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ELEMENTS IN THE THEORY OF INCOMPRESSIBLE SIMPLE FLUIDS

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INTRODUCTION.

The subject I mean to deal with regard some elementary facts in the field of incompressible simple fluids. The incompressible fluids occupy a very special chapter in the study of continuum bodies but they are quite attractive both for we can find there all the general aspects of the modern view in Continuum Mechanics and for we can give, by using a very simple mathematics, explicit solutions to numberous problems that represent common experimental situations.

These talks do not claim to be complete on the subject nor to give a review of it. To give shortly some historical references, I recall the approach by STOKES (1845) who first tried to define the essence of fluidity and studied the case of a fluid where the stress linearly depends on the velocity gradient $\underset{\sim}{D}$; REINER (1945) who considered a polynomial dependence on $\underset{\sim}{D}$; Rivlin - Ericksen (1955) who considered the dependence on the gradients of the displacement, velocity, acceleration, second acceleration . . . , Green - Rivlin (1957) and NOLL (1958) who introduced the idea that the stress could depend on the history of the velocity gradient.

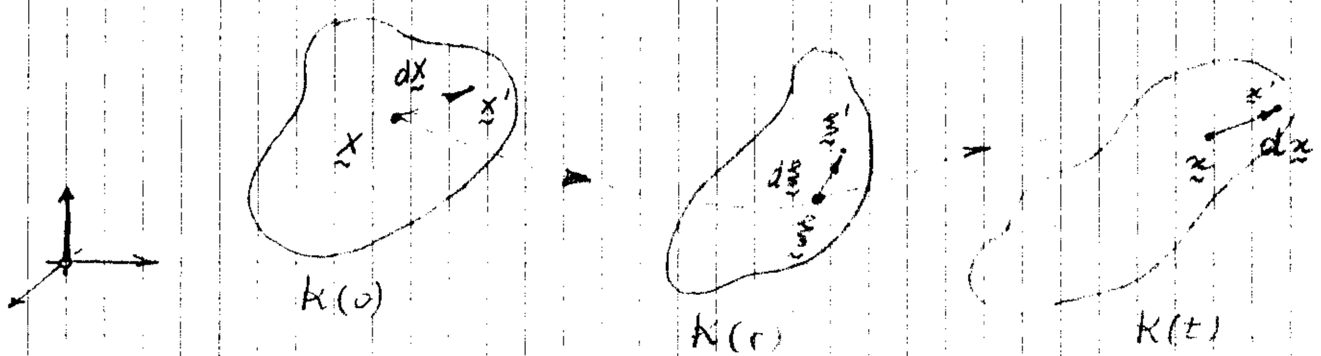
All these researches about more and more sophisticated models have a precise motivation. Since the first experiments certain fluids exhibit departures from the Navier-Stokes model. The experimental anomalies regard some special fluids such as polymer and protein soluweight. Here I mean to present, together with the basic concepts in the study of fluids, the method used to deal with this kind of phenomena. The lines of the exposition follow the ideas and the scheme by Coleman - Noll on the subject.

These authors abandon the methodological way of their predecessors. In fact, they do not investigate the consequences of some specification of the constitutive equations but use general requirements for the constitutive laws in order to deduce informations on the behaviour of fluids in viscometric flows. These results explain quite well the anomalous behaviours (such as the normal stress effects) and give also the starting-point for the construction of suitable models of the fluids at hand.

In the talks of other lecturers you have already met the basic elements of Fluid Mechanics such as kinematics, stress tensor concept and balance equations. Also I presume that you are familiar with the concepts of vectors and tensors and with the usual notations of tensor analysis. Therefore I do not dwell upon these points but make some brief recalls only in order to fix the language and to give a selfcontained exposition.

1. KINEMATICS AND NOTATIONS

In order to overcome differences in notations with respect to other lectures, now I list the symbols I'll use and sketch, once more, the elements of kinematics of a continuum body



τ, t . . . time parameters, t denotes the present time;

$\kappa(0), \kappa(\tau), \kappa(t)$ . . configurations at times $0 \le \tau \le t$;

$\underset{\sim}{X}, \underset{\sim}{\xi}, \underset{\sim}{x}$ . . . positions of the material point $\underline{X}$;

$\underset{\sim}{\xi}(\tau) = \chi(\underset{\sim}{X}, \tau) = \chi_t(\underset{\sim}{x}, \tau)$ . . descriptions of the motion w.r.t. $\kappa(0)$ and $\kappa(t)$;

$\underset{\sim}{F}_0(\underset{\sim}{X}, \tau) = \frac{\partial \chi}{\partial \underset{\sim}{X}}(\underset{\sim}{X}, \tau)$ . . . deformation gradient ;

$\underset{\sim}{F}_t(\underset{\sim}{x}, \tau) = \frac{\partial \chi_t}{\partial \underset{\sim}{x}}(\underset{\sim}{x}, \tau)$ . . relative def. gradient ;

$\underset{\sim}{F}(s) = \underset{\sim}{F}_t(t-s)$, $s \ge 0$ . . rel. def. grad. history ;

$\underset{\sim}{v}(\underset{\sim}{x}, t) = \frac{d}{d\tau}\chi_t(\underset{\sim}{x}, \tau)\Big|_{\tau = t}$ . . velocity ;

$\underset{\sim}{L} = \frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{x}}(\underset{\sim}{x}, t)$ . . . velocity gradient ;

