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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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EVOLUTION EQUATIONS

(Part II)

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These are preliminary lecture notes intended for participants only.
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1-2- <u>an Other Example</u> : <u>Cauchy - Neumann Problem.</u>

We consider $\Omega =]0, L[$, $t \in [0,T]$ and we look for $u(x,t)$ as a solution of

$$(\mathcal{H}_1) \begin{cases} (i) & \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + \rho u = f \\ (ii) & \dfrac{du}{dx}(0,t) = \dfrac{du}{dx}(1,t) = 0 \quad \text{(Neumann Condition)} \\ (iii) & u(x,0) = u_0(x) \quad \text{(Cauchy condition)} \end{cases}$$

We introduce the eingenfunction of the operator $A = -\dfrac{d^2}{dx^2}$ With domain $\mathcal{D}(A) = \{ u \in H^2(\Omega) , \dfrac{du}{dx} \in H_0^1(\Omega) \}$

We have $\operatorname{Ker} A \neq \{0\}$ but $A + \rho I$ $\rho > 0$ is an unbounded self-adjoint operator With domain $\mathcal{D}(A)$ and its inverse is a bounded, self-adjoint, compact operator from $L^2(\Omega)$ to $L^2(\Omega)$

Therefore we obtain the eigenvalues and eingenfunctions of A :

$$(46) \qquad \lambda_k = k^2 \pi^2 \quad , \quad W_k : W_k(x) = \sqrt{2} \cos k\pi x \ , \ k = 0, 1, \dots$$

$\{W_k\}_{k \in \mathbb{N}^*}$ is a orthogonal basis of $L^2(\Omega)$ and $H_0^1(\Omega)$

So we put

$$(47) \qquad a(u,v) = \int_\Omega \frac{du}{dx} \cdot \frac{dv}{dx}\, dx + \rho \int_\Omega u v \, dx$$

~~F~~ ...'s method lead us to the following Variational there exists one function u with:

$$(\Pi_2)\begin{cases} (i) & u \in L^2(0T;\, H^1(\Omega)) \\ (ii) & u \in \mathcal{C}^0([0,T];\, L^2(\Omega)) \\ (iii) & u(0)=u_0 \ , \quad u_0 \in L^2(\Omega) \quad (\text{given}) \\ (iv) & \dfrac{d}{dt}(u(\cdot),v) + a(u(\cdot),v) = (f,v) \\ & \text{in the distribution sense} \quad \forall v \in V \end{cases}$$

As in the Cauchy-Dirichlet Problem f is here given in the space $L^2(0T;\, L^2(\Omega))$ or is a "reasonable" element of $L^2(0T;\, (H^1(\Omega))')$ since $(H^1(\Omega))'$ is not a "concrete Space" (i.e. is not a Distribution space)

We have still a explicit formula for the solution

if $\quad u = \sum_{k=0}^{\infty} \mu_k(t)\, w_k$

We find

$$(48)\begin{cases} \mu_k(t) = \mu_{0k}\, e^{-(\lambda_k-\ell)t} + \int_0^t e^{-(\lambda_k-\ell)(t-\sigma)} \gamma_k(\sigma)\, d\sigma \\ k=0,\ 1,\ 2,\ \dots \end{cases}$$

<u>Remark</u> We can do remarks by analogy with the case of Cauchy-Dirichlet problem.

In particular the boundary condition $[(\Pi_1)-ii]$ is contained

1-3 - Exercises

1) Solve by Fourier's Method and state a variational formulation for problem (P_1) in the following cases:

a) P_1 - (ii) is replaced by $u(0,t)=0$, $\frac{\partial u}{\partial t}(1,t)=0$

b) P_1 - (ii) is replaced by $\frac{\partial u}{\partial t}(0,t)=0$, $\frac{\partial u}{\partial t}(1,t)+K u(1,t)=0$, $K \geq 0$.

