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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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EVOLUTION EQUATIONS
First Order Linear Equations
(Part III)

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2 - Cauchy's Problems.

2 - 1. an one-dimensional example.

We consider the case of an infinite bar, and we look for a function $u(x,t)$ $x \in \mathbb{R}$, $t \geq 0$ which satisfies

$$(\Pi_1) \begin{cases} (1) & \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0 \\ (2) & u(x,0) = u_0(x) \end{cases}$$

Naturally we must precise in what sense we take the derivatives in equation (1) and the spaces in which the solution lies.

Since the operator $A = -\frac{d^2}{dx^2}$ commute with the translations in $\mathbb{R}$ it is indicated to use Fourier's Transform with respect of x.

Therefore we look for $u(t) \in \mathcal{S}'(\mathbb{R})$ for all $t \in]0,T[$ (i.e: $u(t): x \to u(x,t)$, $\mathbb{R} \to \mathcal{S}'(\mathbb{R})$)

$\mathcal{S}'(\mathbb{R}^n)$ $n \geq 1$ is the space of Schwartz of Tempered Distributions that is the dual of $\mathcal{S}(\mathbb{R}^n)$ the space of C^∞ functions which are quickly decreasing at infinity:

If, $x \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$ we put $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, $\alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{N}^n$

$D^\beta u = \dfrac{\partial^{\beta_1+\beta_2+\dots+\beta_n} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \dots \partial x_n^{\beta_n}}$, $\beta = (\beta_1, \dots \beta_n) \in \mathbb{N}^n$ and $|x| = \sqrt{x_1^2 + \dots + x_n^2}$

Then

(1) $u \in \mathcal{S}(\mathbb{R}^n) \Leftrightarrow u \in \mathcal{C}^\infty(\mathbb{R}^n)$, $\forall \alpha, \beta$ $x^\alpha D^\beta u(x) \to 0$ when $|x| \to \infty$

Now if $x \cdot y = x_1 y_1 + \dots + x_n y_n$, and $u \in \mathcal{S}(\mathbb{R}^n)$ the Fourier

(2) $\hat{u}(y) = \int_{\mathbb{R}^n} e^{-i\,x\cdot y} u(x)\,dx \quad , u \in \mathcal{S}(\mathbb{R}^n)$

and we have

$$(3)\begin{cases} (i) & u(x) = \dfrac{1}{(2\pi)^n}\displaystyle\int_{\mathbb{R}^n} e^{+i\,x\cdot y}\,\hat{u}(y)\,dy = \overline{\mathcal{F}}(\hat{u})(x) \\ (ii) & \mathcal{F}(D^\alpha u) = (iy)^\alpha \mathcal{F}u \\ (iii) & D^\beta \mathcal{F}(u) = \mathcal{F}((-ix)^\beta u) \end{cases}$$

We recall also that $\mathcal{F}$ is an Isometry on $L^2(\Omega)$ $\quad L^2(\Omega) \subset \mathcal{S}'(\mathbb{R}$

Now if $T \in \mathcal{S}'(\mathbb{R}^n)$, $\hat{T} = \mathcal{F}(T)$ is defined by

(4) $\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$

Then if $T = \delta$ we obtain :

$$\langle \hat{\delta}, \varphi \rangle = \langle \delta, \hat{\varphi} \rangle = \hat{\varphi}(0) = \int_{\mathbb{R}^n} \varphi(x)\,dx$$

and

(5) $\hat{\delta} = 1$.

Therefore if we take $u_0 \in \mathcal{S}'(\mathbb{R})$ in Problem (Π_1), we look for u and $\frac{\partial u}{\partial t}$ taken their values in $\mathcal{S}'(\mathbb{R})$. By Fourier's Transform with respect to x, we obtain :

$$(6)\begin{cases} \dfrac{\partial \hat{u}}{\partial t}(\cdot,t) + y^2\,\hat{u}(\cdot,t) = 0 \\ \hat{u}(\cdot,0) = \hat{u}_0 \end{cases}$$

by

$$(7) \quad \hat{u}(\cdot,t) = e^{-y^2 t} \cdot \hat{u}_0$$

(The right-hand side of (7) being well-defined because of $y \to e^{-y^2 t}$ is a $\mathcal{C}^\infty$-function, $t > 0$)

Now suppose $u_0 \in L^2(\mathbb{R})$. Then $\hat{u}_0 \in L^2(\mathbb{R})$ and the unique solution of (6) is given by (7) with $\hat{u}(y,t)$, $y^2 \hat{u}(y,t) \in L^2(\mathbb{R})$

Thus Problem (Π_1) has a unique solution with $u(x,t)$ and $Au(x,t) \in L^2(\mathbb{R})$ $t > 0$.

