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FUNCTIONAL ANALYTIC METHODS IN ELASTIC-PLASTIC TORSION OF CYLUNDRICAL BARS
(Part I)

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Functional analytic methods in elastic - plastic torsion of cylindrical bars.

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Introduction. (Motivation)

This problem probably does not interest any participant of this session but it can be a very good exercise about technics of functional analysis applied to a real problem:

We shall describe briefly the physical problem

We shall write down the equations or "inequations" which have to be satisfied by the unknown functions.

We shall put them in a variational writing.

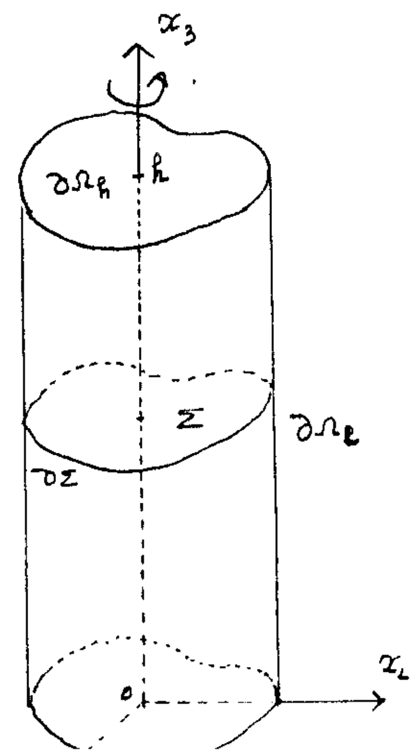
" " shall obtain existence and uniqueness theorem and also the nature of the regularity of the solution.

We shall go back to the concrete problem giving some useful properties of the solution.

We shall exhibit at the end the numerical results

I. Introduction to the concrete problem

I.1. a brief description



- The bar is twisted by opposite terminal couples $\pm \mathcal{M}\vec{n}_3$
- no body forces in Ω : $f = 0$ in Ω
- no forces on the lateral boundary (including the boundaries of cavities if the cross section is supposed multiply connected)

$$F = 0 \quad \text{on} \quad \partial\Omega_L$$

- The material is supposed to be "elastic - plastic perfectly plastic"

I. 2 An associated boundary value problem

Find σ and u defined on Ω

σ; stress tensor field with values in R^9; $\sigma_{ij} = \sigma_{ji}$

u, displacements field with values in R^3; u_i

such that:

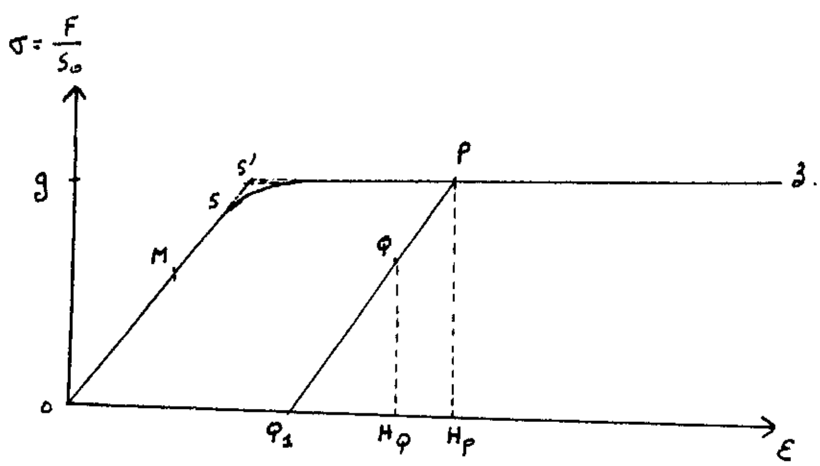
$$\begin{cases} \sigma_{ij,j} = 0 & \text{in } \Omega \quad \text{(equilibrium equations)} \\ \sigma_{ij} n_j = 0 & \text{on } \partial\Omega_L \quad \text{(absence of applied forces on } \partial\Omega_L) \\ u_1 = u_2 = 0 \;;\; \sigma_{33} = 0 & \text{on } \partial\Omega_0 \\ u_1 = -\alpha(t) h x_2 \;;\; u_2 = \alpha(t) h x_1 & \text{and } \sigma_{33} = 0 \text{ on } \partial\Omega_h \end{cases}$$

(In fact, a rotation of the terminal section is imposed; $\alpha(t)$ is the angle of twist. Later on, we shall take $h = 1$)

Furthermore, σ and u has to be related by the constitutive law that we are going to explicite below

I.3 Constitutive law for an elastic-plastic (perfectly plastic) media.

a) Experimental illustration of the elastic plastic behavior.



