INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
MIRAMARE - P.O.B. 586 - 34100 TRIESTE (ITALY) - TELEPHONES: 224281/2/3/4/5/6 - CABLE: CENTRATOM

SMR/23/15

AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

22 September - 26 November 1976

EVOLUTION EQUATIONS

First Order Linear Equations

M. Artola
University of Bordeaux I
Talence, France

1. Artola (Bordeaux - France) (1)

Lectures on Evolution Equations

I. First order linear equations

0 - Introduction

10-03-76 and 10-04-76

We begin with a very simple example of linear evolution equation of first order in time variable. The standard Equation of this type is the equation of heat flow:

$$(E) \qquad \frac{\partial u}{\partial t} = K \Delta u \quad , \; K > 0$$

(if $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, Δ is the Laplacian : $\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}$)

This equation appears in a wide variety of diffusion and equalization processes, some of Which are listed below

1. Conduction of heat in bars or solids
2. Diffusion of concentrations of liquid or gaseous substances in physical chemistry.
3. The diffusion of neutrons in atomic piles
4. The diffusion of vorticity in viscous fluid flow
5. Slow motion in hydrodynamics
6. Equalization of charge in electromagnetic theory
7. The [illegible] dissipative effect in hydromagnetics

– Let us derive the partial differential equation of heat flow in a one-dimensional bar extended along the x axis.
We denote by $u(x,t)$ the temperature function and, for simplicity, we assume a constant specific heat c, so that the heat content per unit length of the bar is $\rho c\, u(x,t)$ (ρ is the density also assumed constant).
We adopt Fourier's law of heat flow, which states that the amount of heat flow at any place x on the bar is proportional to the gradient of the temperature.
By the laws of thermodynamics, this flow is in the direction of decreasing temperature.
Then the total heat in the segment $0 \le \xi \le x$ of the bar is

$$(1) \qquad H(x,t) = c\rho \int_0^x u(\xi,t)\, d\xi$$

The heat flow across the point x is, by Fourier's law:

$$(2) \qquad -\rho c k \frac{\partial u}{\partial x}$$

where $k > 0$ is the conductivity of the bar.
Thus, if an amount of heat $f(x,t)$ is added to the bar by external heating, per unit time, then

$$(3) \qquad \frac{\partial H}{\partial t} = \rho c k \frac{\partial u}{\partial x} + \int_0^x f(\xi,t)\, d\xi$$

On the other hand, we can differentiate with respect to time under the integral sign in (1) so that

$$(4) \qquad \frac{\partial H}{} = \rho c \int^x \frac{\partial u}{}\, d\xi$$

Thus the right-hand sides of (3) and (4) are equal.
If we differentiate both with respect to x, we obtain the heat equation:

$$(5) \quad \frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} + \frac{f}{\rho c}$$

With source density $\frac{f}{\rho c}$.

We shall consider two types of problems.

1. The length of the bar is finite, $0 \leq \xi \leq l$ and the corresponding problem is a Cauchy problem with boundary conditions (end conditions for the bar)
2. The length of the bar is infinite, $-\infty < \xi < +\infty$ and the corresponding problem is a ~~mixed~~ Cauchy problem: that is we expect to assign one datum function at $t = 0$ say $u(x,0) = u_0(x)$

The most common conditions for a bar are:

a) - u given at $\xi = 0$ and $\xi = l$ (Dirichlet Condition)

b) - $\frac{\partial u}{\partial x}$ given " " (Neumann Condition)

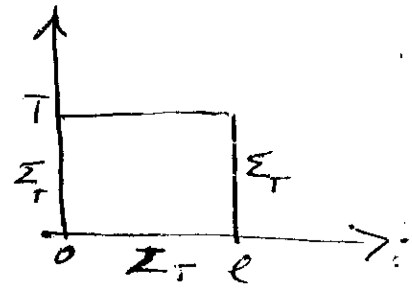
c) - $\frac{\partial u}{\partial x}$ (resp. u) given at $\xi = 0$ and u (resp. $\frac{\partial u}{\partial x}$) given at $\xi = l$
(Mixed Conditions)

Suppose now (for simplicity) $f \equiv 0$ in (5)

By common experience it is known that the highest temperature in a bar or other conducting medium at a given instant
... of the bar.

and the highest temperature get observed at the ends.

The mathematical counterpart of this physical observation is the following Theorem in which we consider only smooth solutions $u(x,t)$ of the heat flow equation.