If we denote by

$$(8) \quad N_t(x) = \mathcal{F}(e^{-y^2 t}) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$$

This solution is given by

$$(9) \quad u(\cdot,t) = N_t(\cdot) \underset{(x)}{*} u_0$$

Where $\underset{(x)}{*}$ denote the convolution with respect of x, that is:

$$(9)' \quad u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-\xi)^2}{4t}} u_0(\xi)\, d\xi$$

It is possible to see that

$$(10) \quad \lim_{t \to 0} N_t(\cdot) \to \delta \quad \text{in } \mathcal{S}'(\mathbb{R})$$

Which implies

$$(11) \quad \lim_{t \to 0^+} u(\cdot,t) = \delta * u_0 = u_0 \quad \text{in } L^2(\mathbb{R}) \text{ topology}$$

If we put:

$$(12) \quad G(t) = N_t(\cdot) * \qquad t \geq 0 \qquad (\text{with } N_0(\cdot) = \delta)$$

(12) defines a family $\{G(t)\}_{t \geq 0}$ of bounded operators in $L^2(\mathbb{R})$ $G(t) \in \mathcal{L}(L^2(\mathbb{R}))$, which satisfies the following properties:

$$(13) \quad \begin{cases} (i) & G(t) \in \mathcal{L}(L^2(\mathbb{R})) \\ (ii) & G(0) = I \\ (iii) & G(t+s) = G(t) \circ G(s) \qquad \forall\, t, s \geq 0 \\ (iv) & \lim\limits_{t \to 0} |G(t)u - u|_{L^2(\mathbb{R})} = 0 \qquad \forall u \in L^2(\mathbb{R}) \end{cases}$$

A such family is called a <u>semi-group in $L^2(\mathbb{R})$</u> of <u>C^0 class</u>.

<u>Remark 2-1</u>

The family $G(t)$ defined by (12) doesn't form a <u>Group</u> because for $t < 0$ $y \to e^{-t|y|^2}$ is not a multiplier in $L^2(\mathbb{R})$. In other words, the problem to find $u \in L^2(\mathbb{R}^n)$

with

$$(14) \quad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u(x,0) = u_0(x) \end{cases}$$

is <u>not</u> a <u>Well-posed problem</u>.

It is not possible to reverse the sense of time in equation (1) !

Remark 2-2.

The function $(x,t) \to E(x,t) = N_t(x)$ is the solution in the two variables (x,t) Distribution sense of the equation:

$$(15) \quad \frac{\partial E}{\partial t} - \frac{\partial^2}{\partial x^2} E = \delta_x \otimes \delta_t$$

In other terms E satisfies the Cauchy's Problem in $\mathcal{S}'(\mathbb{R})$:

$$(16) \begin{cases} \dfrac{\partial E}{\partial t} - \dfrac{\partial^2 E}{\partial x^2} = 0 \\ E(\bullet,0) = \delta_x \end{cases}$$

The function E is called the elementary solution of Problem (Π_1)

2-2- n - dimensional exple.

We look for a function $u(x,t)$ $x \in \mathbb{R}^n$, $x=(x_1, \dots x_n)$, $t>0$

Which satisfies:

$$(\Pi_n) \begin{cases} (i) & \dfrac{\partial u}{\partial t} - \Delta u = 0 \\ (ii) & u(x,0) = u_0(x) \end{cases}$$

By Fourier's transform with respect of x Problem (Π_n) becomes

$$(\hat{\Pi}_n) = \begin{cases} (\hat{i}) & \dfrac{\partial \hat{u}}{\partial t} + |y|^2 \hat{u} = 0 \\ (\hat{ii}) & \hat{u}(y,0) = \hat{u}_0(y) \end{cases}$$

... in $L^2(\mathbb{R}^n)$ (resp. in $\mathcal{S}'(\mathbb{R}^n)$), we obtain one

(17) $\hat{u}(y,t) = e^{-|y|^2 t} \cdot \hat{u}_0$.

