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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS TO MECHANICS

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Elements of the Theory of Elastic Stabilty (Continued - 2)

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6.1 In dealing with the stability of the null solution (equilibrium) in the class of nonlinear elastic perturbations, it is important to realise that classical or strong solutions of the elastodynamic problem cannot be used since these are unlikely to exist for all time. This is certainly true for one-dimensional nonlinear conservation laws and so it must be expected that the same will hold for solutions of nonlinear elastodynamics. Shock waves and other singular surfaces will develop and hence stability must be analysed in the class of generalised or weak solutions. In the case of systems of nonlinear hyperbolic conservation laws in one independent space variable, Glimm has proved global existence of the generalised solution for initial data of sufficiently small oscillation and total variation. However, his results cannot be immediately extended to elasticity since they require the stress to be everywhere concave or convex and it is neither. Furthermore, generalised solutions may be nonunique and therefore lead immediately to instability (c.p. Thm 2.4.1) unless a selection procedure, such as the viscosity method, is used to recover uniqueness

Further details and extended comments on these topics are provided in the excellent survey by Dafermos [in Aldas "Nonlinear Waves", Cornell Univ Press 1974.]

Nevertheless, in a course of this type, it is better to proceed formally and treat classical ~~st~~ solutions. Little loss is involved ... all methods used continue valid for suitably formulated weak solutions. Indeed, it emerges from subsequent computations that the crucial property required of any solution is an energy inequality in the form

$$E(t) \leq E(0).$$

Two problems are of especial concern at the present time. The first, which has already been encountered in the study of linearised elasticity, asks whether a minimum value of the potential energy at equilibrium is sufficient for stability. This appears unlikely in the absence of compactness, although as we shall see, the introduction of a potential well formally overcomes the difficulty. The second problem is to determine whether stability of the equilibrium solution in the class of linearised motions is sufficient to guarantee stability in the class of nonlinear motions.

Finally, we shall also treat instability not only to establish necessary conditions for stability, but also from the point of view of nonexistence of solutions.

6.2. Potential Wells. Stability.

Defn. 6.2.1. (Gurtin, Sattinger). The set $\mathcal{K}^P$ is a potential well for $I(u)$ at 0 if

(a) $\mathcal{K}^P$ is a N^d of 0, i.e., $\exists\ y \in \mathcal{K}^P$ s.t. $\rho(y) < \varepsilon$, $\varepsilon > 0$, where ρ is a +ve def. fn. (e.g., metric, norm)

(b) $I(u)$ has a minimum at 0 over $\mathcal{K}^P$, i.e., for some $\varepsilon_0 > 0$

$$I(u) \geq 0, \qquad 0 \leq \rho(u) < \varepsilon_0 .$$

(c) $$\inf_{\rho(u)=\varepsilon} I(u) = \psi(\varepsilon) > 0, \quad 0 < \varepsilon < \varepsilon_0 .$$

Remark: Cond (c) states that the $\inf I(u)$ is achieved on $\rho(u) = \varepsilon$, and is +ve.

Thm. 6.2.1. The null soln. is stable w.r.t. $E(0)$ and ρ, provided $I(u)$ possesses a potential well at 0 over $\mathcal{K}^P$.

Proof: See Thm 3.5.1. (discrete case).

Remark: 1. The set $\mathcal{K}^P$ for which a potential well exists determines the choices for ρ and hence the measures for stability.

2. The energy test is true (formally, at least) in the modified form expressed by the Thm.

3. We would like to know conditions on the strain energy yielding an energy well.

4. Thm 6.2.1. has been given by Koiter [e.g., Proc. Kron. Ned. Akad. Wet. B $\underline{66}$, 173–177, 1963] and Gurtin [J. Elast. $\underline{3}$, 1–4, 1973; Arch. Rat. Mech. Anal. $\underline{59}$, 63–96, 1975].

One condition producing a potential well is the convexity of $I(u)$, but this is unsuitable since it implies global uniqueness of solutions in the equilibrium problem. On recalling that critical points of $V(z)$ are equilibrium solutions, we see that $I(u)$ should behave as shown in Figs. I and II.

