INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
MIRAMARE - P.O.B. 586 - 34100 TRIESTE (ITALY) - TELEPHONES: 224281/2/3/4/5/6 - CABLE: CENTRATOM

SMR/23 - 13

AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

22 September - 26 November 1976

ELEMENTS OF THE THEORY OF ELASTIC STABILITY

R.J. Knops
Heriot-Watt University
Riccarton, Edinburgh (U.K.)

ELEMENTS OF THE THEORY OF ELASTIC STABILITY.

Chapt. I. Basic stability concepts and definitions.

. Heuristic ideas.

A frequently quoted, loose, definition of stability, is the implied generalisation of that due to LAGRANGE AND DIRICHLET:

"An equilibrium soln. belonging to a system is stable if perturbations starting in the neighbourhood of the equilibrium soln. remain in the neighourhood for all time".

Becomes precise only after careful definition of the terms:

(a) System
(b) Duration of examination (is $\forall$ time really necessary?)
(c) Time at which initial disturbance occurs.
(d) Neighourhood
(e) Measurement of perturbations.
(f) Equilibrium posn. (or motion, perhaps?)

1. Dynamical System

Primitive form only needed for definition. more structure is usual.

i) Wish to describe thermodynamic parameters — displacement, velocity, stress, temperature — entering the stability investigation

ii) Example. Displacement $\underset{\sim}{u} \in \mathbb{R}^3$, is a function of

(a) position: $\underset{\sim}{x} \in \Omega \subset \mathbb{R}^3$
(b) time: $t \in T \subseteq \mathbb{R}^+$

As fn. of $\underset{\sim}{x}$, displacement belongs to set X: $\underset{\sim}{u}(\cdot, t) \in X$, t fixed

" " " t, we have $\underset{\sim}{u} : t \longmapsto \underset{\sim}{u}(\cdot, t)$.

(iv) Generally, we consider the functions

$$\varphi : T \longrightarrow X$$
$$t \longmapsto \varphi(t) \in X \qquad t \in T.$$

Example: $\varphi \in C[(0,T); W^{1,2}]$.

Observe: φ generic symbol for displacement, stress, etc ...

(v) Require e.g., displacement to remain in X irrespective of when observations start. Thus, let $\tau \in J \subseteq T$, and impose condition:-

$$\varphi(\tau + t) \in X \qquad \tau \in J, \; t \in T.$$

(vi) THEN functions φ form a dynamical system.

(vii) Example:
$$\left.\begin{aligned} &u_{,xx} = u_{,tt}, \quad 0 \le x \le 1, \quad 0 \le t < \infty \\ &u(0,t) = u(1,t) = 0. \end{aligned}\right\} \qquad (1.2.1)$$

Choose

$$X = \{ \psi \in C^2[0,1] : \psi(0) = \psi(1) = 0 \}.$$

Dynamical system defined by (1.2.1) is set of functions $u(t,\cdot)$ defined on $[0,\infty)$, satisfying $(1.2.1)_1$, and taking values in X.

1.3. Definition of a dynamical system.

X ... arbitrary set containing thermodynamic variables

T ... set of non-negative real numbers (possibly included ∞)

$\mathcal{B}(T,X)$ ... set of functions defined on T taking in values in X

$$\mathcal{B}(T,X) = \{ \varphi : \varphi : T \to X \}$$

J ... subset of T

φ_τ ... translate of φ defined by

$$\varphi_\tau(t) = \varphi(\tau + t), \quad \tau \in J, \; t \in T.$$

Definition 1.3.1. (Dynamical System): A dynamical system corresponding to the triple $(T, \mathcal{T}, X)$ is a set of functions $\mathcal{B}(T,X)$ defined on T taking values in X such that

(i) $\varphi_\tau \in \mathcal{B}(T,X)$, for $\varphi \in \mathcal{B}(T,X)$, $\tau \in \mathcal{T}$.

(ii) $\lim_{t \to 0} \varphi_\tau(t) = \varphi(\tau)$, " $\varphi \in \mathcal{B}(T,X)$, $\tau \in \mathcal{T}$.

Defn. 1.3.2. (a) The function φ is the <u>solution</u>

(b) The translate φ is the <u>motion</u>.

(c) The trajectory is the graph of the motion

(d) The orbit is the set of values of the motion.

Defn 1.3.3. (a) The <u>equil. soln.</u> is defined by

$$\varphi_\tau(t) = \varphi(\tau), \qquad \text{fixed } \tau \in \mathcal{T};\ \forall t \in T,\ \varphi \in \mathcal{B}(T,X).$$

(b) The <u>null soln.</u> is

$$\varphi_\tau(t) = 0. \qquad \text{"} \qquad \text{"} \qquad \text{"}$$

Remark: The null solution is associated with null initial data ($\varphi_\tau(0) = 0$). Other solutions may have same data (see later).

