u,v)$$
$$= (1-\theta)\{a(u,u) + \theta[a(u,v) - a(u,u)]\} + \theta\{a(v,v) + (1-\theta)[a(u,v) - a(v,v)]\}$$

$$A = (1-\theta)\, a(u,u) + \theta\, a(v,v) - \theta(1-\theta)\, a(u-v,u-v)$$
$$\leq (1-\theta)\, a(u,u) + \theta\, a(v,v) - \theta(1-\theta)\alpha \|u-v\|^2$$

if α is the constante >0 of coercivity such that

$$a(u,u) \geq \alpha \|u\|^2 .$$

Whence

$$J[(1-\theta)u + \theta v) \leq (1-\theta) J(u) + \theta J(v) - \theta(1-\theta)\alpha\|u-v\|^2.$$
$$< (1-\theta) J(u) + \theta J(v) \qquad si\ u \neq v \qquad q.e.d.$$

J is differentiable and

$$\langle J'(u), v\rangle = 2\,(a(u,v) - (f,v))$$

The assumption (i) of the theorem 12 is realized because

$$J(v) = a(v,v) - 2(f,v) \geq \alpha\|v\|^2 - 2(f,v)$$
$$\geq \alpha\|v\| \left[\|v\| - 2\|f\|_{*}\right]$$

then $\exists$ an unique $u \in K$ s.t $J(u) \leq J(v) \quad \forall v \in K$

and from the theorem 14, u is characterized by

$$a(u, v-u) \geq \langle f, v-u\rangle \quad \forall v \in K.$$

Remark 23: We obtained the preceding result by direct application of the theorems 12. 13 and 14 but are the hypothesis on the bilinear form absolutely necessary? The answer is no but then the functional inequalities obtained are no longer corresponding to a problem of minimization of a functional.

V. 4 Non Variational Inequalities.

4. 1 absence of symetry for $a(u,v)$

What about the problem

(94) $u \in K, \quad a(u, v-u) \geq \langle f, v-u\rangle \quad \forall v \in K.$

where a, K, f verify the same hypothesis as above, except the symetry of $a(\ ,\ )$? The answer is given in the following statement

Theoreme 16: With the same Hypothesis as in the preceding theoreme 15 except the symetrie of the bilinear form $a(\ ,\ )$. The conclusion is the same the proof of this theorem is non trivial but it uses an interesting method of fixed point. We can no longer use the theorems 12.13.14.

a) Ideas of the proof

the problem (74) is non linear; we can, as we did in the case of the Lax Milgram Theorem, introduce an operator A from V in V' and now A will be non linear.

In the general case of a non linear problem of the type

(75) $\quad Au = f.$

it is sometime useful to resolve the equivalent problem

(76) $\quad Su + \rho(Au - f) = Su \quad$ for $\rho > 0$.

We begin, in supposing u given, to resolve

(77) $\quad Sw = Su - \rho(Au - f)$

S is supposed simple, linear such that w is defined in an unique way, then

$$w = \phi(u)$$

and we have then to look for the fixed points of ϕ

$$\phi(u) = u.$$

b) Preliminary reduction

Let $(,)$ be the scalar product on V. We can then introduce $\mathcal{A}$ by

$$a(u,v) = (\mathcal{A}u, v)$$

$\mathcal{A}$ is a continuous operator from V in V, in fact

$$|(\mathcal{A}u, v)| = |a(u,v)| \leq M \|u\| \|v\|$$

$$\Rightarrow \quad \|\mathcal{A}u\| \leq M \|u\|.$$

and by identification of V and V' we have

$$\langle f, v\rangle = (\tilde{f}, v)$$

The new problem to resolve is

(74 bis) $\quad (\mathcal{A}u - \tilde{f}, v - u) \geq 0 \quad \forall v \in K.$

Let apply the ideas above taking for S the identity

(78) $\quad (w, v - w) \geq (u - \rho(\mathcal{A}u - \tilde{f}), v - w) \quad \forall v \in K.$

Let put: $u - \rho(\mathcal{A}u - f) = g$ (78) is then a variational inequality.