Denote by Σ_T the set:

$$\Sigma_T = \{(x,t)\ ;\ \{t=0,\ 0\le x\le l\} \cup \{x=0,\ 0\le t\le T\} \cup \{x=l,\ 0\le t\le T\}\}$$

$\boxed{\text{Th - 0 - 1}}$. - Maximum principle -

If a function $u:(x,t)\mapsto u(x,t)$ satisfies (6) $K\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ for

$0 \le x \le l$, $0 \le t \le T$,

then the maximum value of $u(x,t)$ is taken for $(x,t) \in \Sigma_T$

We omit the proof and leave it to do in exercise.

The Maximum principle has many interesting consequences.

- For instance we can use it to prove the following uniqueness theorem:

$\boxed{\text{Th - 0 - 2}}$ A solution $u(x,t)$ of (6) subject to the initial condition

$$(7)\quad u(x;0) = u_0(x)$$

and the boundary conditions:

$$(8)\quad u(0,t) = u_1(t)\ ,\quad u(l,t) = u_2(t)$$

is uniquely determined.

exceed zero if.

- an other deduction from the maximum principle is that two solution which differ by less than ε on the set Σ_T will differ by less than ε within the domain.

This enables us to assert the following theorem

$\boxed{\text{Th. 0.3}}$ The solution $u(x,t)$ of the initial and boundary problem ~~defined~~ (6)-(7)-(8) depends continuously on the data.

Thus a smooth solution of this heat flow problem has a certain stability.

For a problem in Partial Differential equations to have a realistic physical interpretation, it should have these properties.

$$(9)\begin{cases} \text{(i) There exist a solution} \\ \text{(ii) The solution is unique} \\ \text{(iii) The solution depends continuously on the data of the problem.} \end{cases}$$

Such a problem is said to be correctly posed or Well-posed

Theorem (0.2) and (0-3) together with construction of a solution would mean that the initial and boundary problem [(6)-(7)-(8)] is well-posed for the heat flow equation.

Now we shall see how to construct solution for the heat flow problem.

1- Fourier's Method

1-1. an Example : Cauchy-Dirichlet Problem

We first shall expose the method of Fourier in the following situation:

The bar is now assimilated to the open set $\Omega =]0,1[$ and we look for $u(x,t)$ $x \in \Omega$, $t \in (0,T)$ $(T>0$ given$)$ as a solution of the following formal Cauchy-Dirichlet problem:

$$(P_1) \begin{cases} (1) \quad \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f & \\ (2) \quad u(0,t) = u(1,t) = 0 & \text{(Dirichlet condition)} \\ (3) \quad u(x,0) = u_0(x) & \text{(Cauchy condition)} \end{cases}$$

where f and u_0 are given functions.

The Fourier's method being essentially an Hilbertian method here f and u_0 will be first choose in the $L^2(\Omega)$ space (See H. Lanchon's Lectures).

We shall denote by $(u,v) = \int_\Omega u(x)\, v(x)\, dx$ the scalar product and by $|u| = \left(\int_\Omega |u(x)|^2 dx\right)^{1/2}$ the norm of $u \in L^2(\Omega)$.

Now we introduce the eigenfunctions of the operator $A = -\dfrac{d^2}{dx^2}$ on Ω with zero boundary conditions.

A is an unbounded operator in $L^2(\Omega)$ with domain

. . 17 . .

$H^m(\Omega)$, $H_0^m(\Omega)$ are usual Sobolev-Spaces. (See H-Lanchon loc.cit)) (7)

Moreover A is a self adjoint operator in $L^2(\Omega)$ and its inverse A^{-1} is a self-adjoint, bounded, compact operator from $L^2(\Omega)$ to $L^2(\Omega)$.

We deduce of this that the spectrum of eingenvalues of A is a discrete countable set and the corresponding eingenfunctions form a complete orthogonal basis of $L^2(\Omega)$ and also of $H_0^1(\Omega)$.

These eigenfunctions satisfies:

$$-W_k'' = \lambda_k W_k \quad , \quad W_k(0) = W_k(1) = 0$$

and we obtain

$$(4) \quad \lambda_k = k^2\pi^2, \quad W_k(x) = \sqrt{2}\sin k\pi x \quad , \quad k = 1, 2, \dots$$

the functions W_k being normalized to unit in $L^2(\Omega)$ by $\int_0^1 |W_k(x)|^2 dx = 1$.