4. <u>Initial data and related mappings.</u>

Defn. 1.4.1. The set of "initial" values of the motion φ_τ is

$$Y(\tau) = \{\varphi(\tau) : \varphi \in \mathcal{B}(T,X),\ \tau \in \mathcal{T}\}, \tag{1.4.1}$$

<u>Remark</u>: $Y(\tau) \subseteq X$.

Defn. 1.4.2. The <u>evolutionary maps</u> are $T_\tau : \varphi(\tau) \to \varphi_\tau$ (1.4.2)

<u>Remark</u> (i) T_τ are important for stability — they relate initial data with ~~motion~~ trajectory.

(ii) T_τ need not be single-valued

5. <u>Positive-definite functions.</u>

These are used to define neighborhoods.

Defn 1.5.1. A positive-definite function on $X \times X$ to $\mathbb{R}$ at y is a function ρ satisfying

(a) $\rho(x,y) \geq 0 \quad \forall x \in X,\ y \in X$.

(b) $\rho(x,y) = 0 \Leftrightarrow x = y$.

Remark: Stability really only requires local behavior.

The positive definite function on $X \times X$ induces one on $\mathcal{B}(T,X)$ by
$$\rho(\varphi,\psi) = \sup_{t\in T} \rho\left(\varphi(t), \psi(t)\right), \qquad \varphi,\psi \in \mathcal{B}(T,X).$$

<u>Defn. 1.5.2</u>. The open ball ~~on~~ in $\mathcal{B}(T,X)$, centre φ, radius r, is
$$S(\varphi,r) = \{\psi \in \mathcal{B}(T,X) : \rho(\varphi,\psi) < r, \ \varphi \in \mathcal{B}(T,X)\}.$$

Remark (i) ρ does not define topology

(ii) positive-definite fns. include metric and norm.

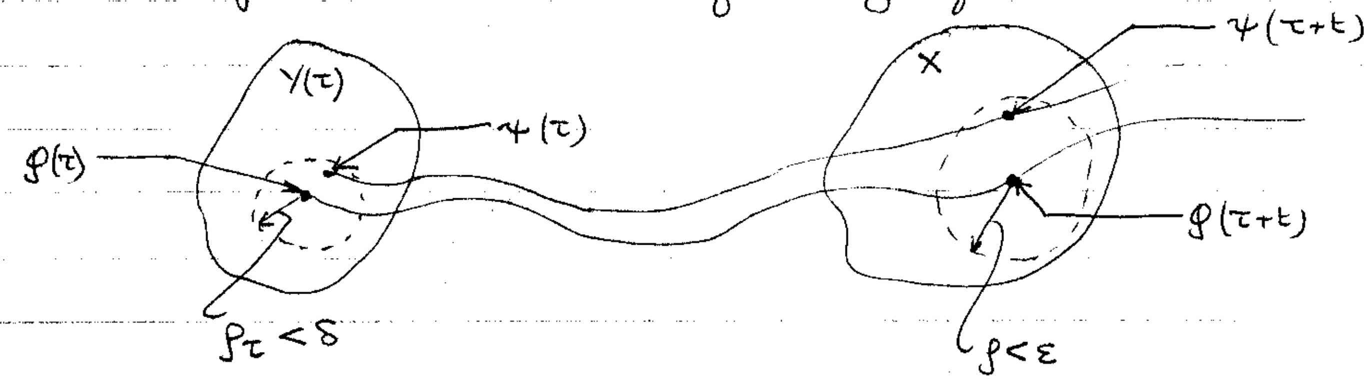
Neighbourhoods on the initial data set $Y(\tau)$ are defined in terms of a <u>second</u> positive-definite definite function ρ_τ. (Movchan).

<u>Remark</u>: While no continuity properties are required on ρ, ρ_τ, trivial instabilities arise if they are not imposed (see later).

Chapt II. Stability definitions.

2.1. Introduction.

Diagrammatic representation of stability definition.



Observe: Stability is continuity of map T_τ at φ.

Defn 2.1. (Liapunov stability). The soln. $\varphi \in \mathcal{B}(T, X)$ is Liapunov stable if and only if for each $\tau \in J$, the mapping T_τ is continuous at φ. That is for each $\varepsilon > 0$, $\exists\ \delta(\tau, \varepsilon) > 0$ s.t. $\rho_\tau(\varphi(\tau), \psi(\tau)) < \delta \Rightarrow \rho(\varphi_\tau, \psi_\tau) < \varepsilon$, where

$$\rho(\varphi_\tau, \psi_\tau) = \sup_{t \in T} \rho(\varphi_\tau(t), \psi_\tau(t))$$

Remarks (i) Generalises Lagrange-Dirichlet defn.; also includes motions.