(79) $w \in K$, $(w, v-w) \geq (g, v-w)$

associated to the functional:

$$J(v) = \|v\|^2 - 2(g, v)$$

hence;

$\forall u \in K$; $\exists\, w = \phi(u)$ such that (78) is satisfied.

c) <u>Research of a fixed point for ϕ.</u>

If we have a strict contraction; that is to say if $\exists\, 0<\theta<1$. such that

$$\|\phi(u_1) - \phi(u_2)\| \leq \theta \|u_1 - u_2\| \qquad \forall u_1 \text{ and } u_2 \in V$$

then, the classic theorem of successive approximations leads us to the conclusion of the existence and uniqueness of a fixed point.

Let show then that $\exists\, \rho > 0$ such that ϕ is a strict contraction

$$\phi(u_1) - \phi(u_2) = w_1 - w_2 \quad \text{but}$$

$(w_1, v - w_1) \geq (u_1 - \rho(\mathcal{A}u_1 - f), v - w_1)$ $\quad \forall v \in K$.

$(w_2, v - w_2) \geq (u_2 - \rho(\mathcal{A}u_2 - f), v - w_2)$ $\quad \forall v \in K$.

let us choose $v = w_2$ in the first inequality and $v = w_1$ in the second one; after addition we obtain

$$-\|w_1 - w_2\|^2 \geq -(u_1 - \rho(\mathcal{A}u_1 - f) - u_2 + \rho(\mathcal{A}u_2 - f),\ w_1 - w_2)$$
$$= -((I - \rho\mathcal{A})(u_1 - u_2),\ w_1 - w_2)$$

$\Longrightarrow$

$$\|w_1 - w_2\|^2 \leq \|(I - \rho\mathcal{A})(u_1 - u_2)\|\,\|w_1 - w_2\|$$

$\Longrightarrow$

$$\|\phi(u_1) - \phi(u_2)\| \leq \|(I - \rho\mathcal{A})\|_* \|u_1 - u_2\|$$

because the continuity of $\mathcal{A}$ and therefore of $I - \rho\mathcal{A}$

we have now to estimate $\|I - \rho\mathcal{A}\|_*$ norm of $I - \rho\mathcal{A}$ in $\mathcal{L}(V, V)$

$$\|(I - \rho\mathcal{A})\varphi\|^2 = \|\varphi\|^2 + \rho^2\|\mathcal{A}\varphi\|^2 - 2\rho(\mathcal{A}\varphi, \varphi)$$

$(\mathcal{A}\varphi, \varphi) = a(\varphi, \varphi) \geq \alpha\|\varphi\|^2$; and $\|\mathcal{A}\varphi\| \leq M\|\varphi\|$

$$\|(I-\rho\mathcal{A})\varphi\|^2 \leq \|\varphi\|^2 + \rho^2 M^2 \|\varphi\|^2 - 2\alpha\rho\|\varphi\|^2$$
$$= (1 - 2\alpha\rho + \rho^2 M^2)\|\varphi\|^2$$

$1 - 2\alpha\rho + \rho^2 M^2 < 1$ as soon as $\rho < \frac{2\alpha}{M^2}$

then.

$$\|\phi(u_2) - \phi(u_1)\| \leq \theta\|u_2 - u_1\| \quad \text{with } \theta < 1 .$$

ϕ is a strict contraction and admits a fixed point which can be approached by the methode of successive approximations.

<u>Example 8</u> : $V = H^1(\Omega)$
$K = \{v \,/\, v \geq 0 \text{ on } \Gamma\}$

- K is a closed convex cône of V.

in fact: Let (φ_n) be a cauchy sequence in K.

$\varphi_n \longrightarrow \varphi$ in $H^1(\Omega)$

$\gamma_0\varphi_n \longrightarrow \gamma_0\varphi$ in $H^{1/2}(\Gamma)$ because the continuity of γ_0.