Now we recall, that if $u \in L^2(\Omega)$, we have

$$(5) \quad u = \sum_{k=1}^{\infty} \mu_k W_k \quad , \quad \mu_k = (u, W_k) \quad k = 1, \dots$$

Where the sequence $\{\mu_k\}_{k\in\mathbb{N}^*}$ is belonging to l^2.

and we have:

$$(6) \quad |u|^2 = \int_\Omega |u(x)|^2 dx = \sum_{k=1}^{\infty} \mu_k^2 < +\infty. \quad \text{(Parseval's equality)}$$

We note also that, $u \in H_0^1(\Omega)$ iff* the sequence $\{\sqrt{\lambda_k}\mu_k\} \in l^2$

and we have

$$(7)\quad \|u\|^2_{H_0^1(\Omega)} = |\operatorname{grad} u|^2 = \sum_{k=1}^{\infty} \lambda_k |\mu_k|^2 < +\infty$$

In the same way, $u \in \mathcal{D}(A)$ iff the sequence $\{\lambda_k \mu_k\}_{k\in\mathbb{N}^*} \in \ell^2$ and we can define $\|u\|_{\mathcal{D}(A)}$ by

$$(8)\quad \|u\|^2_{\mathcal{D}(A)} = \sum_{k=1}^{\infty} \lambda_k^2 |\mu_k|^2 < +\infty .$$

Now if we put

$$(9)\begin{cases} t\in(0,T)\ ,\ f(t) = \sum_{k=1}^{\infty} \gamma_k(t)\, w_k \in L^2(\Omega) \\ \left(\text{with } |f(t)| = \left(\sum_{k=1}^{\infty} |\gamma_k(t)|^2\right)^{1/2} < +\infty \right)\end{cases}$$

(we shall precise the dependence on t later)
it is natural to look for $u(t) = \sum_{k=1}^{\infty} \mu_k(t) w_k$ as a solution of the problem (P_1).

Then Equation (1) formally gives

$$\sum_{k=1}^{\infty} (\mu_k' + \lambda_k \mu_k)\, w_k = \sum_{k=1}^{\infty} \gamma_k w_k$$

and because of the uniqueness of the development on the basis $\{w_k\}_{k\in\mathbb{N}^*}$ in $L^2(\Omega)$, we have:

$$(1)'\quad \begin{cases} \mu_k' + \lambda_k \mu_k = \gamma_k \\ k=1,\dots \end{cases}$$

Let u_0 be an element of $L^2(\Omega)$ with

$$\sum^{\infty}$$

the condition (3) becomes

$$(10) \quad \mu_k(0) = \mu_{0k} \quad , k = 1, 2, \ldots$$

and we have, by an elementary way:

$$(11) \begin{cases} \mu_k(t) = \mu_{0k} e^{-\lambda_k t} + \int_0^t e^{-\lambda_k (t-\sigma)} \gamma_k(\sigma)\, d\sigma \\ k = 1, 2, \ldots \end{cases}$$

We insist on the fact that the previous computations are purely formal for the moment

We must now justify these computations.

a) - <u>Convergence of the serie which defines u.</u>

We deduce from (11) by Cauchy-Schwarz's inequality:

$$|\mu_k(t)|^2 \leq 2 \left[|\mu_{0k}|^2 e^{-2\lambda_k t} + \left(\int_0^t e^{-2\lambda_k (t-\sigma)} d\sigma \right) \left(\int_0^t |\gamma_k(\sigma)|^2 d\sigma \right) \right.$$

as $$\int_0^t e^{-2\lambda_k (t-\sigma)} d\sigma = \frac{1}{2\lambda_k} \left(1 - e^{-2\lambda_k t}\right) < \frac{1}{2\lambda_k}$$

We obtain

$$(12) \quad |\mu_k(t)|^2 \leq 2 \left[|\mu_{0k}|^2 e^{-2\lambda_k t} + \frac{1}{2\lambda_k} \int_0^t |\gamma_k(\sigma)|^2 d\sigma \right]$$

~~iff~~ the integrals $\int_0^t |\gamma_k(\sigma)|^2 d\sigma$ exist.