(ii) When δ indep. of τ, stability is uniform

(iii) Choice of +ve def. fn. crucially affects stability classification.

Example 2.2.1. (Hadamard).

$$u_{,xx} + u_{,tt} = 0 \qquad 0 \le x \le 1, \quad t > 0, \tag{2.2.1}$$

$$u(0,t) = u(1,t) = 0, \qquad t > 0, \tag{2.2.2}$$

$$u(0,x) = 0, \quad \dot{u}(0,x) = \tfrac{1}{n}\sin n\pi x, \quad n = 1, 2, \dots \tag{2.2.3}$$

Set $\tau = 0$.

$Y(\tau)$ .... given by (2.2.3)

T .... compact.

$$X = \{\psi \in C^2(0,1) : \psi(0) = \psi(1) = 0\}.$$

Explicit soln. is

Set

$$\rho_T\left(u(\xi),0\right) = \rho\left(u(\xi),0\right) = \sup_{x\in[0,1]} |u(\xi,x)|.$$

Then null solution is <u>unstable</u> (i.e., it is not stable).

Stability recovered on choosing

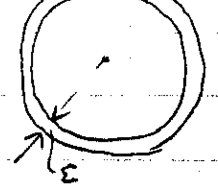
(a) $$\rho_T = \rho(u(t)) = \tfrac{1}{2}\int_0^1 [u_{,t}^2 - u_{,x}^2]\,dx$$

(b) By taking bounded set of solutions on compact sub-intervals of T, and setting $\rho_T, \rho = \| \ \|_{L_2}$. (Lavrentiev; see later).

<u>Example 2.2.2</u>. (Three-dimensional wave equation)

$$\left.\begin{aligned} \nabla^2 u &= u_{,tt} &\quad &\text{in } \Omega\,(\subset \mathbb{R}^3)\times[0,\infty) \\ u &= 0 & &\partial\Omega\times[0,\infty) \end{aligned}\right\}.$$

Is null solution stable?



For initial data, select:-

$u(x,0) = 0$

$u_{,t}(x,0) = \chi(x) \neq 0$ ~~at~~ inside ⊙^{cir} annulus,

$= 0$ outside " "

Then

$$E(0) \equiv \tfrac{1}{2}\int_\Omega \left(u_{,i}u_{,i} + u_{,t}^2\right)dx \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

But $$u(0,t) = \frac{1}{4\pi t}\iint_{\Sigma_t} \chi(x)\,dx,$$ Σ_t surface of sphere of radius t, centre O.

Thus displacement not arb. small at O for $\forall\, t$. Hence, with

$$\rho_T \sim E(0); \qquad \rho = \sup_{x\in\Omega} |u(x,t)|,$$

obtain <u>instability</u>.

quivalent measures.

. Suppose ρ_τ, ρ and $\hat\rho_\tau, \hat\rho$ are two sets of positive-definite functions satisfying

$$\rho_\tau \leq \alpha \hat\rho_\tau \qquad \alpha, \beta \quad +ve.$$
$$\hat\rho \leq \beta \rho$$

Then a solution stable w.r.t. ρ_τ, ρ is also stable w.r.t. $\hat\rho_\tau, \hat\rho$.

In a finite dimensional normed linear space, all norms are ~~equal~~ equivalent, in the sense that there exist constants $\alpha, \beta, \delta > 0$ s.t.

$$\alpha \| \ \|_1 \leq \beta \| \ \|_2 \leq \delta \| \ \|_1$$

for any pair of norms $\| \ \|_1$, $\| \ \|_2$.

Hence, in this sense, measures are superfluous for stability in discrete systems.

3. The result is not true for continuous systems.

3. Some further definitions.

Defn 2.3.1: We say ψ tends to φ at ∞ if $\lim_{t\to\infty} \rho(\varphi(t), \psi(t)) = 0$.

Defn 2.3.2. The soln. $\varphi \in \mathcal{B}(T, X)$ is asymptotically stable if and only if

(i) φ is stable

(ii) $\exists\, \delta > 0$ s.t. $\rho_\tau(\varphi(\tau), \psi(\tau)) < \delta \Rightarrow \psi_\tau \to \varphi_\tau$ at ∞.

Set

$$\rho_{t_1}(\varphi_\tau, \psi_\tau) = \sup_{t\in[0,t_1]} \rho(\varphi_\tau(t), \psi_\tau(t)), \qquad t_1 < \infty,$$

and let $\mathcal{B}_{\rho_1}(T, X)$ denote the set $\mathcal{B}(T, X)$ equipped with the +ve def. fn ρ_{t_1}.