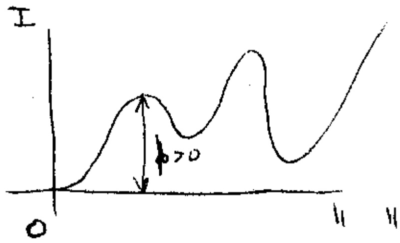


Fig I

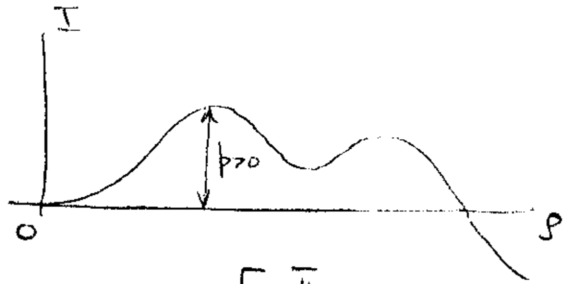


Fig II.

This suggests that provided the 'curvature' of $I(0)$ is sufficiently positive, a potential well ought to exist in the N^d of 0. Analytically, the argument is roughly:-

$$\delta^2 I(0) v \equiv \int_\Omega d_{ijkl} v_{i,j} v_{k,l}\, dx \qquad (6.2.1)$$

$$\geqslant d \int_\Omega v_{i,j} v_{k,l}\, dx, \quad d > 0 \qquad (6.2.2)$$

Suppose $P \sim W_{1,2}(\Omega)$, and I is C^3 in sense of Frechet. Then for $v \in W_{1,2}$, by T.S.,

$$I(u) = I(0) + \delta I(0) u + \tfrac{1}{2} \delta^2 I(0) u^2 + R$$

$$\geqslant d \int_\Omega u_{i,j} u_{k,l}\, dx + R,$$

where

$$R = o(\| \quad \|_{W_{1,2}})$$

$$\therefore\ \underline{I(u) \geqslant k \| \quad \|^2_{W_{1,2}}}$$

Hence, we may establish stability of $u \equiv 0$.

Remark 1. Martini (Proc. Kon. Ned. Wet to appear) has proved that if $I(u)$ is C^2 in the sense of Fréchet wrt. ~~W~~ $\| \quad \|_{W_{1,2}}$, then $W(y)$ must be quadratic. Furthermore, if (6.2.2) holds ~~and $I(u)$ is C^2 wrt.~~ but with the L.H.S. replaced by another norm with respect to which $I(u)$ is twice continuously differentiable, the same result holds.

It appears that the above argument for stability cannot be used without modification.

Remark 2. Grimaldi and Cano (Meccanica, Dec. 1975, 254 – 268) have often offered one solution; Mejlbro (Report of the Matematiske Institut, Danmarks Tekniske Højskole, 1976) another, while Naghdi Trapp (Arch. Rat. Mech. Anal. 51, 165–191, 1973) give a third. All are based on a T.S. expansion of $I(u)$, but possess aspects somewhat unsatisfactory.

6.2.2. An example.

We construct an example in an endeavour to describe an ~~energy~~ potential well. We know that the selected function $I(u)$ cannot be twice cont. diff. in the sense of Frechet. From Figs I, II, however, it seems reasonable that the linearised part of $I(u)$ should be positive definite; but then the nonlinear part must be such that for sufficiently large values of $\| \quad \|$, it dominates the expression and produces ~~to~~ a local maximum.

It is also important to specify the space P over which the potential well is defined.

Thus, let us take a one-dimensional model and set

$$W(\nabla u) = u_x^2 - u_x^3 , \qquad x \in [0,1]$$

Then
$$V(u) = \int_0^1 (u_x^2 - u_x^3)\,dx$$
and
$$\delta^2 V(0)v = 2\int_0^1 v_x^2\,dx. \qquad (6.2.3)$$

Hence, because of (6.2.3) it is naturally to try $P \sim W_{1,2}(\Omega)$, and the problem is to prove the inequality
$$V(u) \geq k_1 \int_0^1 u_x^2\,dx, \qquad u \in W_{1,2}(\Omega), \qquad (6.2.4)$$
whenever $\|u\|_{1,2} = \varepsilon_1$. We show that (6.2.4) can never be satisfied. Set ~~...~~ $u(x) = a(x^\alpha - x)$; then $u_x = a[\alpha x^{\alpha-1} - 1]$, and
$$\|u\|_{1,2}^2 = \int_0^1 u_x^2\,dx = \frac{a^2(\alpha-1)^2}{(2\alpha-1)} = \varepsilon, \text{ say}; \qquad (6.2.5)$$