Let us consider a metal rod whose cross-sectional is S_o; Suppose we apply a tractive force F to the ends of this rod. If we put in plot $\sigma = \frac{F}{S_o}$ against the relative lenthening ε, we find at the beginning that the point (ε, σ) describes a straight line segment through the origine O. Then, at S the line curves after which it tends to the axis $O\varepsilon$. Let $O3$ be the

The variation of σ in a small neighbourhood about some point M of the open segment OS is both linear and reversible and represents the elastic behaviour of the material

Let us now suppose that starting at the point P on the arc S_3, we allow F to decrease to zero; the path now traced out becomes the straight line segment PQ_1, virtually parallel to OS. Then, at the point P, the material behaviour is no longer reversible and the arc O_3 represents its plastic part. However, along the open segment PQ_1, the behaviour of the media is again reversible and linear and hence elastic. We can see that the total relative lengthening at a point φ of PQ_1 is the sum of the elastic strain $Q_1 H_\varphi$, corresponding to the load $\sigma(\varphi)$, and the permanent (or plastic) strain

$$O\varphi_1 = \pi(\sigma(P))$$

Perfect plasticity is characterized by S_3 approximating to a half-line parallel to $O\varepsilon$; the stress then never exceeds a maximum limit g, wich is independant of the strain $g = \sigma(S)$.

b) The Prandtl Reuss Law

In the one-dimentional case described above, we can write

$$d\varepsilon = A\,d\sigma + d\pi(\sigma)$$

$$(1)\quad \text{where} \begin{cases} d\pi(\sigma) = 0 & \text{if } \sigma < g \text{ or if, } \sigma = g \text{ and } d\sigma < 0 \\ d\pi(\sigma) \geq 0 & \text{if } \sigma = g \text{ and } d\sigma = 0 \end{cases}$$

and $\frac{1}{A}$ is the slope of OS

When this infinitesimal variation is carried out during a time intervalle dt, then (1) can be written

$$(2)\quad \begin{cases} \sigma \leq g\ ; \quad \dot{\varepsilon} = A\dot{\sigma} + \lambda\ ; \qquad \left(\lambda = \frac{d\pi}{dt}\right) \\ \text{with } \lambda = 0 \ \text{ if } \sigma < g \text{ or if, } \sigma = g \text{ and } \dot{\sigma} < 0 \\ \qquad \lambda \geq 0 \ \text{ if } \sigma = g \text{ and } \dot{\sigma} = 0. \end{cases}$$

another equivalent expression of (2) is

$$(3)\quad \begin{cases} \sigma \leq g\ ; \quad \dot{\varepsilon} = A\dot{\sigma} + \lambda \\ \text{with } \lambda \text{ such as } \lambda(\tau - \sigma) \leq 0 \qquad \forall\, \tau \leq g. \end{cases}$$

In the three dimensional case, the criterion of elasticity: $\sigma \leq g$ can be replaced by an inequality of the type.

$$(4) \qquad \mathcal{F}[\sigma(x)] \leq 0 \qquad \forall\ x \in \Omega$$

Ω being the domain of R^3 occupied by the media, $\mathcal{F}$ being a scalar function, continuous and convexe with respect to σ_{ij} in a such a way that the inequality (4) definites a closed convex set in R^6. ($\sigma(x) \in R^6$ because the symetry $\sigma_{ij} = \sigma_{ij}$)

The analogue of (3) can be written.

$$(5) \begin{cases} \mathcal{F}(\sigma) \leq 0 \quad \text{in } \Omega \\ \dot{\varepsilon}_{ij}(u) = A_{ijkh}\,\dot{\sigma}_{kh}(u) + \lambda_{ij}. \\ \lambda_{ij}(\tau_{ij} - \sigma_{ij}) \leq 0 \quad \forall\ \tau \in R^9 \text{ such that } \tau_{ij} = \tau_{ji} \text{ and } \mathcal{F}(\tau) \leq 0. \end{cases}$$

(the equivalent of $\lambda\dot{\sigma} = 0$ can be deducted of the preceeding inequality when σ is derivable in t)

$\varepsilon(u)$ of components $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the linearised strain tensor.