Defn. 2.3.5. The soln. $\varphi \in \mathcal{B}(T, X)$ depends continuously on the initial data if the map

$$T_\tau : Y(\tau) \longrightarrow \mathcal{B}_{\rho_1}$$

is continuous at φ.

Defn 2.3.9. A soln. $\varphi \in \mathcal{B}(T,X)$ is Liapunov unstable if and only if for some $\tau \in J$, the mapping T_τ from $Y(\tau)$ to $\mathcal{B}(T,X)$ is discontinuous at φ. That is, $\varphi \in \mathcal{B}(T,X)$ is unstable $\Leftrightarrow$ for one instant $\tau \in J$, $\exists\ \varepsilon > 0$ s.t. $\forall\ \delta > 0$,

$$\rho_\tau(\varphi(\tau), \psi(\tau)) < \delta \Rightarrow \rho(\varphi_\tau, \psi_\tau) \geq \varepsilon.$$

Remarks (i) must hold for all δ, given ε.

(ii) alternatively, could define instability, as not stable.

2.4. Some immediate consequences of the definition of Liapunov stability.

2.4.1. Stability implies uniqueness (Movchan).

Defn 2.4.1. The soln. $\varphi \in \mathcal{B}(T,X)$ is unique for each $\tau \in J$ if $\psi(\tau) = \varphi(\tau) \Rightarrow \psi_\tau(t) = \varphi_\tau(t)$, $\tau \in J$, I

Thm. 2.4.1. If $\varphi \in \mathcal{B}(T,X)$ is Liapunov stable it is unique.

Proof. Suppose φ is not unique. Thus, $\exists\ \psi \in \mathcal{B}(T,X)$ s.t. $\psi(\tau) = \varphi(\tau)$ but $\psi_\tau(t_1) \neq \varphi_\tau(t_1)$ for $\tau \in J$ and some $t_1 \in T$. $\therefore \rho(\psi_\tau(t_1), \varphi_\tau(t_1)) \neq 0$. But φ is stable. Thus, for $\varepsilon > 0\ \exists\ \delta > 0$ s.t. for $\chi \in \mathcal{B}(T,X)$, $\rho_\tau(\chi(\tau), \varphi(\tau)) < \delta \Rightarrow \rho(\chi_\tau, \varphi_\tau) < \varepsilon$, and thus there is a contradiction $\square$.

2.4.2. Stability and boundedness.

Defn. 2.4.2. The map T_τ, $\tau \in J$, is bounded w.r.t. ρ_τ, ρ at $\varphi \in \mathcal{B}(T,X)$ if $\exists$ +ve consts. A, B s.t.

$$\rho_\tau(\varphi(\tau), \psi(\tau)) \leq A \Rightarrow \rho(\varphi_\tau, \psi_\tau) \leq B$$

Clearly, Liapunov stab. of φ $\not\Rightarrow$ boundedness of T_τ at φ.

A linear functional is continuous $\Leftrightarrow$ it is bounded. Hence, we have:—

Thm. 2.4.2. Let X be a normed linear space and let each map T_τ satisfy $T_\tau(\alpha\varphi) = \alpha T_\tau(\varphi)$ for $\varphi \in X$, $\alpha > 0$. Then, if ρ_τ, ρ are <u>norms</u>, a soln. $\varphi \in \mathcal{B}(T, X)$ is Liapunov stable if and only if each T_τ is bounded at φ; that is, if and only if, $\exists$ a positive constant B_τ s.t. $\rho_\tau(\varphi(\tau) - \psi(\tau)) \leq 1 \Rightarrow \rho(\varphi_\tau - \psi_\tau) \leq B_\tau$, $\psi \in \mathcal{B}(T, X)$.

Thm. 2.4.3. Let X be a set with the property that "$\varphi \in X \Rightarrow \alpha\varphi \in X$, $\alpha > 0$"; and let $T_\tau(\alpha\varphi) = \alpha T_\tau(\varphi)$ for each T_τ. Then, if ρ_τ, ρ satisfy

$$\rho_\tau(\alpha\psi, 0) \leq \alpha \rho_\tau(\psi, 0), \quad \rho(\alpha\psi, 0) = \alpha\rho(\psi, 0), \ \psi \in X,$$

the <u>null soln</u>. of $\mathcal{B}(T, X)$ is Liapunov stable if and only if each T_τ is bounded at the null soln.

<u>Proof</u>: Left as exercise.

<u>Remark</u>: Linear systems have unique solns. in class of bounded solns.