2) Solve by Fourier's method and state a variational formulation for Cauchy-Dirichlet Problem in a rectangular domain

See also other situations in Exercise Lecture -

1-4- The Cauchy-Dirichlet Problem in a Disc

Let Ω be the open set

$$\Omega = \{ (x,y) \in \mathbb{R}^2 ; \ \rho = \sqrt{x^2+y^2} < 1 \}$$

It is indicated to consider the analogue of Problem (P_1) for instance in polar coordinates

$$x = \rho \cos\theta , \quad y = \rho \sin\theta , \quad 0 \leq \theta \leq 2\pi .$$

We look for a function $u(\rho,\theta,t)$ which satisfies

$$(49) \begin{cases} (i) & \frac{\partial u}{\partial t} - \Delta_{\rho,\theta} u = f \\ (ii) & u(1,\theta,t) = 0 \qquad 0 \leq \theta \leq 2\pi , \ t \in [0,T] \\ (iii) & u(\rho,\theta,0) = u_0(\rho,\theta) \qquad (\rho,\theta) \in \Omega \end{cases}$$

Where $\Delta_{\rho,\theta} = \dfrac{\partial^2}{\partial\rho^2} + \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2}{\partial\theta^2}$

We introduce the eigenfunctions of $-\Delta_{\rho,\theta}$ in the Dirichlet problem for a ~~circle~~ Disc.

The eigenvalues are $\lambda_{n,m}$ such that

(50) $\quad \forall n = 0, 1, \dots, \quad J_n(\lambda_{n,m}) = 0, \quad m = 1, 2, \dots$

Where $x \to J_n(x)$ is the n-order Bessel function Which is the regular solution of ~~the~~ Bessel's Equation

(51) $$Z'' + \frac{1}{x} Z' + \left(1 - \frac{n^2}{x^2}\right) Z = 0$$

(See Courant-and Hilbert's Book or Sneddon - Fourier's Transform book)

The corresponding eigenspace $V_{n,m}$ associated with $\lambda_{n,m}$ is a two dimensional space spanned by the functions:

(52) $$\begin{cases} \Phi_{n,m}(\rho,\theta) = k_{n,m} J_n(\lambda_{n,m}\rho) \cos n\theta \\[2ex] \Psi_{n,m}(\rho,\theta) = k_{n,m} J_n(\lambda_n \rho) \sin n\theta \end{cases}$$

With $\quad k_{n,m}^{-2} = \frac{\pi}{2} J_n'^{\,2}(\lambda_{n,m})$ such that

(53) $$|\Phi_{n,m}|^2 = \int_\Omega \rho \cdot \Phi_{n,m}^2(\rho,\theta)\, d\rho\, d\theta = |\Psi_{n,m}|^2 = 1$$

... complete orthonormal system in $L^2(\Omega)$

$$(54)\quad (u,v) = \int_{\Omega} \rho\, u(\rho,\theta)\, v(\rho,\theta)\, d\rho\, d\theta$$

(this is obvious in this case because of $(-\Delta)^{-1}$ is a bounded, self-adjoint compact operator from $L^2(\Omega) \to L^2(\Omega)$.)

If $f \in L^2(0,T; L^2(\Omega))$, $u_0 \in L^2(\Omega)$ are given by

$$(59)\quad \begin{cases} f(\rho,\theta,t) = \displaystyle\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \left(a_{n,m}(t)\cos n\theta + b_{n,m}(t)\sin n\theta\right) J_n(\lambda_{n,m}\rho) \\ u_0 = \displaystyle\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \left(\alpha_{0,n,m}\cos n\theta + \beta_{0,n,m}\sin n\theta\right) J_n(\lambda_{n,m}\rho) \end{cases}$$

We look for u defined by

$$(60)\quad u(\rho,\theta,t) = \sum_{n,m} \left(\alpha_{n,m}(t)\cos n\theta + \beta_{n,m}(t)\sin n\theta\right) J_n(\lambda_{n,m}\rho).$$

The functions $\alpha_{n,m}$, $\beta_{n,m}$ are determined by

$$(61)\quad \begin{cases} \alpha'_{n,m} + \lambda_{n,m}\alpha_{n,m} = a_{n,m} \\ \beta'_{n,m} + \lambda_{n,m}\beta_{n,m} = b_{n,m} \\ \alpha_{n,m}(0) = \alpha_{0,n,m}\,, \quad \beta_{n,m}(0) = \beta_{0,n,m} \end{cases}$$

We let to the participants to show in what sense Prob. $(\widehat{P_1})$ is solved.

