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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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Elements of the Theory of Elastic Stability
(Continued)

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3.1. Introduction.

We describe theorems for establishing Liapunov stability. They are

(i) Maximum principles

(ii) Direct (second) method of Liapunov.

3.2. Maximum principles

Thm 3.2.1. The soln. $\varphi \in \mathcal{B}(T,X)$ is Liapunov stable if there is a bounded real function $M_\tau(t)$ on T such that

$$\rho(\varphi_\tau(t), \psi_\tau(t)) \leq M_\tau(t)\, \rho_\tau(\varphi(\tau), \psi(\tau)), \quad \forall\, t \in T$$

and for each $\tau \in J$ and $\psi \in \mathcal{B}(T,X)$.

Proof: obvious.

Corollary (i) $M_\tau(t)$ indep. of $\tau \in J$ ... uniform stability

Corollary (ii) $M_\tau(t) \to 0$ as $t \to \infty$ ... asymptotic stability.

Remarks (i) C.p. fluid mechanics and use of (kinetic) energy when $M_\tau(t) \sim e^{\sigma t}$ (see for instance D.D. Joseph.)

(ii) Theorem is that essentially used in establishing Hölder stability (see later.)

3.3 Direct method.

This approach requires no explicit knowledge of the solution, but instead uses an auxiliary function. It generalises the Lagrange-Dirichlet energy test for ~~[illegible]~~ discrete systems; this test, which states that a minimum potential energy at equilibrium implies stability, has been uncritically extended to continuous systems often without proper justification.

To understand the direct method, we introduce some further notation:

$F_{\tau,t} : X \times X \to \mathbb{R}$ is a positive-definite function, where $\tau \in J$, $t \in T$.

$B_{F_\tau}(T,X)$ .... the set $B(T,X)$ equipped with measure

$$F_\tau(\varphi_\tau, \psi_\tau) = \sup_{t \in T} F_{\tau,t}(\varphi_\tau(t), \psi_\tau(t)), \quad \varphi, \psi \in B(T,X).$$

N^{ds} on $Y(\tau)$ still defined by ρ_τ

" " $B(T,X)$ " " " ρ.

Regard $B_{F_\tau}(T,X)$ as set intermediate to $Y(\tau)$ and $B(T,X)$. Consider the maps:

(i) $T_\tau : Y(\tau) \to B_{F_\tau}(T,X)$

(ii) identity mapping $i : B_{F_\tau}(T,X) \to B(T,X)$

$$\begin{tikzcd} & B_{F_\tau}(T,X) \arrow[dr, "i"] & \\ Y(\tau) \arrow[ur, "T_\tau"] \arrow[rr, "T_\tau"'] & & B(T,X) \end{tikzcd}$$

For stability, we require continuity at φ of $T_\tau : Y(\tau) \to B(T,X)$.

Hence, by composition law,

[Cont. at φ of $T_\tau : Y(\tau) \to B_{F_\tau}(T,X)$] + [Cont. at φ of $i : B_{F_\tau}(T,X) \to B(T,X)$]

$\Rightarrow$ [Cont. at φ of $T_\tau : Y(\tau) \to B(T,X)$].

Conversely, if φ is stable, then fns. $F_{\tau,t}$ can be found, ~~such~~ since

$$\rho \equiv F_{\tau,t}.$$

We have thus proved:

<u>Thm 3.3.1.</u> The soln. $\varphi \in B(T,X)$ is stable if and only if there exist positive-definite functions $F_{\tau,t}$, where $t \in T$, $\tau \in J$, defined on $X \times X$ for which

(i) given real $\varepsilon > 0$ $\exists$ $\delta(\tau, \varepsilon) > 0$ s.t.

$$\rho_\tau(\varphi(\tau), \psi(\tau)) < \delta \Rightarrow F_\tau(\varphi_\tau, \psi_\tau) < \varepsilon, \quad \psi \in B(T,X)$$

(ii) given real $\eta > 0$, $\exists$ real $\xi(\eta, \tau) > 0$ s.t.

$$F_\tau(\varphi_\tau, \psi_\tau) < \xi \Rightarrow \rho(\varphi_\tau, \psi_\tau) < \eta, \quad \psi \in B(T,X).$$

<u>Corollary</u> 1. The soln. is <u>uniformly stable</u> if conds (i), (ii) hold uniformly in τ

<u>Corollary</u> 2. The soln. φ is asymptotically stable if cond (i) hold and

Alternative formulations.

Movchan replaces Cond (i) by: (cf. also Zubov)

(iii) given real $\varepsilon > 0$ $\exists$ real $\delta(\varepsilon, \tau) > 0$ s.t.

$$\rho_\tau(\varphi(\tau), \psi(\tau)) < \delta \Rightarrow F_{\tau,0}(\varphi(\tau), \psi(\tau)) < \varepsilon,$$

and

(iv) $F_{\tau,t}(\varphi_\tau(t), \psi_\tau(t))$ is non-increasing w.r.t. t.

Remarks

(1) Unless continuity requirements are imposed on ρ_τ, ρ and $F_{\tau,t}$, trivial instabilities arise.

(2) Cond (ii) of Thm. 3.3.1. may be replaced by

(v) $$F_{\tau,t}(\varphi_\tau(t), \psi_\tau(t)) \geq \lambda / \rho(\varphi_\tau(t), \psi_\tau(t)), \quad t \in T,$$

where

$$\lambda : \mathbb{R} \to \mathbb{R} \quad \text{is monotone increasing, s.t}$$

$$\lambda(0) = 0, \qquad \lambda(x) \to \infty, \quad \text{as } x \to \infty.$$

When $\lambda(x) = \lambda x$, λ +ve const, $F_{\tau,t}$ is "positive-definite bounded below" (cp. Mikhlin).