$\underset{\sim}{D} = \frac{1}{2}(\underset{\sim}{L} + \underset{\sim}{L}^T)$ . . . stretching tensor ;

$\underset{\sim}{W} = \frac{1}{2}(\underset{\sim}{L} - \underset{\sim}{L}^T)$ . . . spin ;

$\underset{\sim}{T}$ . . . . Cauchy stress tensor ($\underset{\sim}{T} = \underset{\sim}{T}^T$ is assumed

a. It is known that to describe the motion of a continuum body we must specify the positions of any material point $\underline{X}$'s at any time τ. Often it is expedient to use the position occupied at the times 0 or t as a label of the material points. χ an χ_t, respectively, perform exactly this task.

b. Consider $\underset{\sim}{F}_0(\tau)$. The deformation gradient $\underset{\sim}{F}_0(\underset{\sim}{X}, \tau)$ is, by definition, the <u>linear approximation</u> of $\chi(\underset{\sim}{X}, \tau)$ at $\underset{\sim}{X}$ (under suitable smoothness hypotheses):

(1) $$\underset{\sim}{\xi}' = \chi(\underset{\sim}{X}', \tau) = \chi(\underset{\sim}{X}, \tau) + \left.\frac{\partial \chi}{\partial \underset{\sim}{X}}\right|_{\underset{\sim}{X}} d\underset{\sim}{X} + o(d\underset{\sim}{X}^2) .$$

(1) expresses the Taylor's expansion of χ at $\underset{\sim}{X}$ and allows us to describe the geometry of a neighbourhood of $\underline{X}$ in the configuration $K(\tau)$ in terms of the geometry of it in $K(0)$. In fact, (1) reads also

$$d\underset{\sim}{\xi} = \left.\frac{\partial \chi}{\partial \underset{\sim}{X}}\right|_{X} d\underset{\sim}{X} ,$$

that gives the image of the material vector $d\underset{\sim}{X}$ at the time τ as a linear function of $d\underset{\sim}{X}$. Thus, it depends on the change of configuration from $K(0)$ to $K(\tau)$ and it is independent when two coordinate systems are given for $K(0)$ and $K(\tau)$ and the coordinates of the points $\underset{\sim}{\xi}$ and $\underset{\sim}{X}$ are denoted by latin and greek indices, the linear operator $\underset{\sim}{F}(\tau)$ is represented by the matrix of its components w.r. to the local bases at $\underset{\sim}{X}$ and $\underset{\sim}{\xi}$

(1')
$$d\xi^i = \left.\frac{\partial \chi^i}{\partial X^\alpha}\right|_{\underset{\sim}{X}} dX^\alpha \qquad (\text{summed on } \alpha) \,,$$

that allows us to calculate explicitly the change of the geometry at X after the deformation. Thus, if a pair of cartesian coordinate systems is assumed for sake of shortness, the new magnitude of the material element $d\underset{\sim}{X}$ is given by

$$|d\underset{\sim}{\xi}|^2 = d\xi^i d\xi^i = \left.\frac{\partial \chi^i}{\partial X^\alpha}\frac{\partial \chi^i}{\partial X^\beta}\right|_X dX^\alpha dX^\beta$$

and the new angle φ between a couple of material vectors $d\underset{\sim}{Y}$ and $d\underset{\sim}{Z}$ is given by

$$\cos\varphi = \frac{1}{|d\underset{\sim}{\xi}|\,|d\underset{\sim}{\eta}|}\left.\frac{\partial \chi^i}{\partial X^\alpha}\frac{\partial \chi^i}{\partial X^\beta}\right|_X dY^\alpha dZ^\beta \,.$$

Analogously, the change of volume occupied in $K(0)$ and $K(\tau)$ by a given subset of material particles can be calculated by

$$V(\tau) = \int_{V(\tau)} d\xi^1 d\xi^2 d\xi^3 = \int_{V(0)} \left|\frac{\partial \underset{\sim}{\xi}}{\partial \underset{\sim}{X}}\right| dX^1 dX^2 dX^3$$

where $\left|\frac{D\xi}{DX}\right|_X$ stands for the absolute value of the determinant of $\underset{\sim}{F}_o(\tau)$.

Exactly the same as above can be repeated for the relative deformation gradient $\underset{\sim}{F}_t(\tau)$ if we substitute the reference to the geometry of $K(0)$ by that of $K(t)$.

Remark: As the injectivity of $x(X, \tau)$ and $x_t(x, \tau)$ is assumed, by a theorem about invertibility of functions it follows that

$$\det \underset{\sim}{F}_o(\tau)\,,\ \det \underset{\sim}{F}_t(\tau) \neq 0$$

and, on noting that $x_t(x, t) = x \Rightarrow \underset{\sim}{F}_t(t) = \underset{\sim}{1} \Rightarrow \det \underset{\sim}{F}_t(t) = 1$, by continuity arguments it is also possible to conclude that

$$\det F_o(\tau)\,,\ \det \underset{\sim}{F}_t(\tau) > 0$$

for any τ.

c. $\underset{\sim}{F}(s) = F_t(t-s)$, $s \geq 0$ is a special notation to make reference to the relative deformation gradient $\underset{\sim}{F}_t(\tau)$ when we want to stress its dependence on the time. The distinction between $\underset{\sim}{F}_t(\tau)$ and $\underset{\sim}{F}(s)$ is the same as between the values of a function and the function itself. Of course the history $\underset{\sim}{F}(s)$ depends not only on the choice of the material point X