$\Rightarrow (\gamma_0\varphi_n, v) \rightarrow (\gamma_0\varphi, v) \quad \forall v \in H^{-1/2}(\Gamma)$.

and therefore $\forall v \in \mathcal{D}(\Gamma)$

but $(\gamma_0\varphi_n, v) \geq 0 \quad \forall v \in \mathcal{D}(\Omega)$

$\Rightarrow (\gamma_0\varphi, v) \geq 0 \quad \forall v \in \mathcal{D}(\Omega)$

- $a(u,v) = \int_\Omega u.v\,dx + \int_\Omega u_{,i} v_{,i}\,dx + \int_\Omega (u_{,1} v - u v_{,1})\,dx$.

a is a non symetric bilinear form.

but a is coercive: $a(u,u) = \|u\|^2_{H^1(\Omega)}$

and a is continuous:

$$|a(u,v)| \leq \|u\|_{H^1}\|v\|_{H^1} + |u_{,1}|_{L^2}|v|_{L^2} + |u|_{L^2}|v_{,1}|_{L^2}.$$

$$\leq \|u\|_{H^1}\|v\|_{H_1} + \|u\|_{H^1}\|v\|_{H_1} + \|u\|_{H_1}\|v\|_{H_1}.$$

$$\leq 3\|u\|_{H_1}\|v\|_{H_1}.$$

We can therefore apply the theorem 16.

4.2 absence of the symetry for $a(u,v)$ and replacement of the coercivity by a property of positivity

What about the problem.

$$(80) \quad u \in K, \quad a(u, v-u) \geq \langle f, v-u \rangle \quad \forall v \in K.$$

$$\begin{cases} \text{Where} & a(u,v) \neq a(v,u) \\ \text{and} & a(v,v) \geq 0 \quad \forall v \in K. \end{cases}$$

in some particular cases we are able to prove the existence of at least one solution for (80). For instance if

$$\left.\begin{array}{l} a(u,v) \text{ is symetric} \\ K \text{ bounded} \end{array}\right\} \Rightarrow \text{there exist some solutions.}$$

the Theory is therefore non Empty, and we are able to obtain some information about the set of eventual solutions.

Theorem 17: Let us suppose we have the same Hypothesis than in the theorem (15) except the symetry of a. The coercivity being replaced by.

$$a(v,v) \geq 0 \quad \forall v \in K.$$

Suppose that there exist at least one solution for.

$$(80) \quad u \in K \quad a(u, v-u) \geq (f, v-u) \quad \forall v \in K.$$

let X be the set of solutions of (80) then X is a closed convex of V.

Proof:

1) $(80) \Leftrightarrow (81) \quad a(v, v-u) \geq (f, v-u) \quad \forall v \in K.$

in fact: $a(v, v-u) = a(v-u, v-u) + a(u, v-u) \geq a(u, v-u)$

Reciprocally: Let suppose that u is solution of (81)

$$\forall w \in K, \qquad v = (1-\theta)w + \theta u \in K \quad \text{if} \quad \theta \in [0,1] \subset \mathbb{R}$$

$$\text{and} \qquad v - u = (1-\theta)(w-u)$$

then $(1-\theta)\, a\left[(1-\theta)w + \theta u, w-u\right) \geq (1-\theta) \langle f, w-u \rangle \quad \forall w \in K$

$$\Rightarrow \quad a\left[(1-\theta)w + \theta u, w-u\right] \geq \langle f, w-u\rangle \quad \forall w \in K \text{ and } \theta \in [0,1[$$

When $\theta \to 1$, thanks to the continuity of $a(\,,\,)$ we obtain

$$a(u, w-u) \geq \langle f, w-u \rangle \quad \forall w \in K.$$

That is to say: u is solution of (80) q.e.d.

b) Let suppose now that u_1 and u_2 are solutions of (80), they are also solutions of (81) and $u_1 \in X$, $u_2 \in X$.