(3) The alternative conds. (iii), (iv) and cond. (v) are only sufficient for stability.

(4) Instead of (iv) could have $F_{\tau,t}$ any "stable" soln. of O.D.E. "$\dot{x} = f(t, x)$".

3.4. <u>Example. The one-dimensional string.</u>

$$u_{,xx} = u_{,tt}, \qquad 0 \le x \le 1, \quad t > 0$$

$$u(t,0) = u(t,1) = 0,$$

Energy conserved:

$$E(t) = \frac{1}{2}\int_0^1 (u_{,t}^2 + u_{,x}^2)\,dx = E(0)$$

Moreover, easily shown that

$$u^2(t,x) \le \int_0^1 u_{,x}^2\,dx \le 2E(t) \qquad \text{(Schwarz)}$$

$$\therefore \sup_t \sup_x u^2 \le 2E(t) = 2E(0) \le \sup_x \left[u_{,x}^2(0,x) + u_{,t}^2(0,x) \right], \qquad (3.4.1)$$

and there is stability w.r.t. to the indicated measures. Observe that the Liapunov function, $F_{\tau,t}$, is here taken as the total energy $E(t)$ which is constant in time.

<u>Exercise</u>. Rework using the potential energy as Liapunov function. The P.Energy is neither constant in time nor time-decreasing, emphasizing that these properties are not essential for a Liapunov function.

To find stability for higher derivatives (existence?), we consider the <u>1st order total energy</u>

$$E_1(t) \equiv \frac{1}{2}\int_0^1 (\ddot{u}^2 + \dot{u}_{,x}^2)\,dx = E_1(0)$$

and since

$$\ddot{u}^2(t,x) \le \int_0^1 \dot{u}_{,x}^2\,dx$$

we obtain similar results to (3.4.1), but with u replaced by $\dot{u}$.

Moreover,

$$u_{,x}^2(t,x) \le \int_0^1 u_{,xx}^2\,dx, \qquad \text{as} \quad u_{,x}(\xi) = 0 \text{ for some } \xi \in [0,1]$$

and a similar result holds. (Hint: $\int_0^1 u_{,xx}^2\,dx = \int_0^1 \ddot{u}^2\,dx \le 2E_1(0)$).

<u>However</u>, (3.4.1) and the direct method here impose more smoothness on the initial data than is necessary; because, from D'Alembert's soln.

$$u(t,x) = \frac{1}{2}\left[f(x-t) + f(x+t) + \int_{x-t}^{x+t} g(\xi)\,d\xi \right],$$

f ... odd periodic extension of $u(0,x)$

g even " " " $\dot{u}(0,x)$

3.5. Discussion of Thm 3.3.1. The discrete case.

To establish cond (i) of Thm 3.3.1. involves the intrinsic props. of system together with continuity (topological) properties of $Y(\tau)$ and $\mathcal{B}_{F_\tau}$.

In the alternative conds, (iii) is "topological", while (iv) reflects behaviour of the system.

Cond (ii) is entirely "topological".

The "topological" conditions are all equivalent to the continuous embedding of one space in another, and must be established separately either a priori, or using e.g., the Sobolev embedding theorems. They are crucial steps in the proof for stability, characterising the continuous nature of the system, and are precisely conditions overlooked in earlier applications of elastic stability criteria.

We examine further this last point by considering the proof of Liapunov's theorem as usually given for discrete systems.

Thm 3.5.1. Let $X = \mathbb{R}^n$ (n-dimensional vector space), and let ρ_τ, ρ both be (equivalent to) the euclidean norm. The soln. φ is stable if and only if $\exists$ a continuous, positive definite functions $F_{\tau,t}$ $(\tau \in J, t \in T)$ s.t. $F_{\tau,t}(\varphi_\tau(t), \psi_\tau(t))$, $\varphi, \psi \in \mathcal{B}(X,T)$, are non-increasing w.r.t. t.

Proof: In $\mathbb{R}^n$ (and only in $\mathbb{R}^n$) the frontier of the unit ball is compact. Hence, for $\varepsilon > 0$, and by the cont. of $F_{\tau,t}$, we have

$$0 < k = \min_{\|x-y\|=\varepsilon} F_{\tau,t}(x,y) \qquad x, y \in \mathbb{R}^n \qquad (3.5.1)$$

where, w.l.o.g., $\rho_\tau \equiv \rho \equiv \| \quad \|^2$.

Fix φ_τ. By cont., $\exists$ $\delta(k,\tau) > 0$ s.t

$$F_{\tau,0}\big(\psi(\tau), \varphi(\tau)\big) < k \quad \text{whenever} \quad \|\varphi(\tau) - \psi(\tau)\| < \delta.$$

By monotonicity of $F_{\tau,t}$

$$\|\varphi(\tau)-\psi(\tau)\| < \delta \quad \Rightarrow \quad F_{\tau,t}\big(\psi_\tau(t), \varphi_\tau(t)\big) < k, \quad t \in T.$$

For $\|\varphi(\tau) - \psi(\tau)\| < \delta$, set

$$g(t) = \sup_{s \in [0,t]} \|\varphi_\tau(s) - \psi_\tau(s)\|$$

Let $t_1 \in T$ be first instant s.t.

$$g(t) < \varepsilon \qquad t \in [0, t_1)$$
$$g(t_1) = \varepsilon.$$

Then either t_1 is infinite or finite. W.l.o.g., assume second alt.

$$\therefore F_{\tau,t_1}\big(\psi_\tau(t_1), \varphi_\tau(t_1)\big) \geq k$$

Thus have contradiction

$$\therefore g(t) < \varepsilon \qquad \forall t \qquad \square.$$

Remarks.