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<u>Exercise</u>: Prove that:

i. $\det F_t(\tau) \equiv 1 \iff$ isochoric motion

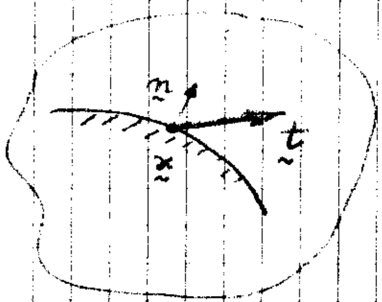
ii. $\underset{\sim}{F}_t^T(\tau)\,\underset{\sim}{F}_t(\tau) \equiv \underset{\sim}{1} \iff$ rigid motion

iii. $\underset{\sim}{L} = \frac{\partial}{\partial \tau}\underset{\sim}{F}_t(\tau)\Big|_{\tau = t} = \frac{\partial}{\partial s}\underset{\sim}{F}(s)\Big|_{s=0}$

iv. $\det \underset{\sim}{F}_t(\tau) \equiv 1 \iff \operatorname{tr} \underset{\sim}{D} = 0$

v. $\operatorname{tr} \underset{\sim}{L} = \operatorname{tr} \underset{\sim}{D}$

d. The Cauchy stress is the linear operator (tensor) that gives the tension on a material surface as a function of the ~~unit~~ unit normal.



k(t)

$$\underset{\sim}{t}(\underset{\sim}{x}, \underset{\sim}{n}) = \underset{\sim}{T}(\underset{\sim}{x}, t)\,\underset{\sim}{n}\,.$$

A frame is a set of objects embedded in the Euclidean space whose mutual distances remain unchanged in the time. The choice of a frame allows us to describe mechanical facts, such as the configurations and the motions of bodies. In Physics, however, it is usually accepted that the role of the frame is confined to description only and doesnot interfere with phenomena at all: hence certain invariance requirements (<u>Frame indifference</u>) with respect to the changes of frame follow for the equations of any physical theory. For any pair of frames a correspondence between points of the Euclidean space is stated which preserves distances and has the general form

$$(2) \qquad \underset{\sim}{x}^* = \underset{\sim}{c} + \underset{\sim}{Q}\,(\underset{\sim}{x} - \underset{\sim}{q})$$

where $\underset{\sim}{Q}$ is an orthogonal tensor and $\underset{\sim}{c}, \underset{\sim}{q}$ are a whatever pair of points in the Euclidean space. Thus a time dependent change of frame is such transformation $\underset{\sim}{x} \mapsto \underset{\sim}{x}^*$ when $\underset{\sim}{c}$, $\underset{\sim}{q}$ and $\underset{\sim}{Q}$ are some given functions of the time.

With regard to the consequences of <u>Frame indifference</u> in our theory, it is useful to calculate the changes of $\underset{\sim}{F}_t(\tau)$ and $\underset{\sim}{T}$ when a time dependent change of frame is assumed. Consider the motion

$$(3) \qquad \underset{\sim}{\xi}^*(\tau) = \underset{\sim}{c}(\tau) + \underset{\sim}{Q}(\tau)\left(\underset{\sim}{\xi}(\tau) - \underset{\sim}{q}(\tau)\right),$$

on calculating $\underset{\sim}{F}^*(\tau) = D\underset{\sim}{\xi}^*$ we have from the derivation

(4) $$\underset{\sim}{F}_t^{*}(\tau) = \underset{\sim}{Q}(\tau)\, \underset{\sim}{F}_t(\tau)\, \underset{\sim}{Q}^{T}(t) \, .$$

From (2) it follows that material vectors change according to

$$\underset{\sim}{Q}(\underset{\sim}{x} - \underset{\sim}{y}) = \underset{\sim}{x}^{*} - \underset{\sim}{y}^{*}$$

and so do the normal $\underset{\sim}{n}$ to a material surface and the tension vector if we wish the mechanical action acting on the surface to be the same in the motions $\underset{\sim}{\xi}^{*}(\tau)$ and $\underset{\sim}{\xi}(\tau)$. There follows then

(5) $$\underset{\sim}{T}^{*}(x^{*}, t) = \underset{\sim}{Q}(t)\, \underset{\sim}{T}(\underset{\sim}{x}, t)\, \underset{\sim}{Q}^{T}(t) .$$

<u>Exercise</u> : Prove that : i. $\underset{\sim}{D}^{*} = \underset{\sim}{Q}(t)\, \underset{\sim}{D}\, \underset{\sim}{Q}^{T}(t)$,

ii. $\operatorname{tr} \underset{\sim}{D}^{*} = \operatorname{tr} \underset{\sim}{D}$.