What about $(1-\theta)u_1 + \theta u_2$?

$$a(v, v-u_1) - \langle f, v-u_1\rangle \geq 0 \qquad \forall v \in K$$

$$a(v, v-u_2) - \langle f, v-u_2\rangle \geq 0 \qquad \forall v \in K.$$

$$\Rightarrow \quad a(v, v-(1-\theta)u_1 - \theta u_2) - \langle f, v-(1-\theta)u_1 - \theta u_2\rangle$$

$$= (1-\theta)\left[a(v, v-u_1) - \langle f, v-u_1\rangle\right] + \theta\left[a(v, v-u_2) - \langle f, v-u_2\rangle\right] \geq 0.$$

$$\Rightarrow \quad (1-\theta)u_1 + \theta u_2 \in X.$$

X is therefore a convex set

(the idea to take (81) in place of (80) is to make the inequality linear in u)

c) X is closed

That is because a Cauchy sequence in X is converging toward an element of V (which is a Hilbert space, therefore a complete space)

$$\{u_n\} \subset X \qquad\qquad u_n \to u \in V.$$

and $\lim\left[a(v, v-u_n) - \langle f, v-u_n\rangle\right] \geq 0 \quad \forall v \in K.$

$$\Rightarrow \quad a(v, v-u) - \langle f, v-u\rangle \geq 0. \quad \forall v \in K \Rightarrow u \in X.$$

<u>Remark 24</u>: If now we suppose that X is non empty, it is interesting to be able to approach an element of X by a converging sequence in K. That is the object of the following Theorem. For that, we perturb the inequality (80) in adding to it a coercive one multiplied by a small parameter ε and we study the convergence of the solution of the new inequality.

Theorem 18 : Let V be an Hilbert space.
$a(\,,\,)$ a continuous bilinear positive form :
(82) $a(v,v) \geq 0$
f given element of V'
Let X the set of the solutions u of (80) supposed non empty

Let u_0 be the unique solution of

$$(83) \quad \begin{cases} u_0 \in X \\ b(u_0, v-u_0) \geq \langle g, v-u_0 \rangle \quad \forall v \in X \end{cases}$$

where b is a continuous, coercive bilinear form on V.
g a given element of V'

Let u_ε be the eventual solution of:

$$(84) \quad \begin{cases} u_\varepsilon \in K \\ a(u_\varepsilon, v-u_\varepsilon) + \varepsilon\, b(u_\varepsilon, v-u_\varepsilon) \geq \langle f+\varepsilon g, v-u_\varepsilon \rangle \quad \forall v \in K. \end{cases}$$

for $\varepsilon > 0$

Then : 1/ u_ε is well defined and unique.
2/ $u_\varepsilon \longrightarrow u_0$ in V when $\varepsilon \to 0$.

Proof. a) (84) has an unique solution.
in fact: $\tilde{f}_\varepsilon = f + \varepsilon g \in V'$
$\tilde{a}_\varepsilon(u,v) = a(u,v) + \varepsilon\, b(u,v)$ is coercive and continuous
$\Rightarrow$ $\exists\, u_\varepsilon$ unique solution of (84) (theorem 16).

b) $\lim\limits_{\varepsilon \to 0} u_\varepsilon = u_0 \in V$

in fact: we have first weak convergence of u_ε toward an element $w \in V$

$u_0 \in X \Rightarrow a(u_0, v-u_0) - \langle f, v-u_0 \rangle \geq 0 \quad \forall v \in K.$
Let us choose here $v = u_\varepsilon$
Let us choose $v = u_0$ in (84)
then, after adding the two inequalities, we obtain:
$- a(u_\varepsilon - u_0, u_\varepsilon - u_0) + \varepsilon\, b(u_\varepsilon, u_0 - u_\varepsilon) - \varepsilon \langle g, u_0 - u_\varepsilon \rangle \geq 0$

$$\Rightarrow \beta \|u_\varepsilon\|^2 \le b(u_\varepsilon, u_\varepsilon) \le b(u_\varepsilon, u_0) - \langle g, u_0 - u_\varepsilon\rangle \le c'(1 + c\|u_\varepsilon\|)$$

$$\Rightarrow \|u_\varepsilon\| \le c''$$

therefore, $\exists$ a subsequence $u_{\varepsilon'}$ weakly converging in V (theorem 9)

$u_{\varepsilon'} \longrightarrow w$ and $w \in K$ closed convex of V.