(a) 'Topological' requirements of Thm 3.3.1 are here automatically satisfied by:—
 (i) Cont. fn. on compact set achieves its bounds.
 (ii) Frontier of sphere is compact $\Leftrightarrow$ space is finite-dimensional.

(b) Especially because of (ii) require "topological" cond. in continuous systems.

(c) Introduction of such assumptions like (3.5.1) found in several alternative proofs of elastic stability (Koiter, Gurtin; see later). They are related to Cond (v) of Thm 3.3.1, and also to the concept of a potential well, which enables an easy formal extension of Thm 3.5.1 to nonlinear elasticity.

However, apart from linear elasticity, verification of the conditions is difficult (see later.)

36. Instability Theorems

Observe that (i) could use necessary conditions for stability.

(ii) can immediately reject non-unique and unbounded solns.

<u>Theorem 3.6.1.</u> The soln. $\varphi \in \mathcal{B}(T,X)$ is unstable if and only if $\exists$ +ve def. $F_{\tau,t}$ such that

(i) $T_\tau : Y(\tau) \longrightarrow \mathcal{B}_{F_\tau}(T,X)$ is discontinuous at $\varphi(\tau)$ for some τ.

(ii) $i : \mathcal{B}(T,X) \longrightarrow \mathcal{B}_{F_\tau}(T,X)$ is continuous at $\varphi(\tau)$.

$$\begin{tikzcd} & \mathcal{B}_{F_\tau}(T,X) & \\ Y(\tau) \arrow[ur] \arrow[rr] & & \mathcal{B}(T,X) \arrow[ul] \end{tikzcd}$$

That is, (a) for at least one τ $\exists$ $\varepsilon > 0$ s.t. $\forall \delta > 0$

$$\rho_\tau(\varphi(\tau), \psi(\tau)) < \delta \Rightarrow F_\tau(\varphi_\tau, \psi_\tau) \geq \varepsilon$$

(b) for given $\eta > 0$ $\exists$ $\xi(\eta,\tau) > 0$ s.t.

$$\rho(\varphi_\tau, \psi_\tau) < \xi \Rightarrow F_\tau(\varphi_\tau, \psi_\tau) < \eta .$$

<u>Remarks</u> (i) Can achieve (a) by requiring

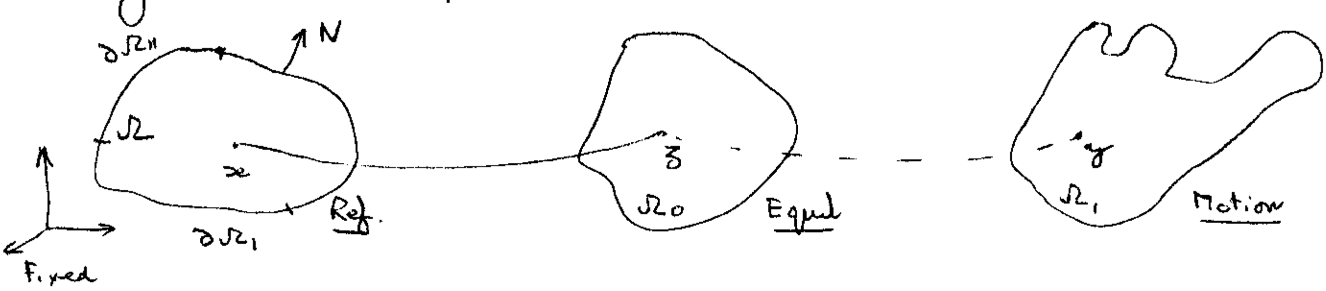
$$\dot{F}_{\tau,t} \geq \alpha > 0 \qquad \text{(Movchan - Zubov)}$$

(ii) Because, in general thermodynamics, suitable Liapunov fns. have rates that are ~~non~~ merely non-negative, the Movchan-Zubov condition is unsuitable for studying instability in such systems.

Chapt IV. Basic Equations of Nonlinear and Linear Elastodynamics.

4.1 Consider a purely mechanical, nondissipative and conservative theory. Thermal and other effects may be included, but at this stage tend to obscure developments. We also neglect body-forces.

A deformable body occupies in its unstressed (natural configuration) a bounded region $\Omega \in \mathbb{R}^3$, with smooth boundary $\partial\Omega$ on which the unit outward normal is N. Ω is the reference configuration. The body is deformed by application of surface displacement and traction to the equilibrium configuration Ω_0 whose stability is of interest. With the surface data held fixed at the equilibrium values, the body is instantaneously disturbed from the configuration Ω_0 by perturbations in the displacement and velocity, so that at time t afterwards it occupies the configuration Ω_1, which clearly is time-dependent.



Displacement is specified on $\partial\Omega_I$ }
Traction " " " $\partial\Omega_{II}$ } where $\partial\Omega = \partial\Omega_I \cup \partial\Omega_{II}$, and either $\partial\Omega_I = \phi$, or $\partial\Omega_{II} = \phi$.

We take a fixed set of cartesian axes, and identify each particle in the body by its position x in Ω. The particle which is at x in Ω, has the position z in Ω_0 (at time $t=0$) and the position y in Ω (at a general time instant $t>0$), where x, y, z are measured w.r.t. the same fixed cartesian axes.