3. EQUATIONS OF THE THEORY.

a. <u>Field equations</u>

(6)
$$\dot{\varrho} + \varrho \operatorname{div} \underset{\sim}{v} = 0 \qquad \text{(mass balance)}$$

$$\operatorname{div} \underset{\sim}{T} + \varrho \underset{\sim}{b} = \varrho \underset{\sim}{a} \qquad \text{(momentum balance)}.$$

where $\underset{\sim}{b}$ is for the body force and ϱ for the mass density. The eq. (6)$_1$ expresses the conservation of mass for any subset of material particles and the second the balance of linear momentum for that part of the body. The eqs (6) are general at all, i.e. they hold for all the continuum bodies, but are not sufficient to

Apart from mathematical motivations, that undeterminacy has a definite mechanical origin: it reflects the lack in the theory of the mechanical characterisation of the continuum body at hand. In order to complete the problem, in fact, we must add some functional relations involving the forces acting in the body and the motions it can undergo (<u>Constitutive equations</u>) that is the dependence of the present Cauchy stress from the <u>past history</u> of the material till the time t:

$$(4) \qquad \underset{\sim}{T}(\underset{\sim}{x}, t) = \mathop{\mathcal{F}}_{\substack{s=0 \\ \underset{\sim}{y} \in K(t-s)}}^{s=\infty} \left(K(t);\ \chi_t(\underset{\sim}{y}, t-s) \right) .$$

Here the presence of $K(t)$ among the arguments of $\mathcal{F}$ denotes that there may be different responses to the same deformation history depending on the present configuration and takes account, in a general form, of the elementary idea sketched in fig. 2.

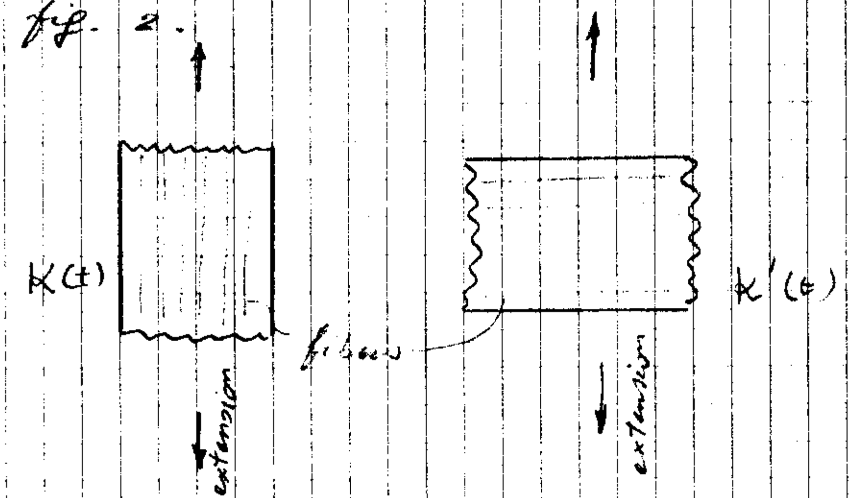


fig 2.

(4) defines an ideal material; that is it furnishes a mathematical model of it. Of course there are restrictions to

ary value problem) and Mechanics. Mathematical restrictions go beyond the plane of these talks and give, however, poor informations if one maintains the generality level of (2).

The restrictions of the second kind come from meaningful mechanical requirements (Constitutive Axioms) and allow for deduction of properties of the class of admissible constitutive functionals by means of simple mathematical tools. The modern view on the subject can be found in the HANDBUCH DER PHYSIK Vol III/3. On conforming here to a list of the Constitutive axioms:

i. AXIOM of LOCAL ACTION : the stress at each point $\underset{\sim}{x}$ depends only on the restriction of the past history to a whatever material neighbourhood of $\underset{\sim}{x}$, that is, under suitable smoothness hypotheses, it depends on the history of the gradients $\overset{n}{\underset{t}{F}}(\tau) = \frac{\partial^n}{\partial \underset{\sim}{x}^n} \underset{t}{F}(\tau)$ at $\underset{\sim}{x}$;

ii. REDUCED AXIOM of DETERMINISM : the stress is determined by constitutive equations to within a term that does not work in the class of motions compatible with internal constraints ;

iii. AXIOM of MATERIAL FRAME-INDIFFERENCE : The material properties described by a constitutive axiom must be indifferent to the choice of the frame. This implies, from (5), that the admissible functionals must satisfy

(6) $Q(\tau) \overset{s=\infty}{\mathcal{F}}(K(t)\ \chi(\ldots)) Q^T(\tau) = \overset{s=\infty}{\mathcal{F}}(K^*(t)\ \chi^*(\ldots))$

4. INCOMPRESSIBLE SIMPLE FLUIDS.

Definition 1. A *fluid* is a material whose constitutive law is

$$\underset{\sim}{T}(\underset{\sim}{x},t) = \underset{s=0}{\overset{s=\infty}{\mathcal{F}}}\left(\rho(x,t),\, \chi_t(y,t-s)\right).$$

Definition 2. An *incompressible simple fluid* is a material that obeys to the constitutive assumptions:

i. $\det \underset{\sim}{F}(s) = 1$ (isochoric motions only),

ii. $\underset{\sim}{T} = -p\,\underset{\sim}{1} + \underset{s=0}{\overset{s=\infty}{\mathcal{F}}}\left(\underset{\sim}{F}(s)\right)$.