If we assume that the functions $t \to \gamma_k(t)$ $(k = 1, 2, \ldots$

are measurable ~~over~~ $(0, T)$ and that

$$(13) \quad \int_0^T |f(t)|^2 dt < +\infty$$

the existence of $\int_0^T |\gamma_k(\sigma)|^2 d\sigma$ follows from (13)

Moreover because of the Lebesgue's theorem on dominated

(14) $\int_0^T |f(t)|^2 dt = \int_0^T \left(\sum_{k=1}^{\infty} |\gamma_k(t)|^2\right) dt = \sum_{k=1}^{\infty} \int_0^T |\gamma_k(t)|^2 dt.$

If we denote by $L^2(0T; L^2(\Omega))$ the space (of classes) of functions f : $t \to f(t)$; $(0,T) \to L^2(\Omega)$ such that

(15) $\begin{cases} \text{i)} & \forall h \in L^2(\Omega),\ t \to (f(t), h) \text{ is Lebesgue measurable over } (0,T) \\ \text{ii)} & \int_0^T |f(t)|^2 dt < +\infty \end{cases}$

the hypotesis on f implies

(16) $\quad f \in L^2(0T; L^2(\Omega))$

It is possible to show that $L^2(0T; L^2(\Omega))$ equipped with the norm $f \to \left(\int_0^T |f(t)|^2 dt\right)^{1/2}$ is an <u>Hilbert space</u>.

Now with hypothises on f we obtain (from (12)):

(17) $\sup_{t\in[0,T]} \sum_{k=m}^{k=m+p} |\mu_k(t)|^2 \leq 2\left[\sum_{k=m}^{k=m+p} |\mu_{0k}|^2 + \sum_{k=m}^{k=m+p} \int_0^T |\gamma_k(\sigma)|^2 d\sigma\right],$

then if we introduce the space of continuous functions on $[0,T]$ with values in $L^2(\Omega)$ namely $\mathcal{C}^0([0,T]; L^2(\Omega))$ (Equipped with the norm $g \to \|g\|_\infty = \sup_{t\in[0,T]} |g(t)|$ this space is a Banach space)

(17) means that the sequence : $\{u_m\}_{m\in\mathbb{N}^*}$ defined by

(18) $\quad u_m(t) = \sum_{k=1}^{m} \mu_k(t) w_k$

is a Cauchy sequence in $\mathcal{C}^0([0,T]; L^2(\Omega))$.

... the sequences $\{\sum^m |\mu_{0k}|^2\}$ and $\sum^m \int^T |\gamma_k(\sigma)|^2 d\sigma$

Thus $u(t) = \sum_{k=1}^{\infty} \mu_k(t) w_k$ belongs to the space $L^2(\Omega)$ (11)
for all $t \in [0,T]$ and the function $t \to u(t)$ is a continuous function on $[0,T]$.

Moreover from (17) (with $m=0$ and $p \to \infty$) we deduce that the map $(u_0, f) \longrightarrow u$ is continuous from the space $L^2(\Omega) \times L^2(0T; L^2(\Omega))$ to $\mathcal{C}^0([0,T]; L^2(\Omega))$.
Thus u depends continuously in a certain sense on the data.

6). On the other hand from equality (11) we deduce

$$|\mu_k(t) - \mu_{0k}|^2 \leq 2|\mu_{0k}|^2 (1 - e^{-\lambda_k t})^2 + \frac{1}{\lambda_k} \int_0^t |f_k(s)|^2 ds$$

and if $m \in \mathbb{N}^*$ we have

$$(19) \quad \sum_{k=1}^{\infty} |\mu_k(t) - \mu_{0k}|^2 \leq 2(1 - e^{-\lambda_m t})^2 \sum_{k=1}^{m} |\mu_{0k}|^2 + 2 \sum_{k=m+1}^{\infty} |\mu_{0k}|^2 + \frac{1}{\lambda_k} \int_0^t |f(s)|^2 ds$$

for $\varepsilon > 0$ given, we can choose m sufficiently large to have
$\sum_{k=m+1}^{\infty} |\mu_{0k}|^2 < \varepsilon$, and we see that

$$(20) \quad \lim_{t \to 0} |u(t) - u_0| = 0$$

As $u \in \mathcal{C}^0([0,T]; L^2(\Omega))$, in fact we obtain

$$(21) \quad u(0) = u_0$$

Consequently the sense of condition (3) is (21) that is

c) In order to show in what sense condition (2) is satisfied, we denote by $L^2(0,T; H_0^1(\Omega))$, the space (of classes) of functions $u : t \to u(t)$ from $(0,T)$ to $H_0^1(\Omega)$ such that:

i) $\forall v \in H_0^1(\Omega)$, $t \to (\operatorname{grad} u, \operatorname{grad} v)$ is measurable over $(0,T)$

ii) $\int_0^T |\operatorname{grad} u(t)|^2 \, dt < +\infty$

Equiped with the norm $u \to \left(\int_0^T |\operatorname{grad} u(t)|^2 dt\right)^{1/2}$, $L^2(0T; H_0^1(\Omega))$ is an Hilbert space.