<u>Notation</u>. No distinction is made between scalars, vectors and tensors, the meaning being clear from the context. Juxtaposition of symbols ... their multiplication in an appropriate sense. E.g. $\alpha u = (\alpha u_i)$,

4.2. The *deformation* of the body in configurations Ω_0 and Ω_1 is determined respectively by the maps

$$\left.\begin{aligned} z:\ \Omega &\longrightarrow \Omega_0 \\ x &\longmapsto z(x) \end{aligned}\right\} \qquad (4.2.1)$$

and

$$\left.\begin{aligned} y:\ \Omega &\longrightarrow \Omega_1 \\ x &\longmapsto y(t,x) \quad \text{at fixed } t \end{aligned}\right\} \qquad (4.2.2)$$

N.B. At this stage, computations are formal, and we assume maps z and y possess sufficient smoothness to justify subsequent calculations.

Defn. 4.2.2. The *deformation gradients* are given by

$$Z = \left(\frac{\partial z_i}{\partial x_\alpha}\right) = \nabla z \ ; \quad Y = \left(\frac{\partial y_i}{\partial x_\alpha}\right) = \nabla y, \qquad (4.2.3).$$

$$i = 1,2,3;\ \alpha = 1,2,3.$$

To avoid local interpenetration of material and to preserve orientation we require maps (4.2.1), (4.2.2) to be locally 1-1, and therefore invertible, and impose the *constraints*:

$$\det Z > 0 \ ; \qquad \det Y > 0, \qquad (4.2.4)$$

which, depending upon the smoothness of z and y may hold e.g., only p.p.

Defn. 4.2.3. The *displacement* is defined by

$$u(x,\cdot) = y(x,\cdot) - z(x),$$

and the *displacement gradient* by

$$U = \nabla u = \left(\frac{\partial u_i}{\partial x_\alpha}\right) = (u_{i,\alpha}) = \nabla y - \nabla z = Y - Z.$$

Defn. 4.2.4. The velocity is

$$\dot{y} = \frac{\partial y}{\partial t}, \qquad x \text{ fixed}, \quad \text{material time derivative.}$$

with higher time derivatives similarly defined.

4.3 The material is assumed (i) homogeneous in Ω and (ii) hyperelastic. The latter assumption is characterised by the existence of a strain-energy functional per unit volume

$$W : M^{3\times 3} \longrightarrow \mathbb{R}$$
$$F \longmapsto W(F), \qquad F = \nabla f.$$

The total strain-energy is defined by

$$V(f) = \int_{\Omega} W(\nabla f)\,dx \qquad (4.3.1)$$

4.4. The formulation of the boundary value problem and the initial boundary value problem.

Set

$$\frac{\partial W}{\partial F} = \left(\frac{\partial W(\nabla f)}{\partial f_{i,\alpha}}\right), \qquad i, \alpha = 1, 2, 3,$$

$$\mathrm{Div}\,\frac{\partial W}{\partial F} = \frac{\partial}{\partial x_\alpha}\left(\frac{\partial W}{\partial f_{i,\alpha}}\right). \qquad (4.4.1)$$

The equations governing the deformation of the body in the equilibrium configuration Ω_0 are

$$\left.\begin{aligned} \mathrm{Div}\,\frac{\partial W}{\partial Z} &= 0, & x &\in \Omega \\ z(x) &= g(x), & x &\in \partial\Omega_I \\ N^T\frac{\partial W}{\partial Z} &= f(x), & x &\in \partial\Omega_{II} \end{aligned}\right\} \qquad (4.4.2)$$

It may be shown that the critical points of the potential energy:

$$V(z) = \int_{\Omega} W(\nabla z)\,dx - \int_{\partial\Omega_{II}} f\,z\,dS \qquad (4.4.3)$$

occur at the solutions to (4.4.2).

Of course, g and f are specified in $(4.4.2)_{2,3}$.

Equations governing the motion of the body (in Ω_1) are

$$\left.\begin{aligned} \operatorname{Div} \frac{\partial W}{\partial Y} &= \rho \ddot{y}, & (x,t) &\in \Omega \times (0,T) \\ y(x,t) &= g(x), & (x,t) &\in \partial\Omega_I \times (0,T) \\ N^T \frac{\partial W}{\partial Y} &= f(x), & (x,t) &\in \partial\Omega_{II} \times (0,T) \\ y(x,0) = y_0(x),\ \dot{y}(x,0) &= \dot{y}_0(x), & x &\in \Omega \end{aligned}\right\} \quad (4.4.4)$$

where $(0,T)$ (T different notation to before!) is the maximal interval of existence, and y_0, $\dot{y}_0$ are specified.

Multiplication of (4.4.4) by $\dot{y}$ and integration over $\Omega \times (0,t)$ yields

$$\mathcal{E}(t) \overset{\text{def}}{=} \frac{1}{2}\int_\Omega \rho \dot{y}\dot{y}\,dx + V(y) - \int_{\partial\Omega_{II}} f y\,dS = \mathcal{E}(0), \qquad (4.4.5)$$

In terms of the displacement, equations (4.4.4) become

$$\left.\begin{aligned} \operatorname{Div}\left(\frac{\partial W}{\partial Y} - \frac{\partial W}{\partial Z}\right) &= \rho \ddot{u}, & (x,t) &\in \Omega \times (0,T), \\ u(x,t) &= 0, & (x,t) &\in \partial\Omega_I \times (0,T), \\ N^T\left(\frac{\partial W}{\partial Y} - \frac{\partial W}{\partial Z}\right) &= 0, & (x,t) &\in \partial\Omega_{II} \times (0,T) \\ u(x,0) = u_0(x),\quad \dot{u}(x,0) &= v_0(x), & x &\in \Omega. \end{aligned}\right\} \quad (4.4.6)$$

Furthermore,

$$E(t) \overset{\text{def}}{=} \frac{1}{2}\int_\Omega \rho \dot{u}\dot{u}\,dx + I(u) = E(0) \qquad (4.4.7)$$

where

$$I(u) = V(y) - V(z), \qquad (4.4.8)$$

and $E(t)$ is the <u>total energy</u>.