Here $p\underset{\sim}{1}$ is an undetermined hydrostatic pressure coming from the assumption i. because of the reduced axiom of determinism. Thus the same material is defined by a class of constitutive functionals which differ for a hydrostatic pressure. The undetermination is removed if in that class we choose $\mathcal{F}$ such that

$$(9)\qquad \operatorname{tr} \underset{s=0}{\overset{s=\infty}{\mathcal{F}}}\left(\underset{\sim}{F}(s)\right) = 0$$

By imposing the H. F. I. axiom there follows also the restriction

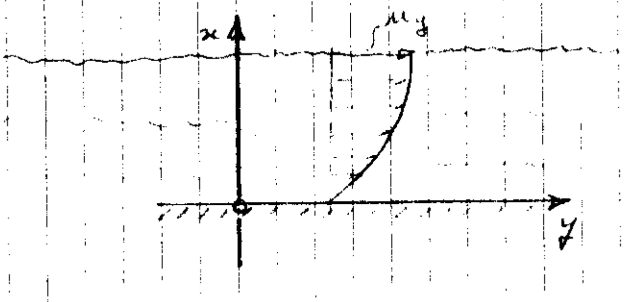
$$(10)\qquad \underset{\sim}{R}(0)\, \underset{s=0}{\overset{s=\infty}{\mathcal{F}}}\left(\underset{\sim}{F}(s)\right) \underset{\sim}{R}^T(0) = \underset{s=0}{\overset{s=\infty}{\mathcal{F}}}\left(\underset{\sim}{R}(s)\,\underset{\sim}{F}(s)\,\underset{\sim}{R}^T(0)\right)$$

for $\forall\, \underset{\sim}{F}(s)$ and $\forall\, \underset{\sim}{R}(s) = \underset{\sim}{Q}(t-s)$, $s \geq 0$.

Remark: Navier-Stokes fluids correspond to the constitutive equation

$$(11) \quad \mathcal{F}(\underset{\sim}{F}(s)) = -\eta_0 \frac{d}{ds}\left(\underset{\sim}{F}(s) + \underset{\sim}{F}^T(s)\right) = 2\eta_0 \underset{\sim}{D}$$

where η_0 is a material constant dependent on thermodynamical variables. (11) describes a fluid where shear forces act between contiguous layers flowing at different speeds.



$$T^{\langle xy \rangle} = \eta_0 \frac{\partial u_y}{\partial x}$$

(11) fits in with the behaviour of many real fluids such as water or fluids with low molecular weight.

Remark: Two elegant results show the physical goodness of the scheme developed above. Starting from the consequence (10) of the M.F.I axiom, by means of algebraic arguments only it is possible to prove that

i. $\underset{\sim}{F}(s) = \underset{\sim}{1} \Rightarrow \overset{s=\infty}{\underset{s=0}{\mathcal{F}}}(\underset{\sim}{F}(s)) = \underset{\sim}{0}$.

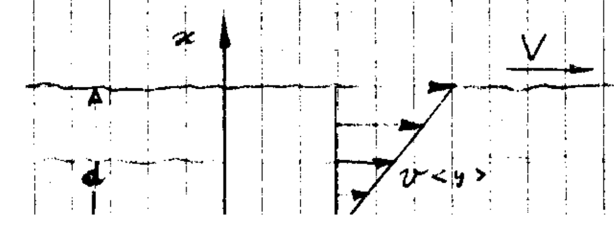
ii. $\underset{\sim}{F}(s)$ orthogonal $\Rightarrow \overset{s=\infty}{\underset{s=0}{\mathcal{F}}}(\underset{\sim}{F}(s)) = \underset{\sim}{0}$

These results read: "A simple fluid which has experienced a rest-history or rigid motions only, exhibits a hydrostatic pressure".

5_ VISCOMETRIC FLOWS _

Rather than specifying the constitutive functional in order to study problems regarding some simple fluid, we can obtain much more interesting results if we confine our attention to a special class of motions. When we properly chose this class, meaningful sequences can be obtained from the general constitutive equation and from the M.F.I. property (10). In this regard, the course developed along in simple fluid theory is illuminating for the methodological aspect and also because it gives the key to understand certain experimental behaviours (<u>normal stress effects</u>) that are anomalous in most simple fluid theories such as Navier Stokes'. In this method the simplest case is studied first. Then, the results are found to hold for a larger class of motions (<u>viscometric flows</u>) in which the behaviour of any simple fluid is described by three constitutive functions (<u>viscometric functions</u>). That class covers a lot of cases in fluid flow experiments. Then, the comparison between experimental measures and the theoretical results, performed by a semi-inverse method, can be used to state the suitable mathematical model of the fluid at hand.

a. <u>Simple shearing motion</u>.



Consider the motion

$$\underset{\sim}{\xi}(\tau) = \left\{ \begin{array}{l} x \\ y + kx(\tau - t) \end{array} \right.$$

The history of $\underset{\sim}{F}_t(\tau)$ is

(11) $$\underset{\sim}{F}(s) = \underset{\sim}{1} - s\underset{\sim}{M}$$

where $\underset{\sim}{M}$, in the coordinate system $\{x, y, z\}$, is given by the matrix

(12) $$[\underset{\sim}{M}] = \begin{bmatrix} 0 & 0 & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As the dependence on s of the history is given, whatever the fluid is its stress is a function of $\underset{\sim}{M}$

(13) $$\underset{\sim}{T} + p\underset{\sim}{1} = \underset{s=0}{\overset{\infty}{\mathcal{F}}}(1 - s\underset{\sim}{M}) = \underset{\sim}{h}(\underset{\sim}{M})$$

Of course, by (10)