(We note that if $u(t) = \sum_{k=1}^{\infty} \mu_k(t) w_k$, we have

$$\int_0^T |\operatorname{grad} u(t)|^2 dt = \int_0^T \left(\sum_{k=1}^{\infty} \lambda_k |\mu_k(t)|^2\right) dt = \sum_{k=1}^{\infty} \lambda_k \int_0^T |\mu_k(t)|^2 dt \quad)$$

Then from (12) by integration with respect of t we obtain

$$\int_0^T |\mu_k(t)|^2 dt \leq \frac{1}{\lambda_k} |\mu_{0k}|^2 + \frac{T}{\lambda_k} \int_0^T |\gamma_k(\sigma)|^2 d\sigma$$

thus

$$(22) \begin{cases} \int_0^T \left(\sum_{k=m}^{k=m+p} \lambda_k |\mu_k(t)|^2\right) dt \leq \sum_{k=m}^{k=m+p} \left(|\mu_{0k}|^2 + T \int_0^T |\gamma_k(\sigma)|^2 d\sigma\right) \\ m \geq 1, \; p \geq 0. \end{cases}$$

and (22) means that $u_m = \sum_{k=1}^{m} \mu_k w_k$ is a Cauchy's sequence in $L^2(0T; H_0^1(\Omega))$ and $u \in L^2(0T; H_0^1(\Omega))$.

This result implies

$$(23) \quad u(t) \in H_0^1(\Omega) \quad \text{a.e.} \quad t \in (0,T)$$

Moreover from (22) (With $m=0$ and $j\to\infty$) we obtain (1)
that the map : $(u_0, f) \to u$ is also continuous from
the product space $L^2(\Omega)\times L^2(0\,T; L^2(\Omega)) \to L^2(0,T; H_0^1(\Omega))$)

d) Now we have to see What is the sense of Equation (1)

First we observe that if the sequence

$$\frac{\partial u_m}{\partial t} = \sum_{k=1}^{m} \mu_k' w_k = \sum_{k=1}^{m} (\lambda_k \mu_k + \gamma_k) w_k$$

converges uniformly on $[0,T]$ then

$$\sum_{k=1}^{\infty} (\lambda_k \mu_k + \gamma_k) w_k = \frac{\partial u}{\partial t}$$

But, (for $\varepsilon > 0$ given) $\sum \lambda_k \mu_k w_k$ converges uniformly in the $L^2(\Omega)$-topology only on $[\varepsilon, T]$ except if $u_0 \in H_0^1(\Omega)$ or if $u_0 = 0$.

However it is necessary to have smoothness properties of function f if we want the uniform convergence with respect of t for the sequence which defines f-

Also, for fixed t, we don't Know if the sequence

$$\sum_{k=1}^{m} \lambda_k \mu_k(t) w_k = A u_m(t) = -\frac{d^2}{dx^2} u_m(t)$$

Converge in the $\mathcal{C}^0(\Omega)$ topology -

Th... observation show that it is not possible to have

derivatives, Except smoothness hypothesis upon u_0 and f (
Which are too restrictives for the natural Hilbertian charact
of the problem

Now we are going to see that the function u satisfies the equation (1) in a weak sense.

If we put

$$(24) \quad a(u,v) = \int_\Omega \frac{du}{dx}\,\frac{dv}{dx}\,dx \quad , \; u, v \in H_0^1(\Omega)$$

Equations (1)' are equivalent to

$$(25) \quad \begin{cases} (u_m'(t), w_k) + a(u_m(t), w_k) = (f_m(t), w_k) \\ 1 \le k \le m \end{cases}$$

where

$$(26) \quad f_m = \sum_{k=1}^{m} \gamma_k \cdot w_k$$

Let k be fixed, $m > k$ and let φ be a smooth function
$\varphi \in \mathcal{D}(]0,T[)$ (Schwartz's space)

Let us Take in (25) the product by φ and integrate over $(0,T)$. We obtain (after an integration by part in the first term of the right-hand side of (25)):

$$-\int_0^T (u_m(t), w_k)\varphi'(t)\,dt + \int_0^T a(u_m(t), w_k)\,\varphi(t)\,dt = \int_0^T (f_m(t), w_k)\,\varphi(t)\,dt$$

The uniform convergence of $u_m \to u$ in $[0,T]$ implies and from the following inequality: (Cauchy-Schwartz) (15)

$$\left|\int_0^T a(u_m(t)-u(t), w_k)\varphi(t)\,dt\right| \le C(\varphi)\cdot\left(\int_0^T |\operatorname{grad} u_m - \operatorname{grad} u|^2 dt\right)^{1/2}$$