<u>Remarks</u> (i) Derivations formal

(ii) $\mathcal{E}(t)$ or $E(t)$ possible Liapunov's functions.

(iii) <u>Cauchy elasticity</u>: The (Piola-Kirchhoff) stress is

$$T = \frac{\partial W}{\ldots} \qquad (*)$$

/cont

so that equations for the motion of the body, written in terms of the stress, may be taken as basic. Then the hypothesis of hyperelasticity (G. Green) may be abandoned in favour of Cauchy elasticity, provided (*) is replaced by the constitutive relation

$$T = T(\nabla u).$$

Observe, however, that the total (and indeed, potential) energy cannot be defined, and so there are no obvious candidates for Liapunov functions.

The treatment of Liapunov stability for Cauchy elasticity is still largely open; c.f. Green and Naghdi, Arch. Rat. Mech. Anal. 40, 37–49, 1971.

(iv). For sufficiently smooth W, we may rewrite $(4.4.4)_1$ as

$$A_{\alpha\beta}\frac{\partial^2 y}{\partial x_\alpha \partial x_\beta} = \rho \ddot{y} \qquad (x,t)\in \Omega\times(0,T) \qquad (4.4.9)$$

where $A_{\alpha\beta}$ are the matrices

$$\begin{cases} A_{\alpha\beta}(y) = \dfrac{\partial^2 W}{\partial y^2}, \\ (A_{\alpha\beta})_{ij} = A_{i\alpha j\beta}(\nabla y) = \dfrac{\partial^2 W}{\partial y_{i,\alpha}\partial y_{j,\beta}}, \end{cases} \qquad (4.4.10)$$

Eqn. (4.4.9) represents a quasi-linear, formally self-adjoint system, since

$$A_{i\alpha j\beta} = A_{j\beta i\alpha}. \qquad (4.4.10)$$

(v). Precise form of W unknown, apart from restrictions imposed by invariance. It cannot, however, be strictly convex w.r.t. F (i.e.,

$$W(\lambda F + (1-\lambda)G) < \lambda W(F) + (1-\lambda)W(G), \quad \lambda\in[0,1],\ \forall F, G), \qquad (4.4.11)$$

since this implies global uniqueness of the equil. soln., and hence buckling impossible. Nor can W be merely convex ('≤' for '<' in (4.4.11)) since the equil. solns. then form a convex (and hence connected) set.

4.5. Equations of Linearised Elasticity.

Assume, in the usual way, that the displacement u is sufficiently small for second and higher order terms to be neglected. Then, can show that linearised equations are

$$\left.\begin{aligned}
\frac{\partial}{\partial x_\alpha}\left(D_{i\alpha j\beta}\frac{\partial u_j}{\partial x_\beta}\right) &= \rho\,\frac{\partial^2 u_i}{\partial t^2}\,, && (x,t)\in \Omega\times(0,T)\\
u_i(x,t) &= 0 && (x,t)\in \partial\Omega_I\times(0,T)\\
N_\alpha D_{i\alpha j\beta}\frac{\partial u_j}{\partial x_\beta} &= 0 && (x,t)\in\partial\Omega_{II}\times(0,T)\\
u_i(x,0) = u_{i0}(x), &\quad \dot u_i(x,0) = v_{i0}(x), && x\in\Omega\,,
\end{aligned}\right\}\qquad (4.5.1)$$

where

$$D_{i\alpha j\beta} = \frac{\partial^2 W}{\partial z_{i,\alpha}\,\partial z_{j,\beta}} = D_{j\beta i\alpha}$$

Alternatively, upon referring quantities to Ω_0 instead of Ω, the same formal set of eqns as (4.5.1) is obtained but with $D_{i\alpha j\beta}$ replaced by

$$d_{ijkl} = \sigma_{jl}\,\delta_{ik} + c_{ijkl} = d_{klij} \qquad (4.5.2)$$

where

σ_{ij} ... (Cauchy) stress in Ω_0

c_{ijkl} .... elasticities in Ω_0.

Provided there is sufficient smoothness, we readily obtain

$$E(t) \overset{\text{def}}{=} \frac{1}{2}\int_{\Omega_0}\rho_0\,\dot u_i\dot u_i\,dx + \frac{1}{2}\int_{\Omega_0} d_{ijkl}\,u_{i,j}\,u_{k,l}\,dx = E(0),$$

$$E_m(t) \overset{\text{def}}{=} \frac{1}{2}\int_{\Omega_0}\rho_0\,\overset{(m+1)}{u_i}\,\overset{(m+1)}{u_i}\,dx + \frac{1}{2}\int_{\Omega_0} d_{ijkl}\,\overset{(m)}{u}_{i,j}\,\overset{(m)}{u}_{k,l}\,dx = E_m(0), \quad \overset{(m)}{u} = \frac{d^m u}{\partial t^m}.$$

Remark: Existence theorems are available, stating that provided d_{ijkl} is C^∞ and satisfies $d_{ijkl}\,\xi_{ij}\,\xi_{kl} \geq d\,\xi_{ij}\xi_{ij}$, $d>0$, with $\sigma_{ij}\equiv 0$, then the solution to (4.5.1) is (roughly) as smooth as the initial data.
(Cp Fichera, Handbuch der Physik VI/a/2.
Hughes and Marsden, Unpublished.)