(14) $$\underset{\sim}{Q}\,\underset{\sim}{h}(\underset{\sim}{M})\,\underset{\sim}{Q}^T = \underset{\sim}{h}(\underset{\sim}{Q}\underset{\sim}{M}\underset{\sim}{Q}^T)$$

for every orthogonal $\underset{\sim}{Q}$ must hold. If I choose

$$[\underset{\sim}{Q}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

it follows that

(15) $$\underset{\sim}{Q}\,\underset{\sim}{M}\,\underset{\sim}{Q}^T = \underset{\sim}{M}$$

and hence, by (14),

$$= \begin{bmatrix} T\langle 11\rangle & T\langle 12\rangle & 0 \\ \end{bmatrix}$$

Therefore the matrix of the extra stress $\underset{\sim}{T}_E = h(\underset{\sim}{T}) = \underset{\sim}{T} + p\underset{\sim}{1}$ is

$$[\underset{\sim}{T}_E] = \begin{Vmatrix} T\langle 11\rangle + p & T\langle 12\rangle & 0 \\ T\langle 21\rangle & T\langle 22\rangle + p & 0 \\ 0 & 0 & T\langle 33\rangle + p \end{Vmatrix}$$

and is determined by the viscometric functions

$$T\langle 12\rangle = \tau(\kappa) \qquad \text{(shear stress function)}$$

$$(17) \qquad \left.\begin{aligned} T\langle 11\rangle - T\langle 33\rangle &= \sigma_1(\kappa) \\ T\langle 22\rangle - T\langle 33\rangle &= \sigma_2(\kappa) \end{aligned}\right\} \text{(normal stress functions)}$$

Remark

i. If the velocity is reversed, by symmetry reasons it follows

$$\tau(\kappa) = -\tau(-\kappa), \quad \sigma_1(\kappa) = \sigma_1(-\kappa), \quad \sigma_2(\kappa) = \sigma_2(-\kappa).$$

ii. As $h(\underset{\sim}{1}) = \underset{\sim}{0}$, it follows also

$$\tau(0) = \sigma_1(0) = \sigma_2(0) = 0$$

b. Generalization of the result.

In the above deduction no use was made of the global geometry of the problem. Thus the same conclusions hold for the stress at the material element, occupying the position $\underset{\sim}{x}$ at the time t, whose relative gradient history satisfies (11) and (12) with respect to some orthonormal base $\{\underset{\sim}{b}\langle 1\rangle, \underset{\sim}{b}\langle 2\rangle, \underset{\sim}{b}\langle 3\rangle\}$. There the stress would have the form (16) with respect to the new base and would be determined by three viscosity the viscosity functions are the same as before. This reflects the property that, by definition, a fluid has the same material properties in every direction.

The result can be extended further, because if $\underset{\sim}{R}(s)$ is a history such that

$$\underset{\sim}{R}(0) = \underset{\sim}{1} \quad , \quad \underset{\sim}{R}(s) \text{ is orthogonal for } \forall s \geq 0$$

and

(18) $$\underset{\sim}{F}(s) = \underset{\sim}{R}(s)\left(\underset{\sim}{1} - s\underset{\sim}{M}\right) \quad ,$$

from (10) it follows that

$$\underset{\sim}{T} + p\underset{\sim}{1} = \underset{s=0}{\overset{\infty}{\mathcal{F}}}\left(\underset{\sim}{R}(s)\left(1 - s\underset{\sim}{M}\right)\right) = \underset{s=0}{\overset{s=\infty}{\mathcal{F}}}\left(1 - s\underset{\sim}{M}\right) = h\left(\underset{\sim}{M}\right)$$

and the previous results are obtained.

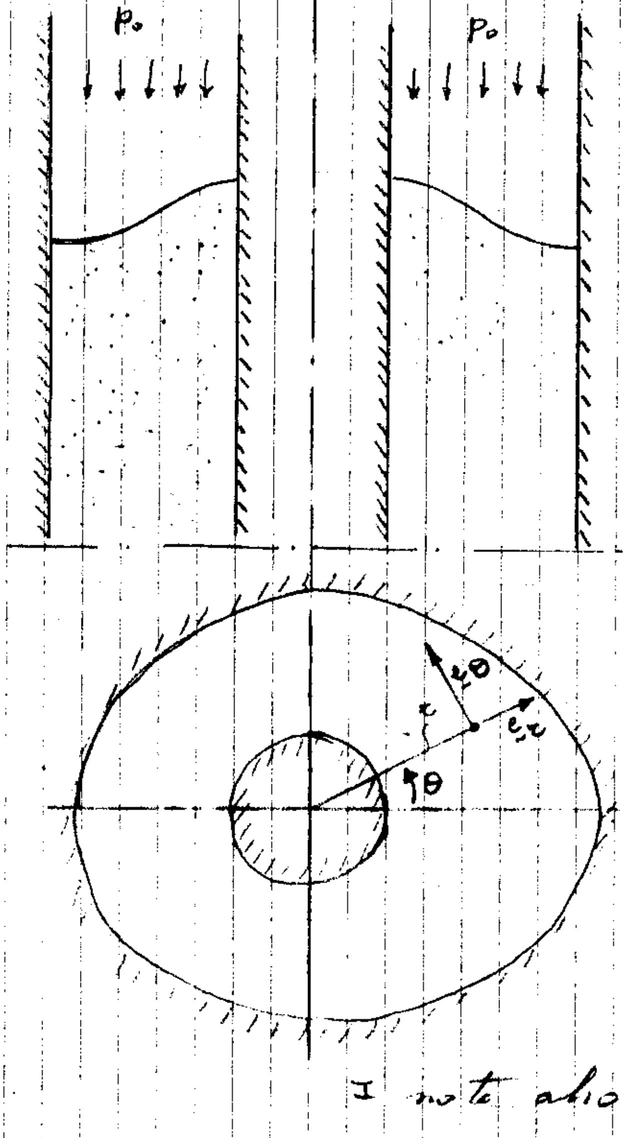
<u>Definition</u>: A viscometric flow is a flow where (18) and (12) hold for any $\underset{\sim}{x}$ and t. $\underset{\sim}{R}$ and $\underset{\sim}{M}$ may depend on the material point.