We have

$$\lim_{m\to\infty}\int_0^T a(u_m, w_k)\varphi(t)\,dt = \int_0^T a(u, w_k)\varphi\,dt$$

Also

$$\lim_{m\to\infty}\int_0^T (f_m, w_k)\varphi\,dt = \int_0^T (f, w_k)\varphi\,dt$$

because of the strong convergence of f_m to f in $L^2(0,T; L^2(\Omega))$

Thus the function $u = \lim_{m\to\infty} u_m$ satisfies:

$$(27)\begin{cases} -\int_0^T (u(t), w_k)\varphi'(t)\,dt + \int_0^T a(u(t), w_k)\varphi(t)\,dt = \int_0^T (f(t), w_k)\varphi(t)\,dt \\ \text{for all } k,\quad k = 1, \dots \end{cases}$$

Because of the density of the set $\{w_k\}_{k\in\mathbb{N}^*}$ in $H_0^1(\Omega)$ (27) is also true for all $v \in H_0^1(\Omega)$, thus u satisfies:

$$(28)\begin{cases} -\int_0^T (u(t), v)\varphi'(t)\,dt + \int_0^T a(u(t), v)\varphi(t)\,dt = \int_0^T (f(t), v)\varphi(t)\,dt \\ \text{for all } v \in H_0^1(\Omega) \text{ and for all } \varphi \in \mathcal{D}(]0,T[) \end{cases}$$

(28) means that u satisfies equation (1) in the following weak sense:

$$(29)\begin{cases} \frac{d}{dt}(u(t), v) + a(u(t), v) = (f(t), v) \quad \text{for all } v \in H_0^1(\Omega) \\ \text{in the sense of } \mathcal{D}'(]0,T[) \text{ (i.e. Distribu...} \end{cases}$$

We obtained the following result:

There exists one* function u with:

$$(P_2)\begin{cases} (i) & u \in L^2(0T;\ H_0^1(\Omega)) \\ (ii) & u \text{ is continuous from } [0,T] \text{ to } L^2(\Omega) \\ (iii) & \frac{d}{dt}(u(\cdot), v) + a(u(\cdot), v) = (f(\cdot), v) \text{ for all } v \in H_0^1(\Omega); \\ & \text{in the Distribution sense on }]0,T[\\ (iv) & u(0) = u_0 \end{cases}$$

where $u_0 \in L^2(\Omega)$ and $f \in L^2(0T;\ L^2(\Omega))$ are given.

The formulation (P_2) of the problem (P_1) is called a Variational formulation for this problem (P_1)

Remark 1. The map $(u_0, f) \longrightarrow u$ is continuous from $L^2(\Omega) \times L^2(0T;\ L^2(\Omega))$ to $\mathcal{C}^0([0,T];L^2(\Omega))$ and to $L^2(0T;\ H_0^1(\Omega))$.

Remark 2

If we know that: $\frac{\partial u}{\partial t} \in L^2(0T;\ L^2(\Omega))$ then $Au \in L^2(0T; L^2$ and we have

$$\frac{\partial u}{\partial t} + Au = f \quad \text{in the sense of } L^2(0T;\ L^2(\Omega))$$

Exercise: Show that it is the case if $u_0 = 0$ or $u_0 \in H_0^1(\Omega)$.

Remark. 3

It is interesting to have a direct proof of the uniqueness of the solution of problem (P_2).

Assume firstly we have $\frac{\partial u}{\partial t}(t) \in L^2(\Omega)$ a.e. in $(0,T)$

if u_1 and u_2 are two solutions of Problem (P_2), $w = u_1 - u_2$ verifies

$$(30)\begin{cases} \left(\frac{\partial w}{\partial t}, v\right) + a(w,v) = 0, \text{ a.e., } \forall v \in H_0^1(\Omega) \\ w(0) = 0 \end{cases}$$

Take $v = w(t)$ in (30) and note that $\left(\frac{\partial w}{\partial t}, w\right) = \frac{1}{2}\frac{d}{dt}|w(t)|^2$

we obtain

$$\frac{1}{2}\frac{d|w(t)|^2}{dt} + a(w(t), w(t)) \leq 0 \quad \text{a.e. in } (0,T)$$

But ~~[illegible]~~ $a(w,w) \geq 0$ implies

$$(31) \quad \frac{1}{2}\frac{d}{dt}|w(t)|^2 \leq 0 \implies w = 0 \quad (\text{because of the continuity of } w)$$

and the uniqueness of the solution follows

- Now generally we have $\frac{\partial u}{\partial t} \in L^2(0,T; H^{-1}(\Omega))$

where $H^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$ (see Handlings Lectures)

We shall see later that $\frac{d}{dt}(w, v) = \left\langle \frac{\partial w}{\partial t}, v \right\rangle$

Where $\langle\ ,\ \rangle$ note the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

We also have $\left\langle \frac{\partial w}{\partial t}, w \right\rangle = \frac{1}{2}\frac{d}{dt}|w(t)|^2$

Remark 4 Positive solutions of (P_e).