Remark: The derivation of eqns. (4.5.1) has imposed no sign-definiteness condition on the quantities d_{ijkl}, even in the case where the equil. posn. Ω_0 is stress-free ($\sigma_{ij} \equiv 0$). Similarly, there is no restriction imposed upon the sign of the total strain energy

$$V(u) = \int_{\Omega_0} d_{ijkl}\, u_{i,j}\, u_{k,l}\, dx.$$

Our aim now is to examine the consequences for stability of imposing such definiteness conditions.

Chpt V

~~4.6~~. Linearised elastic stability

We examine the stability of the null soln. $u_i \equiv 0$, and prove

Thm 5.1.1 Suppose $\partial\Omega_I \neq \phi$, and

(i) $d_{ijkl} \in C^2(\Omega_0)$

(ii) $\int_{\Omega_0} d_{ijkl}\, \xi_{ij}\, \xi_{kl}\, dx \geqslant d \int_{\Omega_0} \xi_{ij}\, \xi_{ij}\, dx, \quad d>0, \quad \xi_{ij} = \xi_{ij}(x).$

Then the null soln. of (4.5.1) (with d_{ijkl} replacing D_{ijkl}) is stable w.r.t.

$$\rho_\tau(u) = \sup_{x\in\Omega_0}\left(|u_{i,j}|^2 + |\dot{u}_{0i}|^2 \right) \quad \text{and} \quad \rho(u) = \int_{\Omega_0} u_i u_i\, dx.$$

Proof: With the help of Poincaré's inequality we have

$$\int_{\Omega_0} u_i u_i\, dx \leq c_1 \int_{\Omega_0} u_{i,j}\, u_{i,j}\, dx \leq c_2 \int_{\Omega_0} d_{ijkl}\, u_{i,j}\, u_{k,l}\, dx \leq 2c_2 E(t) = 2c_2 E(0) \leq c_3 \rho_\tau(u)$$

where c_1, c_2, c_3 are positive constants $\square$.

Remark: Stability of $u_i \equiv 0$ w.r.t. $\rho_\tau(u)$ and $\rho(u) = \sup_{x\in\Omega_0} (u_i u_i)$ may be obtained with sufficiently smooth initial data.

It may be proved: (Shield, Z.A.M.P. 16, 649–686, 1965; K + Wilkes, Int. J. Eng. Sci. 4, 303–329, 1966.

5.2 <u>Sufficient conditions for the positive-definiteness of d_{ijkl}.</u>

We examine conditions for the validity of the inequality

$$\int_{\Omega_0} d_{ijkl}\, u_{i,j} u_{k,l}\, dx \geq d \int_{\Omega_0} u_{i,j} u_{i,j}\, dx,$$

fundamental to in the proof of stability.

(i) $d_{ijkl} \in C^0(\bar\Omega_0)$, $u_i \in C^1(\bar\Omega_0)$, $d_{ijkl}\xi_{ij}\xi_{kl} > 0$, $x \in \bar\Omega_0$, $\xi_{ij} \neq 0$.
(a cont. fn. achieves its bounds on a closed set.)

(ii) d_{ijkl} constant, $\partial\Omega_{II} = \phi$, $d_{ijkl}\xi_i\xi_k\eta_j\eta_l > d_1 \xi_i\xi_i\eta_j\eta_j$, $d_1 > 0$.
(Strong ellipticity.)

Actually, we required for stability the inequality

$$\int_{\Omega_0} d_{ijkl}\, u_{i,j} u_{k,l}\, dx \geq d_2 \int_{\Omega_0} u_i u_i\, dx \qquad (5.2.1)$$

and so we are led naturally to questions connected with the <u>Rayleigh quotient</u>

$$\int_{\Omega_0} d_{ijkl}\, u_{i,j} u_{k,l}\, dx \Big/ \left(\int_{\Omega_0} u_i u_i\right)$$

whose lower bound, which for stability must be positive, is the smallest (minus) eigenvalue of e.g.,

$$\left.\begin{aligned} (d_{ijkl} v_{k,l})_{,j} - \rho\lambda^2 v_i &= 0 \qquad x \in \Omega \\ v_i &= 0 \qquad x \in \partial\Omega_* \end{aligned}\right\}$$

This, in turn, provides a relation with eigenfunction analyses.
[See also Koiter and Trefftz]

We may also establish (5.2.1) using spectral theory: let L be the linear operator defined by

$$Lv \equiv \{(d_{ijkl} v_{k,l})_{,j}\,,\ v_i = 0 \text{ on } \partial\Omega\}$$

<u>Thm.</u> Suppose L is self-adjoint, positive ($= (Lv, v) > 0$) operator defined on $W_{1,2}(\Omega)$. A necessary and sufficient cond. for (5.2.1) is that L^{-1} is compact.

Question: Show L is self-adjoint $(Lv, v) > 0$ but L^{-1} is not compact.