We introduce :

$$(18) \quad N_t(x) = \overline{\mathcal{F}}\left(e^{-|y|^2 t}\right) = \left(\frac{1}{2\sqrt{\pi t}}\right)^n \cdot \exp\left(-\frac{|y|^2}{4t}\right)$$

Thus if $u_0 \in L^2(\mathbb{R}^n)$, (Π_n) has one solution given by:

$$(19) \quad u(x,t) = N_t(\cdot) \underset{x}{*} u_0(x).$$

Where $g * f(x) = \int_{\mathbb{R}^n} g(x_1-\xi_1, x_2-\xi_2, \dots x_n-\xi_n) \cdot f(\xi_1, \dots \xi_n)\, d\xi_1 d\xi_2 \dots d\xi_n$

The family $G(t)$ defined by.

$$(20) \quad G(t) = N_t(\cdot) \underset{(x)}{*}$$

form a semi-group of C^0 class in $L^2(\mathbb{R}^n)$ and remarks 2-1 and 2-2 are still valid in this case-

Remark 2-3:

If we consider the non-Homogeneous problem:

$$(\Pi) \begin{cases} \frac{\partial u}{\partial t} - \Delta u = f \\ u(x,0) = u_0(x) \end{cases}$$

With for example $f \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}))$ and $u_0 \in L^2(\mathbb{R})$ are given.

Problem (Π) has one solution given by the formula:

$$(21) \quad u(t) = G(t) u_0 + \int_0^t G(t-\sigma) \cdot f(\sigma)\, d\sigma$$

the integral of right-hand side being taken in $L^2(\mathbb{R})$

If we introduce:

$$(22)\quad \begin{cases} \tilde{N}_t : \begin{cases} \tilde{N}_t(x) = N_t(x) & \text{if } t \geq 0 \\ \tilde{N}_t(x) = 0 & \text{if } t < 0 \end{cases} \\ \tilde{f} : \begin{cases} \tilde{f}(x,t) = f(x,t) & t > 0 \\ \tilde{f}(x,t) = 0 & t < 0 \end{cases} \end{cases}$$

We can write (21) in the following form:

$$(23)\quad u(x,t) = \tilde{N}_t(x) \underset{x}{*} u_0 + \tilde{N}_t(x) \underset{(x,t)}{*} \tilde{f}(x,t)$$

Now we give some basic ideas on Semi-group theory.

10-11-76

2-3- Some basic ideas on Semi-group theory.

Let X be a complex (or real) Banach Space. We denote by $x \to |x|_X$ the norm of X. We recall that $\mathcal{L}(X)$ is a Banach Space equipped with the norm: $T \in \mathcal{L}(X) \to \|T\| = \sup\limits_{\|x\|=1} \|T(x)\|$

Definition 2-1

We shall said that the family $\{G(t)\}_{t\geq 0}$ is a Semi-group of C^0 Class if this family satisfies:

(i) $G(t) \in \mathcal{L}(X)$, $t \geq 0$.

(ii) $G(0) = I$

(iii) $G(t+s) = G(t) \circ G(s)$ $\quad \forall s,t \geq 0$

(iv) $\lim\limits_{t\to 0} \|G(t)x - x\|_X = 0$

Remark 2-4- If (iii) is true for $s,t \in \mathbb{R}$ the Semi-group

Examples

– 1 – $X = \mathbb{R}^n$ (or $\mathbb{C}^n$) A a (n,n) matrix

$$G(t) = e^{At} = \sum_{k=0}^{+\infty} \frac{(At)^k}{k!}$$

is a group of C^0-class.

– 2 – X B-space. A a bounded operator

$$G(t) = e^{At} = \sum_{k=0}^{+\infty} \frac{(At)^k}{k!}$$

is a group of C^0-class.

– 3 – $X = L^2(\mathbb{R}^n)$, $G(t) = N_t(\cdot) \underset{x}{*}$ where

$$N_t(x) = \frac{1}{(2\sqrt{\pi t})^n} \exp\left(\frac{-|x|^2}{4t}\right)$$

is a semi-group of C^0-class which is not a group.