(32) $\begin{cases} \text{If } f, u_0 \text{ are given as above but with } u_0 \geq 0 \text{ and} \\ f \geq 0 \text{ then also } u \geq 0. \end{cases}$

If v is a function defined on Ω then we can write

(33) $v = v^+ - v^-$

when $v^+ = \sup(v, 0)$, $v^- = \sup(-v, 0)$

v^+ and v^- are both ≥ 0.

We note that

(34) $\begin{cases} (i) \ a(v, v^-) = -a(v^-, v^-) \\ (ii) \left(\frac{\partial v}{\partial t}, v^-\right) = -\frac{1}{2} \frac{d}{dt} |v^-|^2 \end{cases}$

((ii) remains valid if $\frac{\partial v}{\partial t} \in L^2(0,T; H^{-1}(\Omega))$ we must then replace the scalar product in $L^2(\Omega)$ by the bracket in the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.)

Now we can take $v = u^-(t)$ in the equation satisfied by u

we obtain:

$$\frac{1}{2} \frac{d}{dt} |u^-(t)|^2 + a(u^-(t), u^-(t)) + (f(t), u^-(t)) = 0$$

$(f(t), u^-(t))$ is ≥ 0 and also $a(u^-, u^-) \geq 0$.

Thus (35) $\frac{1}{2} \frac{d}{dt} |u^-(t)|^2 \leq 0$

As $u^-(0) = u_0^- = 0$ $\quad (u_0 \geq 0)$

We have $|u^-(t)|^2 = 0 \Rightarrow u^-(t) = 0 \quad \forall t \in [0,T]$

by continuity of u_0^- and (32) is true.

Remark 5 . Non Homogeneous Cauchy-Dirichlet Problem

For instance we look for u with:

$$(P_1') \begin{cases} (i) & \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0 \\ (ii) & u(0,t) = v(t) \ , \ u(1,t) = 0 \\ (iii) & u(x,0) = u_0(x) \end{cases}$$

If Φ is a smooth function with

(36) $\quad w = u - \Phi$

w must verifies

$$(P_1'') \begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial \Phi}{\partial t} + \dfrac{\partial^2 \Phi}{\partial x^2} \\ W(0,t) = W(1,t) = 0 \\ W(x,0) = W_0(x) = u_0(x) - \Phi(x,0) \end{cases}$$

and we have to solve problem (P_1).

Remark 6.

In the previous case the operator A is a self-adjoint operator

Now we look for u which satisfies:

$$\left(P_1^{(iii)}\right)\begin{cases}(i) & \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + c\dfrac{\partial u}{\partial x} + \rho u = f \\ (ii) & u(0,t) = u(1,t) = 0 \\ (iii) & u(x,0) = u_0(x)\end{cases}$$

c and $\rho > 0$ are given constants

a) First suppose $\rho = 0$ and $c \neq 0$

then we take for new unknown function $w = u e^{-(\alpha x + \beta t)}$

If we choose $\alpha = \frac{c}{2}$, $\beta = -\frac{c^2}{4}$, ~~so~~ the function w

must verify

$$\left(P_1^{(iv)}\right)\begin{cases}\dfrac{\partial w}{\partial t} - \dfrac{\partial^2 w}{\partial x^2} = f e^{-(\alpha x + \beta t)} \\ w(0,t) = w(1,t) = 0 \\ w(x,0) = u_0(x) e^{-\alpha x}\end{cases}$$

that is a problem of (P_1) type.

b) <u>If $c = 0$ $\rho \neq 0, \rho > 0$</u>

the corresponding problem is (P_1) with λ_k changed in $\lambda_k + \rho$.