5.3. Linearised elastic instability.

We have seen that a strain energy satisfying the definiteness condition (ii) of Thm 5.1.1 is sufficient for stability of the null soln. in an appropriate sense. We now examine whether the same condition is necessary. More generally, we establish sufficient conditions for instability and in this way establish that a necessary condition for stability is not (ii) of Thm 5.1.1 but instead

$$2V \equiv \int_\Omega d_{ijkl}\,\xi_{ij}\,\xi_{kl}\,dx > 0 \qquad \forall\, \xi_{ij} \neq 0.$$

5.3.1. The Lagrange-Jacobi argument. (Differential inequalities)

(See also Lipschitz, J. reine angew. Math. 78, 329–337, 1874; Caughey and Shield, Z. A. M. P. 19, 485–491, 1968.)

Let

$$F(t) = \int_\Omega \rho\, u_i u_i\, dx$$

$$\therefore \quad \dot F(t) = 2\int_\Omega \rho\, u_i \dot u_i\, dx$$

$$\ddot F(t) = 2\int_\Omega \rho\, \dot u_i \dot u_i\, dx + 2\int_\Omega \rho\, u_i \ddot u_i\, dx$$

$$= 2\int_\Omega \rho\, \dot u_i \dot u_i\, dx - 2\int_\Omega d_{ijkl}\, u_{i,j}\, u_{k,l}\, dx$$

$$= 4\int_\Omega \rho\, \dot u_i \dot u_i\, dx - 4E(0),$$

by cons. of energy.

$$\therefore\ \ddot F(t) \geq -4E(0)$$

$$\therefore\ F(t) \geq F(0) + tF'(0) - 2t^2 E(0). \qquad (5.3.1)$$

Hence, if $E(0) < 0$ ($\Rightarrow V < 0$) then $F(t) \to \infty$ as $t \to \infty$, and the null solution is unstable.

Remarks: 1. (5.3.1) is inconclusive when initial data may be chosen s.t. $E(0) \geq 0$, which may happen when d_{ijkl} is *indefinite*.

2. When d_{ijkl} are strictly negative-definite, exponentially increasing

5.3.2. Logarithmic convexity

This general method has early appearance in the proof of the Hadamard three circle theorem, and has been extensively discussed by Agmon (Unicité et convexité dans les probs. differentials. Sem. Math. Sup. Montreal U.P. 1966). It depends upon the simple differential inequality

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t) \geq 0$$

in a Banach space, and has been fully exploited in continuum mechanics.

Set

$$G(t) = \int_\Omega \rho u_i u_i \, dx + \alpha(t-t_0)^2 = F(t) + \alpha(t+t_0)^2, \qquad (5.3.2)$$

where α, t_0 are +ve consts. to be determined later. Observe that $G(t)$ is positive definite (Defn. 1. .). We prove that

$$\frac{d^2}{dt^2}\left[\ln G(t)\right]\left\{= \frac{GG'' - (G')^2}{G}\right\} \geq 0 \; ; \; t \in [0, t_1] \subseteq [0, T] ; \qquad (5.3.3)$$

that is, $G(t)$ has a convex logarithm.

It easily follows that

$$\dot{G}(t) = 2\int_\Omega \rho u_i \dot{u}_i \, dx + 2\alpha(t+t_0)$$

$$\ddot{G}(t) = 4\int_\Omega \rho \dot{u}_i \dot{u}_i \, dx - 4E(0) + 2\alpha \; , \quad t \in [0, T)$$

$$\therefore G\ddot{G} - (\dot{G})^2 = S^2(t) - 2(\alpha + 2E(0))\,G(t),$$

where

$$S^2(t) = 4\left[\left\{\int_\Omega \rho u_i u_i \, dx + \alpha(t+t_0)^2\right\}\left\{\int_\Omega \rho \dot{u}_i \dot{u}_i \, dx + \alpha\right\} - \left\{\int_\Omega \rho u_i \dot{u}_i \, dx + \alpha(t+t_0)\right\}^2\right]$$

$$\geq 0, \quad \text{by Schwarz's inequality.}$$

$$\boxed{\therefore \; GG'' - (G')^2 \geq -2(\alpha + 2E(0))\,G(t)} \qquad (5.3.4)$$

We examine various sets of initial data.

(i) $\underline{u_{i0} = v_{i0} = 0. \Rightarrow E(0) = F(0) = F'(0) = 0.}$

Set $\alpha = 0$ and then $G(t) = F(t)$, so that by (5.3.4)

$$FF'' - (F')^2 \geqslant 0.$$

We would like to prove that $F(t) \equiv 0$, $t \in [0,T)$. Assume the contrary viz

$$F(t) = 0$$
$$> 0 \qquad 0 \leq t_1 < t < t_2 \leq T$$

$$\therefore \ (\ln F)'' \geqslant 0 \qquad t \in (t_1, t_2)$$

$$\therefore \ F(t) \leq [F(t_1)]^{\frac{t_2-t}{t_2-t_1}} [F(t_2)]^{\frac{t-t_1}{t_2-t_1}} \qquad t \in [t_1, t_2]. \qquad (5.3.5)$$

By cont. of ~~G(t)~~ F(t), we have

$$F(t_1) = 0.$$
$$\therefore F(t) = 0 \quad , \ t \in [t_1, t_2]$$
$$\therefore F(t_2) = 0 \quad , \ \text{by cont.}$$
$$\therefore \ F(t) = 0 \quad , \ t \in [0,T].$$

Hence, the eqns. of ~~elastodynamics~~ linearised elastodynamics possess a unique soln., provided only that $d_{ijkl} = d_{klij}$ (i.e., they are formally self-adjoint). No definiteness assumption is required.
(See L. Brun, J. de Mech. 8, 125 – 192, 1966, for different proof)