– 4 – $X = L^p(\mathbb{R})$, $+\infty > p \geq 1$; if $u \in L^p(\mathbb{R})$

$G(t)u$ is defined by.

$$G(t)u(x) = u(x+t) \quad \text{a.e. in } \mathbb{R}.$$

$G(t)$ is a group of C^0-class. (Translation group in $L^p(\mathbb{R})$)

Proof (in exercise).

Proposition 2.1

If $\{G(t)\}_{t\geq 0}$ is a semi-group of C^0-class then

i) the function $t \to G(t)x$ is a continuous function in X for all $x \in X$

(24) $\|G(t)\| \leq M e^{\omega t}$, $t \geq 0$.

(See the proof in Yoshida's Book. cf Bibliography)

If. $\omega = 0$ in (24), $\{G(t)\}_{t \geq 0}$ is said to be a contraction semi-group of C^0 class

Generally, we can see that $t \to G(t)x$ is not a differentiable function for all $x \in X$, $t \geq 0$.

Definition 2-2

We Denote by $D(A)$ the set of all vectors in X which are "differentiable" for all values of $t \geq 0$.

By semi-group property (iii) we have

(25) $D(A) = \{ x \in X ; \lim_{h \to 0^+} \frac{G(h)x - x}{h}$ exist, $t \geq 0 \}$

let A_h be defined by

(26) $A_h = \frac{G(h) - I}{h} \in \mathcal{L}(X)$ $\forall h > 0$.

We have:

Proposition 2-2 If $\{G(t)\}_{t \geq 0}$ is a semi-group of C^0-class

(i) $D(A)$ is a vector subspace of X which is dense in X.

(ii) if $x \in D(A)$, $G(t)x \in D(A)$, $0 \leq t < +\infty$

(iii) $A_h x \to Ax$ if $h \to 0_+$

generally A is an unbounded operator in X which maps $D(A)$ into X.

We omit the proof

Examples

1- $X = \mathbb{R}^n$, A (n,n) matrix, $G(t) = e^{At}$

Then $D(A) = X$

2 - It is the same with X be a Banach space, A be a bounded operator in X.

3 - $X = L^2(\mathbb{R}^n)$, $G(t) = N_t(\cdot) *$ where

$$N_t(x) = \frac{1}{(2\sqrt{\pi t})^n} \exp\left(-\frac{|x|^2}{4t}\right)$$

Show (by Fourier's transform) that

$$(27) \quad \lim_{h \to +0} \left(\frac{e^{-|y|^2 h} - 1}{h}\right)\hat{u} \longrightarrow -|y|^2 \hat{u} \quad \text{in the } L^2(\mathbb{R}^n) \text{ topology}$$

Thus $$(28) \quad \lim_{h \to +0} \{A_h u\} = \Delta u \quad \text{in the } L^2(\mathbb{R}^n) \text{ topology.}$$

therefore $A = \Delta$ with domain $\mathcal{D}(A) = \{u \in L^2(\mathbb{R}^n), \Delta u \in L^2(\mathbb{R}^n)\}$.

Exercise. Show that

$$(29) \quad D(A) = H^2(\mathbb{R}^n).$$

4 - $X = L^p(\mathbb{R})$, $1 \leq p < +\infty$. $G(t)$ is the group of translations.

Exercise! - Then $A_h u \to Au = \frac{du}{dx}$

- $D(A) = W^{1,p}(\mathbb{R}) = \{u \in L^p(\mathbb{R});\ \frac{du}{dx} \in L^p(\mathbb{R})\}$

We recall that an unbounded operator A in a Banach space X is said closed if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ $\{x_n \in \mathcal{D}(A)\ \forall n\}$ which converge to x in X with $\{y_n = Ax_n\}_{n\in\mathbb{N}}$ a convergent sequence to y in X, there exist $x \in \mathcal{D}(A)$ with $y = Ax$.

Proposition 2-3

If $\{G(t)\}_{t\geq 0}$ is a semi-group of C^0 class in X.

(i) $D(A)$ is a dense subset of X and A is closed.

(ii) $D(A)$ equipped with the norm of the graph of A:
(i.e. $u \to \|u\|_{D(A)} = \|u\|_X + \|Au\|_X$)
is a Banach-Space.