Now, let us show that $w \in X$:

(84) can be written.

$$a(u_\varepsilon, v) + \varepsilon b(u_\varepsilon, v) - \langle f + \varepsilon g, v - u_\varepsilon\rangle \geq a(u_\varepsilon, u_\varepsilon) + \varepsilon b(u_\varepsilon, u_\varepsilon)$$

$a + \varepsilon b$ is a continuous, coercive bilinear form and therefore is semi continuous from below for the weak convergence.

$$\liminf_{\varepsilon \to 0} a(u_\varepsilon, u_\varepsilon) + \varepsilon b(u_\varepsilon, u_\varepsilon) \geq a(w, w)$$

hence:

$$\liminf \left[a(u_\varepsilon, v) + \varepsilon b(u_\varepsilon, v) - \langle f + \varepsilon g, v - u_\varepsilon\rangle\right] = a(w, v) - \langle f, v - w\rangle \geq a(w, w) \geq$$

$\forall v \in K$.

$\Rightarrow w \in X$; let us show that $w = u_0$

in (83), let us choose then $v = w$ we obtain

$$(86) \quad b(u_0, w - u_0) - \langle g, w - u_0\rangle \geq 0$$

but in a other hand from (85) we obtain

$$b(u_\varepsilon, u_0) - \langle g, u_0 - u_\varepsilon\rangle \geq b(u_\varepsilon, u_\varepsilon) \text{ and then.}$$

$$b(w, u_0) - \langle g, u_0 - w\rangle = \liminf \left[b(u_\varepsilon, u_0) - \langle g, u_0 - u_\varepsilon\rangle\right]$$

$$\geq \liminf b(u_\varepsilon, u_\varepsilon) \geq b(w, w)$$

$$\Rightarrow (87) \quad b(w, u_0 - w) - \langle g, u_0 - w\rangle \geq 0.$$

$$(86) + (87) \Rightarrow -b(u_0 - w, u_0 - w) \geq 0.$$

and because b is coercive: $w = u_0$.

We have to finish, to prove that u_ε converges strongly towards u_0.

$$(85) \Rightarrow b(u_\varepsilon - u_0, u_\varepsilon - u_0) \le -\langle g, u_0 - u_\varepsilon\rangle - b(u_0, u_\varepsilon - u_0)$$

$$u_\varepsilon \xrightarrow{\text{weakly}} u_0.$$

$$\Rightarrow \lim_{\varepsilon \to 0} \beta \|u_\varepsilon - u_0\|^2 \le -\lim_{\varepsilon \to 0} \left(\langle g, u_0 - u_\varepsilon\rangle + b(u_0, u_\varepsilon - u_0)\right) = 0$$

$$\Rightarrow u_\varepsilon \longrightarrow u_0 \text{ strongly.}$$

Remark 25 : for $u_\varepsilon \notin X$. the interest of this theorem is that the ...

Example 9 : We can approach the element of X with a minimum norm

For that, we can choose:

$b(u,v) = (u,v)$, scalar product in V

and $g = 0$.

u_0 is then solution of:

$$\langle u_0, v - u_0 \rangle \geq 0 \quad \forall v \in X$$

$$\Rightarrow \quad \|u_0\|^2 \leq \|v\| \|u_0\| \quad \forall v \in X.$$

$$\Rightarrow \quad \|u_0\| \leq \|v\| \quad \forall v \in X. \quad \text{q.e.d}$$

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