$$\begin{aligned}\int_0^1 u_x^3\,dx &= a^3\left[\frac{\alpha^3}{3\alpha-2} - \frac{3\alpha^2}{2\alpha-1} + 2\right] \\ &= \frac{(2\alpha-1)^{3/2}}{(\alpha-1)^3}\varepsilon^{3/2}\left[\frac{\alpha^3}{3\alpha-2} - \frac{3\alpha^2}{2\alpha-1} + 2\right], \text{ from (6.2.5)} \\ &\longrightarrow \infty \text{ as } \alpha \to \infty.\end{aligned}$$

$$\therefore \underline{V(u) < 0 \quad \text{for } \|u\|_{1,2} = \varepsilon.}$$

and hence no potential well exists in $W_{1,2}(\Omega)$.

It may similarly be shown that no potential well exists in $W^{1,p}$, $1 \leq p \leq \infty$.

Similar examples may be constructed in two-dimensions, and we conclude: (!) <u>potential wells are rare</u>.

6.3. A sufficient condition for a potential well (Payne)

In one situation, at least, we may establish a potential well.

Let ~~G be~~ $G : \mathbb{R} \to \mathbb{R}$ be cts. and satisfy $G(0) = 0$.

Set

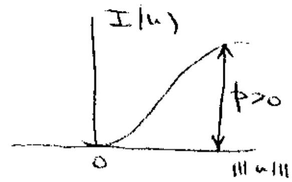
$$|||u|||^2 = \int_\Omega u_{i,j} u_{i,j} \, dx$$

We examine functions $I(u)$ for which

$$I(u) \geq |||u|||^2 \left(k_1 - k_2 \, G\left(|||u|||^2\right)\right) \qquad \text{when } |||u||| < c, \qquad (6.3.1)$$

where k_1, k_2, c are +ve constants.

We seek conditions ensuring that a potential well exists in $W_{1,2}$, with $|||u||| < c_2$, by ~~p~~ establishing that the max. value of the R.H.S. of (6.3.1) has a positive maximum, $p > 0$. Recall that



$$\tilde{V}(u) \stackrel{\text{def}}{=} |||u|||^2 \left(k_1 - k_2 \, G\left(|||u|||^2\right)\right)$$

is cont., by hypoth., and hence assumes all values between $0, p$.

Now $\tilde{V}(u)$ achieves its max. value for $\varphi \in W^{1,2}(\Omega)$ when $w \equiv |||\varphi|||^2$ is a soln. of

$$k_1 - k_2 \left(w \, G(w)\right)' = 0, \qquad (6.3.2)$$

provided $w \leq c$.

Suppose w_1 is the smallest root of this eqn., and <u>assume</u> that $w_1 \leq c^2$. Then

$$p = \max_{|||\varphi|||^2} \tilde{V}(\varphi) = k_2 w_1^2 \, G'(w_1).$$

We require $p > 0$. Hence, we must impose

$$G'(w_1) > 0.$$

<u>Theorem 6.3.1.</u> (K. and Payne). Suppose for $|||u||| < c$, the potential energy $I(u)$ satisfies (6.3.1), where $G \in C^0(\mathbb{R})$, $G(0) = 0$ and $G'(w_1) > 0$ and $w_1 \in \mathbb{R}$ is the smallest +ve root of (6.3.2). Then

(i) for $w_1 \leq c^2$, the soln. on its interval of existence satisfies $|||u(t)|||^2 < w_1$ for initial data

$$(\alpha)\quad |||u(0)|||^2 \leq w_1,$$

$$(\beta)\quad E(0) < k_2 w_1^2 G'(w_1);$$

(ii) for $w_1 > c^2$, the soln. on its interval of existence satisfies $|||u(t)|||^2 < c$ for initial data

$$(\alpha_1)\quad |||u(0)|||^2 < c,$$

$$(\beta_1)\quad E(0) < c^2\left(k_1 - k_2\, G(c^2)\right).$$

Proof. (i) By (α), the soln. begins inside the potential well.

Let t_1 be 1st instance of time s.t.

$$|||u(t_1)|||^2 = w_1$$

Suppose $t_1 < \infty$. Then

$$p = \hat{V}(w_1) \leq V(u(t_1)) \leq E(0) < p$$

Contradiction $\square$.