(iii) $\forall x \in D(A)$

$$(30) \quad \frac{dG(t)x}{dt} = A[G(t)x] = G(t)(Ax)$$

(iv) let a be >0.
$$\sup_{t\in(0,a)} \|G(t)x - e^{tA_h}x\|_X \to 0 \quad \text{when } h \to 0$$

- We omit the proof -

- Sometimes one traduces property (iv) improperly by the formula
$$G(t) = e^{At}, \text{ with } A \text{ unbounded.}$$

Definition 2-3

A is called the infinitesimal generator of the semi-group $\{G(t)\}_{t\geq 0}$

Prop. 2-4

The infinitesimal generator of a semi-group of C^0 class is a bounded operator iff the function $t \to G(t)$ is continuous in the topology of the space $\mathcal{L}(X)$.

(We omit the proof)

The main result of the semi-group theory is a characterisation of closed unbounded operators with dense domain in X which are

Before to give Hille-Yoshida's Theorem we state the Problem (P).

Problem (P) :

We look for u such that:

(31) $\begin{cases} u \in D(A) \\ -Au + \mathfrak{p} u = f \end{cases}$, $\mathfrak{p} = \lambda + i\mu \in \mathbb{C}$, f given in X.

If A is the infinitesimal generator of a Semi-group of C^0-class which satisfies equality (24), then (31) has one solution given by

(32) $u = R(\mathfrak{p}) f$ for $\lambda > \omega$.

Where $R(\mathfrak{p}) \in \mathcal{L}(X)$ is given by the following Laplace's Integral

(33) $R(\mathfrak{p}) f = \int_0^{+\infty} G(t) f . e^{-\mathfrak{p} t} dt$ $(\lambda > \omega)$

$R(\mathfrak{p}) = (\mathfrak{p} I - A)^{-1}$ is called the resolvent of A.

Theorem 2-1 (Hille-Yoshida's theorem)

Let A be a closed unbounded operator in X with dense domain $D(A)$.

Then A is the infinitesimal generator of one Semi-group of C^0 class which satisfies (24) iff

i) the problem (P) has an unique solution $u = R(\mathfrak{p}) f$

(ii) $\| R^k(\mathfrak{p}) \| \leq \dfrac{M}{(\lambda - \omega)^k}$ for $k = 1, 2, \ldots$

Examples of Applications of Hille-Yoshida's Theorem

1 - $X = L^2(\mathbb{R}^4)$, $G(t) = \mathcal{N}_t(x) \underset{x}{*}$

We have seen already that the infinitesimal generator of this Semi-Group is the Laplacian Δ.

Now in order to apply Hille-Yoshida's theorem, we look for u as a solution of

$$(34) \quad -\Delta u + p u = f \quad , \quad p = \lambda + i\mu \quad \lambda, \mu \in \mathbb{R}$$

and f given in $L^2(\mathbb{R}^4)$.

By Fourier's transform, $\hat{u} = \mathcal{F}(u)$, $\hat{f} = \mathcal{F}(f)$ we obtain

$$(35) \quad \left(|y|^2 + \lambda + i\mu\right)\hat{u} = \hat{f}$$

and

$$(36) \quad \hat{u} = \frac{\hat{f}}{|y|^2 + \lambda + i\mu} \qquad \forall \lambda > 0, \ \forall \mu \in \mathbb{R}$$

Therefore if $\underline{\lambda > 0}$, $\hat{f} \in L^2(\mathbb{R}^4) \Rightarrow \hat{u} \in L^2(\mathbb{R}^4) \Rightarrow u \in L^2(\mathbb{R}^4)$

We deduce from that Δu also belongs to $L^2(\mathbb{R}^4)$.

Thus equation (34) posses <u>one</u> solution $u \in D(\Delta) = H^2(\mathbb{R}^4)$

But from equality (36) we deduce

$$(37) \quad |\hat{u}(y)| \leq \frac{1}{\lambda} |\hat{f}(y)| \quad , \quad \lambda > 0 .$$

Therefore

(38) $$\|u\|_{L^2(\mathbb{R}^n)} = \|R(\lambda)f\|_{L^2(\mathbb{R}^n)} \leq \frac{1}{\lambda}\|f\|_{L^2(\mathbb{R}^n)}$$

By Hille-Yoshida's theorem, Δ is the infinitesimal generator of a semi-group of contractions of C^0-class.