Eqn. (5.3.5) also yields, for general initial data for which $E(0) \leq 0$,

$$F(t) \leq [F(0)]^{\frac{T-t}{T}} [F(T)]^{\frac{t}{T}} \qquad t \in [0,T),$$

so that in the class of solutions bounded in the sense that

$$F(T) \leq M, \quad \text{a const,}$$

~~the initial soln~~
there is <u>continuous dependence upon the initial data on compact subintervals of $[0,T]$.</u>

(ii) <u>Initial data for which $E(0) < 0$.</u>

Choose α s.t. $\alpha + 2E(0) \leq 0$, and then from (5.3.4), we have

$$(\ln G)'' \geq 0$$

$$\therefore \quad G(t) \geqslant G(0) \exp\left\{ \frac{G'(0)}{G(0)} t \right\} \qquad (5.3.6)$$

We can always choose t_0 s.t. $G'(0) > 0$, even though u_0, v_0 are s.t. $F'(0) < 0$. Hence, we see that $G(t)$ has an <u>exponentially increasing lower bound</u>. <u>for sufficiently large time.</u>

<u>Remark</u>: If $\lim_{t\to\infty} e^{-\sigma t} G(t) = 0$, $\sigma = \frac{G'(0)}{G(0)}$ (>0), then $G(t) \equiv 0$, $t \geqslant 0$.

(iii) <u>Initial data for which $E(0) = 0$, $F'(0) > 0$.</u>

Set $\alpha = 0$, and recover (ii) above.

Observe that if $F'(t_1) > 0$, for some $t_1 > 0$, then $F(t) \nearrow \infty$ for $t > t_1$.

(iv) <u>$E(0) = 0$, $F'(0) = 0$.</u>

Set $\alpha = 0$; and from (5.3.6) obtain

$$F(t) \geqslant F(0).$$

Hence, $\exists\, t_1$ s.t. $\qquad F(t) = F(0) \qquad 0 \leq t < t_1$

and either

$$F'(t_1) > 0 \qquad (a)$$

or

$$F'(t_1) = 0. \qquad (b)$$

In case (b), we have $F(t) = 0$, $\forall t$.

$$\therefore\ 0 = F''(t) = 4 \int_\Omega \rho \dot{u}_i \dot{u}_i \, dx.$$

$$\therefore \quad u_i(t,x) = w_i(x)$$

and, by direct substitution, $w_i(x)$ is an equilibrium soln.

On the other hand, consider the initial data

$$u_i(0,x) = w_i(x), \qquad \dot{u}_i(0,x) = 0 \qquad (c)$$

Set

$$p_i(t,x) = u_i(t,x) - w_i(x)$$

Since $p_i(0,x) = \dot{p}_i(0,x) = 0$, we recover Case (i), and conclude

We have therefore shown:

"Suppose the initial data is such that $E(0) = F'(0) = 0$. Then the solution measured by $F(t)$ tends ~~to~~ exponentially to infinity, <u>unless</u> in addition the initial data corresponds to (c); that is, the body is placed initially at rest in an equilibrium position."

(v) <u>$E(0) > 0, \quad F'(0) > 0$.</u>

Set $\alpha = 0$.

$$FF'' - (F')^2 \geqslant -4E(0)F. \qquad (a)$$

(vi) Bounds for the kinetic and strain energies

In cases (i) – (v), we have proved that inequalities of the type

$$\left(\frac{F'}{F}\right)' \geq 0 , \qquad t > 0$$

are satisfied. Thus, we may integrate to get

$$\frac{\dot F(0)}{F(0)} \leq \frac{\dot F(t)}{F(t)}$$

$$\therefore F(t)\frac{\dot F(0)}{F(0)} \leq 2\int_\Omega \rho u_i \dot u_i \, dx \leq 2\left(\int_\Omega \rho u_i u_i \, dx\right)^{1/2}\left(\int_\Omega \rho \dot u_i \dot u_i \, dx\right)^{1/2},$$

(Schwarz)

$$\therefore F(t)\left[\frac{\dot F(0)}{F(0)}\right]^2 \leq 4\int_\Omega \rho \dot u_i \dot u_i \, dx, \qquad \text{if } \dot F(0) > 0.$$

Hence, the kinetic energy $\nearrow \infty$ whenever $F(t) \nearrow \infty$.

Furthermore, since

$$\frac{1}{2}\int_\Omega \rho \dot u_i \dot u_i \, dx + \frac{1}{2}\int_\Omega d_{ijkl} u_{i,j} u_{k,l} \, dx = E(0),$$

the ~~?~~ strain energy $\searrow -\infty$ whenever $F(t) \nearrow \infty$.

Remark 1. Initial data in Cases (i) – (v) all imply that the potential (strain) energy

$$V(t) = \frac{1}{2}\int_\Omega d_{ijkl} u_{i,j} u_{k,l} \, dx$$

is initially non-positive: $V(0) \leq 0$. Cases (i) – (iv) the result is obvious.
Case (v) we have in I.

$$0 < A^2 \Rightarrow$$

$$\therefore 8E(0)F(0) < [F'(0)]^2 \leq 4F(0)\int_\Omega \rho \dot u_i \dot u_i \, dx$$

$$\therefore 4V(0)F(0) < 0 \qquad \square$$

Similarly for II.

Remark 2. Further details, and applications of log. convexity to weak solutions of an abstract linear eqn. $(M\ddot u = Nu)$ may be found in K. & Payne, Arch.