<u>Remark</u>: Navier-Stokes fluids correspond to the assumptions:

$$\tau(K) = \eta_0 K \quad ,$$
$$\sigma_1(K) = \sigma_2(K) = 0 \quad .$$

6. COUETTE FLOW.

Consider the steady flow of an incompressible simple fluid between two concentric cylinders when they rotate with angular velocities Ω_1 and Ω_2 about their common axis. The fluid fills the container up to a certain height and in the steady conditions its free surface is some equilibrium configuration. I suppose there are no body force but gravity and denote by M the couple per unit height that it is necessary to apply to the cylinders in order to maintain the motion.



Field equations:

$$\operatorname{div} \underset{\sim}{v} = 0 \,, \qquad (1.19)$$
$$\operatorname{div} \underset{\sim}{T} + \rho \underset{\sim}{b} = \rho \underset{\sim}{a} \,.$$

Boundary conditions:

The fluid adheres to the walls without slipping. A pressure p_0 acts on the free surfaces.

If cylindrical coordinates $\{r, \theta, z\}$ are chosen the kinematical conditions read

$$v\langle r \rangle = v\langle z \rangle = 0$$
$$v\langle \theta \rangle = R_1 \Omega_1 \,, \qquad v\langle \theta \rangle = R_2 \Omega_2 \,.$$

I note also that in cylindrical coordinates the physical components of $\operatorname{div} \underset{\sim}{v}$ and $\operatorname{div} \underset{\sim}{T}$ are

$$\operatorname{div} v = \frac{\partial v\langle r \rangle}{\partial r} + \frac{1}{r}\frac{\partial v\langle \theta \rangle}{\partial \theta} + \frac{\partial v\langle z \rangle}{\partial z} + \frac{1}{r} \ldots$$

$$\partial_r T\langle rr\rangle + \frac{1}{r}\partial_\theta T\langle r\theta\rangle + \partial_z T\langle rz\rangle + \frac{1}{r}\left(T\langle rr\rangle - T\langle\theta\theta\rangle\right) ,$$

(21) $$\partial_r T\langle r\theta\rangle + \frac{1}{r}\partial_\theta T\langle\theta\theta\rangle + \partial_z T\langle\theta z\rangle + \frac{2}{r} T\langle r\theta\rangle ,$$

$$\partial_r T\langle rz\rangle + \frac{1}{r}\partial_\theta T\langle\theta z\rangle + \partial_z T\langle zz\rangle + \frac{1}{r} T\langle rz\rangle .$$

To solve the problem I assume that the motion is described by

(22) $$\left[\underset{\sim}{\xi}(\tau)\right] = \begin{Bmatrix} \bar{r} \\ \bar{\theta} + \omega(\bar{r})(\tau - t) \\ \bar{z} \end{Bmatrix}$$

where $\bar{r}, \bar{\theta}, \bar{z}$ are the coordinates at the time t of the material point at hand and $\omega(r)$ is some radial distribution of velocity to be determined. Then the velocity and acceleration are

(23) $$\dot{\underset{\sim}{\xi}}(\tau) = \omega(\bar{r})\, \underset{\sim}{e}_\theta\left(\bar{r}, \bar{\theta} + \omega(\bar{r})(\tau - t), \bar{z}\right) ,$$
$$\ddot{\underset{\sim}{\xi}}(\tau) = -\bar{r}\omega^2(\bar{r})\, \underset{\sim}{e}_r\left(\bar{r}, \bar{\theta} + \omega(\bar{r})(\tau - t), \bar{z}\right) .$$

On using $(23)_1$ in (20) it is soon seen that the mass balance is satisfied whatever the distribution of velocity is. Consider now $\underset{\sim}{F}(s)$ and in particular its components w.r. to the local bases $\{\underset{\sim}{e}^i(t-s)\}$ and $\{\underset{\sim}{e}_\alpha(t)\}$ at the times $t-s$ and t, respectively:

(24) $$\underset{\sim}{e}^i \cdot \underset{\sim}{F}(s)\, \underset{\sim}{e}_\alpha = \frac{\partial \xi^i}{\partial \bar{x}^\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ -s\omega'(\bar{r}) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

from which, on passing to the physical components, it follows

(25) $$\underset{\sim}{e}\langle i\rangle \cdot \underset{\sim}{F}(s)\, \underset{\sim}{e}\langle\alpha\rangle = [\underset{\sim}{1}] - s\begin{bmatrix} 0 & 0 & 0 \\ \bar{r}\omega'(\bar{r}) & 0 & 0 \end{bmatrix}$$

As $\underset{\sim}{e}\langle i\rangle$ are orthonormal bases there exist an orthogonal tensor $\underset{\sim}{R}(s)$: $\underset{\sim}{e}\langle i\rangle(t-s) = \underset{\sim}{R}(s)\,\underset{\sim}{e}\langle\beta\rangle$ for $\beta = i$.
It follows then