The A_{ijkh}, non necessarily constants, are the elasticity coefficients, satisfying the usual properties:

$$(6) \qquad A_{ijkh} = A_{jikh} = A_{khij}.$$

$$(7) \qquad A_{ijkh}X_{ij}X_{kh} \geq c\,X_{ij}X_{ij}\ ; \quad c > 0 \quad \forall\ X \in R^9 \text{ such that } X_{ij} = X_{ji}$$

$A_{ijkh}\,\dot{\sigma}_{kh}$ is therefore the elastic part of the rate of train tensor $\dot{\varepsilon}_{ij}(u)$ and λ_{ij} is called by definition the plastic rate of strain

The relations (4) (5) (6) (7) expresses the "Prandtl-Reuss law"

c) The Hencky law

If we go back now to the one dimensional example excluding any possibility of unloading for the rod Then

$$(8) \qquad \dot{\sigma} = \frac{\dot{F}}{S_0} \geq 0.$$

expect a law of the type.

$$\sigma = \text{function of } \varepsilon,$$

the reciprocal relation being impossible because the hypothesis of perfect plasticity. For $P \in OSg$, we shall have then:

$$(9)\quad \begin{cases} \sigma(P) = \dfrac{\varepsilon(P)}{A} & \text{if } \dfrac{\varepsilon(P)}{A} < g. \\ \sigma(P) = g & \text{if } \dfrac{\varepsilon(P)}{A} \geq g \end{cases}$$

Which can be still written:

$$(10)\quad \begin{cases} \sigma(P) \leq g \;;\quad \varepsilon(P) = A\,\sigma(P) + \lambda(P) \\ \text{with } \lambda \text{ scalar function such that.} \\ \lambda(P) = 0 \quad \text{if } \sigma(P) < g. \\ \lambda(P) \geqslant 0 \quad \text{if } \sigma(P) = g. \end{cases}$$

equivalent to

$$(11)\quad \begin{cases} \sigma(P) \leq g \;;\quad \varepsilon(P) = A\,\sigma(P) + \lambda(P) \\ \text{with } \lambda \text{ scalar function such that.} \\ \lambda(P)\,[\tau - \sigma(P)] \leq 0 \quad \forall \tau \leq g. \end{cases}$$

If again we generalize these observations for a tridimentional problem, we obtain. the Hencky-law

$$(12).\quad \begin{cases} \mathcal{F}(\sigma) \leq 0 \quad \text{on } \Omega. \\ \varepsilon_{ij}(u) = A_{ijhk}\,\sigma_{hk} + \lambda_{ij}. \\ \lambda_{ij}(\tau_{ij} - \sigma_{ij}) \leq 0 \quad \forall \tau \in R^9,\ \tau_{ij} = \tau_{ji},\ \mathcal{F}(\tau) \leq 0. \end{cases}$$

This law, simpler that the preceding one, constitutes a good mathematic model and allows to study the boundary value problems at a given time t without take account of any derivatives with respect to t. However, the Hencky law is definitively inconnect as soon as an unloading is possible. (for instance vibrations of propagation of waves)

d) Precise constitutive law for the problem of torsion described above

We shall consider successively the Hencky and Prandtl Reuss law with:

- The Von mises' criterion:

$$(13) \qquad \mathcal{F}(\sigma) = \frac{1}{2}\sigma_{ij}\sigma_{ij} - \frac{1}{6}\sigma_{ii}^2 - g^2$$

(g, constant or non constant scalar, >0 ; yield stress in simple shear)

- The elasticity tensor of Hooke's law:

$$(14) \qquad A_{ijkh} = \frac{1+\nu}{2E}\left[\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}\right] - \frac{\nu}{E}\delta_{ij}\delta_{kh}.$$

($E>0$; $0<\nu<\frac{1}{2}$ constants or non constants scalars)

I.4. Precis object of our research.

We are, for each value of $\alpha(t)$, interested in

- Existence and uniqueness of the solution $(\sigma_\alpha, u_\alpha)$
- The determination of

$$\mathcal{E}_\alpha = \{x \,/\, x \in \Omega \quad \text{s.t} \quad \mathcal{F}[\sigma_\alpha(x)] < 0\},$$ elastic region of Ω,

(in fact, in this region $\lambda_{ij} = 0$ and the media is then ruled by the Hooke's law)

and **also of**

$$\mathcal{P}_\alpha = \{\underline{x} \,/\, \underline{x} \in \Omega \quad \text{s.t.} \quad \mathcal{F}(\sigma_\alpha(x)) = 0\},$$ plastic region of Ω

I.5 Ideas of the method - Sheme for the following lectures

i) The first step consists in the formulation of a general boundary value problem with Hencky's Law:

Ω is not necessarily a cylendar

The media can be non homogeneous and non isotrope.

Let us call (A) this general problem. We obtain for it a theorem of existence and uniqueness for a "possible" stress field solution

ii) The second step consists in applying the preceding results to three particular cases

(B) Torsion of a homogeneous cylindrical bar

(B*) Torsion of a cylindrical bar with cylindrical cavities

(B') Torsion of a fiber reinforced cylindrical bar

In the three cases $\mathcal{F}(\sigma)$ is given by the Von Mises criterium

iii) The third step will show that, at least for the problem (B), The Prandtl Reuss law (much more realistic but more complex) leads to the same results that the Hencky's one. when $\alpha(t)$ is increasing.

iv) Some comments will be given about the numerical results.

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