Here $G(t)$ is given by:

(39) $$\mathcal{F}(G(t)u) = e^{-|y|^2 t}\mathcal{F}(u)$$

and it is possible to verify directly the result.

Now we give other examples in which it is not possible to obtain directly the result.

2 - Let Ω be an open set of $\mathbb{R}^n$, we consider

(40) $$\begin{cases} A = -\displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) \\ \text{where } a_{ij} \in L^\infty(\Omega) \quad i,j = 1,\dots n \end{cases}$$

We also assume that

(41) $$\operatorname{Re}\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\bar{\xi}_j \geq \alpha\sum_{i=1}^{n}|\xi_i|^2, \quad \alpha > 0 \quad \xi_i \in \mathbb{C}, i=1,\dots$$

We define A as an unbounded operator in $L^2(\Omega)$ with domain

$$\ldots H^1(\Omega),\ Au \in L^2(\Omega)\}$$

Then

Proposition 2-5

$(-A)$ is the infinitesimal generator of a contraction semi-group of C^0 class in $L^2(\Omega)$

Proof:

Let $\rho = \lambda + i\eta$ we must solve

(43) $\begin{cases} Au + \rho u = f \quad , \quad f \text{ given in } L^2(\Omega) \\ u \in D(A) \end{cases}$

This problem is equivalent (See Lanchon's lecture) to the following variational problem:

(44) $\begin{cases} \text{To find } u \in H_0^1(\Omega) \text{ which satisfies} \\ a(u,v) + \rho(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega) \end{cases}$

Where

$$(u,v) = \int_\Omega u(x)\,\overline{v}(x)\,dx \quad , \quad a(u,v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\overline{\partial v}}{\partial x_i}\,dx$$

From (41) we have

$$\operatorname{Re}\left(a(v,v) + \rho|v|^2 \right) \geq \inf(\alpha,\lambda)\, \|v\|^2_{H_0^1(\Omega)} \quad , \quad \lambda > 0$$

Thus by Lax-Milgram theorem (44) possess one solution

If we take $v = u$ in (44) we obtain

(45) $\quad Re\, a(u,u) + \lambda |u|^2 = Re\, (f, u)$

Now by Cauchy-Schwarz's inequality and since $Re\, a(u,u) \geq 0$ we obtain

(46) $\quad |u| \leq \frac{1}{\lambda} |f|$.

The result of proposition 2-5 follows.

3- Sesquilinear forms and Semi-groups

We consider two separable complex hilbert spaces V and H with $V \hookrightarrow H$. V is dense in H.

We denote by $\| \ \|$ the norm in V and by $(\, ,\,)$, $|\ |$ respectively the scalar product and the norm in H.

Let $a(u,v)$ be a continuous sesquilinear form on V such that

(47) $\begin{cases} \text{There exists } \lambda_0 \text{ with} \\ Re\, a(v,v) + \lambda_0 |v|^2 \geq \alpha \|v\|^2 \quad \alpha > 0,\ v \in V \end{cases}$

Let A be the unbounded operator in H with domain $D(A)$ (dense in H) defined by

(48) $D(A) = \{ u \in V;\ v \to a(u,v)$ is continuous on V with respect to the H-topology$\}$

... in H (Exercise)

le state the

Theorem 2.2 If equality (47) lies, $-A$ is the infinitesimal generator of a semi-group $\{G(t)\}_{t\geq 0}$ of C^0 class in H which satisfies

$$(49) \quad \|G(t)\|_{\mathcal{L}(H)} \leq e^{\lambda_0 t}, \quad t \geq 0.$$

Proof: Let $\rho = \lambda + i\mu \quad \lambda, \mu \in \mathbb{R}$.

We look for u as a solution of problem:

$$(50) \begin{cases} u \in \mathcal{D}(A) \\ (A+\rho I)u = f, \quad f \text{ given in } H \end{cases}$$

(50) is equivalent to

$$(51) \begin{cases} u \in V \\ a(u,v) + \rho(u,v) = (f,v) \quad \forall v \in V \end{cases}$$

Because of (51) implies $u \in D(A)$ by definition of $D(A)$)

From (47) we deduce:

$$\begin{cases} \operatorname{Re}\left[a(v,v) + \rho|v|^2\right] = \operatorname{Re}\left[a(v,v) + \lambda_0|v|^2 + (\lambda-\lambda_0)|v|^2\right] \geq \\ \qquad \geq \alpha\|v\|^2 + (\lambda-\lambda_0)|v|^2 \geq \alpha\|v\|^2 \text{ if } \lambda > \lambda_0 \end{cases}$$

By Lax-Milgram's theorem there exists one solution of (51).