<u>Remark 7</u> <u>A limitation for Fourier's method</u>

if $c = c(x)$, $\rho = \rho(x)$ then we cannot use the Fourier's method. We shall use an other method. (Variational method)

<u>Remark 8</u> <u>the case when f takes its values in $H^{-1}(\Omega)$</u>

As an example we consider the case of the bar the case with (for simplicity) the intensity of the heat = 1 (21

Then (36) $f = \delta_b$. (Dirac measure at the point b, $0<b<1$)

And $\delta_b \notin L^2(\Omega)$

But δ_b is the derivative in the distribution sense of the function Y defined by

$$(37) \quad \begin{cases} Y(x) = 0 & 0 < x < b \\ Y(x) = 1 & b < x < 1 \end{cases}.$$

As $Y \in L^2(\Omega)$, it follows from the characterisation of the elements of $H^{-1}(\Omega)$ (see Exercise lecture) that:

$$(38) \quad \delta_b \in H^{-1}(\Omega)$$

By lecture's lecture we have

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$$

and the set $\{w_k\}_{k\in\mathbb{N}^*}$ which is dense in $L^2(\Omega)$ so does in $H^{-1}(\Omega)$

If now we still denote by $(\,,\,)$ the bracket in the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, we can see (cf. Exercise lecture) that

$$(39) \quad \begin{cases} f_m = \sum_{k=1}^{m} (\delta_b, w_k)\, w_k \;\rightarrow\; f = \delta_b \quad \text{in } H^{-1}(\Omega) \\ \text{when } m \rightarrow +\infty \end{cases}$$

Since (40) $(\delta_b, w_k) = w_k(b) = \sqrt{2} \sin k\pi b$ (2.

We find that

(41) $f_m \in L^2(\Omega)$ $\forall m$ (also $f_m \in H_0^1(\Omega)$)

and (39) means

$$(42) \quad \begin{cases} \int_0^1 f_m(x)\varphi(x)\,dx \longrightarrow \varphi(b) \text{ when } m \to \infty \\ \forall \varphi \in \mathcal{D}(\Omega) \end{cases}$$

(See Exercise lecture to verify (42))

Now we return to problem (P_1) where $f \equiv \delta_b$.

We find

$$(43) \quad \begin{cases} \mu_k'(t) + \lambda_k \mu_k(t) = w_k(b) \\ 1 \le k \le m \end{cases}$$

and we obtain

$$(44) \quad \mu_k(t) = \mu_{0k} e^{-\lambda_k t} + \frac{1}{\lambda_k}\left(1 - e^{-\lambda_k t}\right) w_k(b)$$

It is easy to show that ~~still~~ the sequences $\{\mu_k\}_{k\in\mathbb{N}^*}$ $\{\sqrt{\lambda_k}\,\mu_k\}_{k\in\mathbb{N}^*}$ still belong to ℓ^2.

And the solution $u \in L^2(0,T; H_0^1(\Omega)) \cap \mathcal{C}^0(0,T; L^2(\Omega))$

$$f = \sum_{k=1}^{\infty} \gamma_k w_k \in H^{-1}(\Omega)$$ iff the sequence (23

$\left\{ \frac{1}{\sqrt{\lambda_k}} \gamma_k \right\}_{k \in \mathbb{N}^*}$ belongs to l^2 and we have

(45) $$|||f|||^2_{H^{-1}(\Omega)} = \|f\|_*^2 = \sum_{k=1}^{\infty} \frac{(\gamma_k)^2}{\lambda_k}.$$

Now if we define $L^2(0T; H^{-1}(\Omega))$ by analogy with the definition of $L^2(0T; L^2(\Omega))$ or $L^2(0T; H_0^1(\Omega))$ we obtain the following result:

There exists one u with the properties:

(5.2)
$$\begin{cases} \text{(i)} & u \in L^2(0T; H_0^1(\Omega)) \\ \text{(ii)} & u \in \mathcal{C}^0(0T; L^2(\Omega)) \\ \text{(iii)} & u(0) = u_0 \qquad u_0 \text{ given } \in L^2(\Omega) \\ \text{(iv)} & \begin{cases} \frac{d}{dt}(u(\cdot), v) + a(u(\cdot), v) = (f(\cdot), v) & \text{in the Distribution} \\ \text{sense on }]0,T[\quad \forall v \in H_0^1(\Omega). \end{cases} \end{cases}$$

Here (,) note the scalar product in $L^2(\Omega)$ as the bracket in the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

By the consideration of (44) we can see that in this situation we cannot have $\frac{\partial u}{\partial t}$ (or Au) $\in L^2(0T; L^2(\Omega))$

We only have $\frac{\partial u}{\partial t} \in L^2(0T; H^{-1}(\Omega))$