1-5- n-dimensional Case

Let Ω be a bounded open set of $\mathbb{R}^n$. $(n=1,2,3,\dots)$

With a boundary Γ which is sufficiently regular

If $x=(x_1,\dots,x_n) \in \mathbb{R}^n$ we put

$$(62) \quad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \qquad (\text{Laplacian in } \mathbb{R}^n)$$

~~the~~ Problems (P_1) and $(\hat{P}_1)$ are particular cases of following Problem

To find $u(x,t)$ which satisfies

$$(P) \quad \begin{cases} (i) \ \dfrac{\partial u}{\partial t} - \Delta u = f \\ (ii) \ u|_\Gamma = 0 \qquad \text{Dirichlet Boundary condition} \\ (iii) \ u(x,0) = u_0(x) \end{cases}$$

With u_0, f convenient given functions.

As Ω is bounded and Γ sufficiently regular then the injective map from $H^1(\Omega)$ to $L^2(\Omega)$ is compact.

If we define $A=-\Delta$ as an unbounded operator in $L^2(\Omega)$ with domain $D(A)=H^2(\Omega)\cap H^1_0(\Omega)$, then A admit a bounded, self-adjoint, compact inverse in $L^2(\Omega)$.

Therefore we can apply Fourier's method to problem (P).

We introduce the Eigen functions of $-\Delta$ in the Dirichlet

$$(63)\begin{cases} -\Delta W_k = \lambda_k W_k \\ W_k|_\Gamma = 0 \\ \int_\Omega W_k^2(x)\,dx = 1 \end{cases}$$

$\{W_k\}_{k\in\mathbb{N}^*}$ being an Hilbertian basis of $H_0^1(\Omega)$ and $L^2(\Omega)$.

It is fundamental to note that the determination of (λ_k, W_k) depend on the geometry of Ω and that except very simple geometrical situations (Cube, Sphere in $\mathbb{R}^n$...) these élements are ~~out~~ out of reach.

Fourier's method would lead us generally to an <u>explicit formula</u> for the solution u which is function of éléments <u>numerically out of reach</u>.

thus It is <u>not</u> generally a <u>constructive method</u>.

Theoritically this method leads to a variational formulation for the Problem (P_1).

Introducing $V = H_0^1(\Omega)$ and $a(u,v) = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx$

we shall obtain the result:

There exists one ~~function~~ function u with the following properties:

$$(64)\begin{cases} (i) & u \in L^2(0T; V) \\ (ii) & u \in \mathcal{C}^0([0,T]; L^2(\Omega)) \\ (iii) & u(0) = u_0 \qquad u_0 \text{ given } \in L^2(\Omega) \\ (iv) & \frac{d}{dt}(u(\cdot), v) + a(u(\cdot), v) = (f(\cdot), v) \quad \forall v \in V \end{cases}$$

Remarks

1) There are analogeous considerations for other boundary condition than ((P)- ii).

2) In particular case of Cauchy - Neuman problem we shall replace A by $A + \rho I$ with $\rho > 0$.

3) Like in the problem (P_1), we have a direct method to show uniqueness of the solution of Problem (P)

1-6- To sum up.

i) We possess an explicit method for the solution of a linear evolution problem.

ii) In a n- dimensional space $(n > 1)$ except in the particular cases of a simple geometrical domain the eigenfunctions (except the first amongst by numerical approximation) are out of reach and the method is no longer constructive.

iii) we have a variational formulation for the stated problems and also a direct method to stablish the uniqueness of the solution.