From (52) it follows

$$\ldots \; a(\ldots) \leq |f||u|$$

and we have :

$$(53) \qquad |u| \leq \frac{1}{\lambda-\lambda_0} |f|$$

The statement of theorem 2-2 follows from Hille-Yoshida's theorem.

2-4. Abstract Cauchy's Problem and Semi-group.

Let X be a Banach space (real or complex) and let A be an unbounded closed operator in X with domain $\mathcal{D}(A)$ which is dense in X.

We look for a function $u: t \to u(t)$ with values in X as a solution of the following abstract Cauchy's problem :

$$(54) \begin{cases} \dfrac{du(t)}{dt} - A u(t) = f(t) \\ u(0) = u_0 \end{cases}$$

Where $u_0 \in X$ is given and f is a given function which take values in X.

Naturally it is necessary to precise the sense of the derivative in (54).

For a deeper discussion of problem (54) we send to Yoshida's book, Kato's book and Pazy's lecture.

Here we note only that if A is the infinitesimal generator of a semi-group of C^0 class we can obtain

u_0 is given in $D(A)$ and for instance f is a convenient given function.

$\boxed{\text{Theorem (2.3)}}$ Homogeneous abstract Cauchy's Problem

Let A be the infinitesimal generator of a semi-group of C^0-class $\{G(t)\}_{t\geq 0}$.

Let u_0 be given in $D(A)$ and $f(t) \equiv 0$.

Then there exists one solution of the problem

$$(55) \begin{cases} \dfrac{du(t)}{dt} = Au(t) \\ u(0) = u_0 \in D(A) \end{cases}$$

This solution is given by

$$(56) \quad u(t) = G(t)u_0 \qquad t \in [0, +\infty[$$

---

mark 2-5 The solution obtained is a strong solution in the following sense:

$$\begin{cases} u \in C^0([0,+\infty[; X), \quad u \in C^1(]0,+\infty[; X) \\ \text{and } u(t) \in D(A) \text{ if } t > 0. \end{cases}$$

mark 2-6

We can see that a strong solution is unique if f is given in $C^0([0,+\infty[; X)$.

in the non homogeneous problem (51)

We can state the following theorem:

Theorem 2-4 - Non homogeneous abstract Cauchy's problem

Let A be the infinitesimal generator of a semi-group of C^0-class $\{G(t)\}_{t\geq 0}$

Let u_0 be given in $\mathcal{D}(A)$ and f given in $\mathcal{C}^0([0,+\infty; X)$

Then the function u given by:

$$(58) \quad u(t) = G(t)u_0 + \int_0^t G(t-s) f(s)\, ds$$

is the unique solution of Problem (51) iff the function

$$(59) \quad t \to v(t) = \int_0^t G(t-s) f(s)\, ds \quad \text{is differentiable}$$

We omit the proof.

Remark 2-7 - Condition (59) is implied by $f \in \mathcal{C}^1[0,+\infty; X)$.

Exercise:

Apply theorems 2-3- and 2-4 to the three examples considered in Section: "Applications of Hille Yosida's Theorem" page 44 → 49.

Remark 2-8

We note that these examples cover cases of Boundary values problems. (Cf Exples 2 and 3)

2-5- <u>To sum up.</u>

– We have a method (Semi-group theory) which allows us to solve Cauchy's problems and also Boundary-value problem.

– Its domain of applications is larger than Fourier's method.

– We can consider Banach Spaces and operator A which are not only self-adjoint. (Cf Exple 2-3 p. 45 -> 49)

– The case $A = A(t)$ can be considered (See Yoshida's book)

– The method lead us to an explicit formula for the solution –

– But it is not a <u>constructive method</u> since generally the semi-group $\{G(t)\}_{t\geq 0}$ <u>is out of reach.</u>

Note However there exists numerical approach for semi-groups (See Trotter's formula in Yoshida's book)