(ii) Same.

6.4. Global nonexistence and instability.

The methods described here continue to apply ~~for~~ to suitably formulated weak solutions. Fuller details may be found in K., Levine and Payne, Arch. Rat. Mech. Anal., 55, 52-72, 1974; K. and Straughan, Trends in Applications of Pure Mathematics to Mechanics. Ed. G. Fichera, Pitman 1976.

We here formally treat (strong) solutions to

$$\frac{\partial}{\partial x_i}\left(\frac{\partial W}{\partial u_{i,j}}\right) = \rho \ddot{u}_i \qquad (x,t) \in \Omega \times (0,T), \qquad (6.4.1)$$

$$u_i = 0 \qquad (x,t) \in \partial\Omega \times (0,T) \qquad (6.4.2)$$

where $(0,T)$ is the maximal interval of existence. Observe that

$$E(t) \equiv \frac{1}{2}\int_\Omega \rho \dot{u}_i \dot{u}_i dx + \int_\Omega W(u)dx \left(= T + V\right) = E(0) \qquad (6.4.3)$$

although in the weak formulation, we cannot derive (6.4.3) from (6.4.1), (6.4.2) and instead ~~to~~ must separately postulate the inequality

$$E(t) \le E(0) \qquad (6.4.4)$$

Set

$$F(t) = \int_\Omega \rho u_i u_i dx.$$

Then

$$\dot{F}(t) = 2\int_\Omega \rho u_i \dot{u}_i dx$$

$$\ddot{F}(t) = 2\int_\Omega \rho \dot{u}_i \dot{u}_i dx + 2\int_\Omega \rho u_i \ddot{u}_i dx$$

$$= 2\int_\Omega \rho \dot{u}_i \dot{u}_i dx - 2\int_\Omega u_{i,j}\frac{\partial W}{\partial u_{i,j}} dx \qquad (6.4.5)$$

$$= 4(1+\alpha)\int_\Omega \rho \dot{u}_i \dot{u}_i dx - 4(1+2\alpha)E(0) + 2\int_\Omega \left[2(1+2\alpha)W - u_{i,j}\frac{\partial W}{\partial u_{i,j}}\right]dx$$

(6.4.6)

where (6.4.3) and (6.4.1) have been used.

Consider (6.4.5) and suppose

$$\int_\Omega u_{i,j} \frac{\partial W}{\partial u_{i,j}}\, dx \le 0; \qquad (6.4.7)$$

Then $\ddot{F}(t) \geq 0$ and hence

$$F(t) \geq F(0) + tF'(0), \qquad t \geq 0,$$

so that when $\dot{F}(0) > 0$, $F(t)$ grows linearly in t. Observe further, that on retaining the K.E. in (6.4.5), we have

$$F\ddot{F} - \tfrac{1}{2}\dot{F}^2 \geq \int_\Omega \rho u_i u_i\, dx \int_\Omega \rho \dot{u}_i \dot{u}_i\, dx - \left(\int_\Omega \rho u_i \dot{u}_i\, dx\right)^2 \geq 0.$$

$$\therefore (F^{1/2})'' \geq 0$$

Hence, if $\dot{F}(0) > 0$,

$$F(t) \geq F(0) + t\dot{F}(0) + \frac{1}{4}\frac{\dot{F}(0)}{F(0)}t^2, \qquad t \geq 0,$$

and so $F(t)$ grows quadratically in t.

We next strengthen (6.4.7) to

$$-\int_\Omega u_{i,j}\frac{\partial W}{\partial u_{i,j}}\, dx \geq k F^{1+\varepsilon}, \qquad k > 0,\ \varepsilon > 0 \qquad (6.4.8)$$

and employ a convergent integral technique used by Fujita. From (6.4.5) the K. Energy may be discarded to give

$$\ddot{F}(t) \geq kF^{1+\varepsilon}(t) \geq 0 \qquad (6.4.9)$$

Assume $\dot{F}(0) > 0$. There exists t_1 such that by continuity

$$\dot{F}(t) > 0 \qquad t \in [0, t_1)$$

Multiplying $(6.4.9)_2$ by $\dot{F}(t)$ and integrating over $[0,t_1]$ produces

$$\dot{F}(t_1)^2 \geq \dot{F}(0)^2 > 0.$$

$$\therefore \dot{F}(t) > 0 \qquad t \in [0, \infty)$$

Multiplying $(6.4.9)_1$ by $\dot{F}(t)$ and integrating gives

$$\dot{F}^2(t) \geq \alpha^2 + a^2 F^{2+\varepsilon}(t), \qquad t \in [0, \infty) \qquad (6.4.10)$$

where

Observe

$$Q^2 > 0 \qquad \text{for sufficently } k.$$

Suppose now $0 \le F(t) < \infty$, $t \in [0, \infty)$. Then from (6.4.10)

$$t \le \int_{F(0)}^{F(t)} \frac{dF}{[Q^2 + a^2 F^{2+\varepsilon}]^{1/2}} \le \int_{F(0)}^{\infty} \frac{dF}{[Q^2 + a^2 F^{2+\varepsilon}]^{1/2}} < \infty$$

Hence, we have a contradiction, and so the solution cannot exist for all time, subject to $\dot F(0)$ and (6.4.8)

Let us next examine (6.4.6) and assume

$$\int_\Omega \left[2(1+2\alpha)W - u_{i,j}\frac{\partial W}{\partial u_{i,j}}\right] dx \ge 0 \qquad (6.4.11).$$

It may easily be proved using Schwarz's inequality (Cp. p.26) that

$$F\ddot F - (1+\alpha)\dot F^2 \ge -4(1+2\alpha)E(0)F(t)$$

Indeed, ~~or~~ instead of $F(t)$ let us use

$$G(t) = F(t) + \beta(t+t_0)^2, \qquad t_0, \beta > 0$$

Then, in similar manner, it may be shown that

$$G\ddot G - (1+\alpha)\dot G^2 \ge -2(1+2\alpha)(\beta + 2E(0))G(t) \qquad t \in [0,T). \qquad (6.4.12)$$

We note that

$$(G^{-\alpha})'' \equiv -\alpha G^{-(\alpha+2)}[G\ddot G - (1+\alpha)\dot G^2]$$

and hence, for $G(t) > 0$, (6.4.12) may be replaced by

$$\underline{(G^{-\alpha})'' \le 2\alpha(1+2\alpha)(\beta + 2E(0))G^{-(1+\alpha)}} \qquad (6.4.13)$$

The inequality (6.4.13) may be discussed by methods analogous to those used for the log. convexity arguments. For instance, on using the property that a concave function lies above the join of any two points on its curve, we may establish the uniqueness and cont. dependence on the initial data of the null solution. Inequality (6.4.13) may also be used for nonexistence. We indicate the argument by considering initial data for which $E(0) < 0$. Set $\beta + 2E(0) = 0$, and use the property that a concave function lies below the tangent to any point of the curve. We obtain

$$G^{\alpha}(t) \geq G^{\alpha}(0) \Big/ \left[1 - \alpha t \frac{G'(0)}{G(0)}\right] \qquad (6.4.14)$$

But t_0 may always be chosen to make $G'(0) > 0$, and so the R.H. of (6.4.14) becomes unbounded in finite time. Thus, there can only be local existence of the solution.

6.5. Interpretation of condition (6.4.11).

Andreussi (Report of Maths. Dept., Univ. of Pisa, 1975) has examined consequences of (6.4.11) for particular elastic materials. We, however, offer another brief interpretation not based upon special constitutive assumptions. We explicitly re-introduce the equilibrium config. Ω_0, in which a particle at x in Ω has position z, and then it may be easily proved that (6.4.11) becomes

$$H(y) \equiv 2(1+2\alpha)\int_{\Omega}\left[W(y_{i,j}) - W(z_{i,j})\right]dx - \int_{\Omega}(y_{i,j} - z_{i,j})\frac{\partial W}{\partial y_{i,j}}(y_{i,j})\,dx \geq 0.$$

Hence

$$H(z) = 0,$$

and so $H(y)$ has a minimum at $y = z$. Therefore

$$\delta^2 H(z) w^2 \geq 0$$

or

$$4\alpha \int_{\Omega} \frac{\partial^2 W}{\partial z_{i,j}\,\partial z_{k,l}} w_{i,j} w_{k,l}\, dx \geq 0;$$

since $\alpha > 0$, we may further conclude that

$$d_{ijkl}\, \xi_i \xi_k \eta_j \eta_l \geq 0$$