(26) $$\underset{\sim}{e}\langle i\rangle \cdot \underset{\sim}{F}(s)\,\underset{\sim}{e}\langle\alpha\rangle = \underset{\sim}{e}\langle\beta\rangle \cdot \underset{\sim}{R}^T(s)\,\underset{\sim}{F}(s)\,\underset{\sim}{e}\langle\alpha\rangle =$$

$$= [1] - s\begin{bmatrix} 0 & 0 & 0 \\ \bar{r}\omega'(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

that is

$$\underset{\sim}{F}(s) = \underset{\sim}{R}(s)\left(\underset{\sim}{1} - s\,\underset{\sim}{M}\right)$$

where $\underset{\sim}{M}$ satisfy (12) when it is represented in the local base at the time t (<u>the Couette flow is viscometric</u>).

From the general results then the physical components of the stress are given by

(27) $$\begin{aligned} T\langle rr\rangle &= \sigma_1(\bar{r}\,\omega'(\bar{r})) + T\langle zz\rangle, \\ T\langle\theta\theta\rangle &= \sigma_2(\bar{r}\,\omega'(\bar{r})) + T\langle zz\rangle, \\ T\langle r\theta\rangle &= T\langle\theta r\rangle = \tau(\bar{r}\omega'(\bar{r})), \\ \frac{1}{3}\left(T\langle zz\rangle - p\right) &= -\left[\sigma_1 - \sigma_2\right](\bar{r}\,\omega'(\bar{r})), \\ T\langle rz\rangle &= T\langle\theta z\rangle = 0. \end{aligned}$$

<u>Remark</u>: By symmetry reasons the dependence on θ of the solution can be excluded.

<u>Equilibrium equations</u>

$$\partial_r T\langle rr\rangle + \frac{1}{r}\left(T\langle rr\rangle - T\langle\theta\theta\rangle\right) = -\rho\, r\,\omega^2(r)$$

(28) $$\partial_r T\langle r\theta\rangle + \frac{2}{r}\, T\langle r\theta\rangle = 0$$

On integrating (28) we have

$$T\langle zz\rangle = \rho g z + H(r)$$

(29) $$T\langle z\theta\rangle = \frac{c}{r^2}$$

$$T\langle rr\rangle = \int_{R_1}^{r} \left\{ -\frac{1}{2}\,[\sigma_1 - \sigma_2] - \rho r \omega^2(r) \right\} dr + K(z)$$

(from $(27)_{1,2}$).

The unknown functions $H(r)$ and $K(z)$ are determined by using $(27)_{1,2}$

(30) $$\int_{R}^{r} \left\{ -\frac{1}{2}\,[\sigma_1 - \sigma_2] - \rho r \omega^2(r) \right\} dr + K(z) =$$

$$= \sigma_1(r\omega(r)) + \rho g z + H(r) \quad ,$$

that implies

(31) $$K(z) = \rho g z + b$$

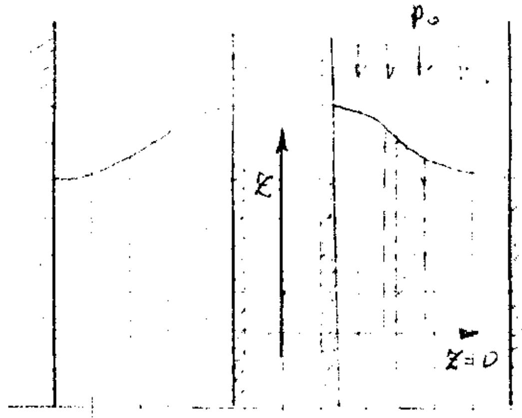
$$H(r) = \int_{R_1}^{r} \left\{ -\frac{1}{2}\,[\sigma_1 - \sigma_2] - \rho r \omega^2(r) \right\} dr + b - \sigma_1(r\omega(r))$$

where b is some constant b.

a. Free surfaces.

If $z = Z(r)$ is the equation of the free surface, it is soon obtained by imposing equilibrium for a cylindrical slice between $z=0$ and $z = Z(r)$.

A. There are no shear stresses on the lateral



(32) $$T\langle zz\rangle = H(r) = -p_0 - \rho g Z(r) ,$$

that is, if one properly redefines $H(r)$,

(33) $$Z(r) = -\frac{1}{\rho g}\left[p_0 + H(r)\right] .$$

<u>Remark</u>. There follows that

$$\frac{dZ}{dr} = \frac{1}{\rho g}\left\{\frac{1}{r}(\sigma_1-\sigma_2) + \rho r\omega^2(r) + \frac{d}{dr}\sigma_1\right\} .$$

The surface climbs at the outer cylinder if

$r\omega^2(r) \geqslant -\frac{1}{r}(\sigma_1-\sigma_2) - \frac{d}{dr}\sigma_1$

and viceversa if the inequality is reversed. So the behaviour is ruled by the <u>normal stress functions</u>. In the Navier-Stokes fluids $\sigma_1=\sigma_2=0$ and hence there is never the <u>climbing</u> (see the picture above).

<u>Remark</u>. The value of b places the plane $z=0$ w.r. to the free surface (cf. 33). As it is the position w.r. to the free surface the only physically meaningful parameter to label the material points, different values of b only change the labelling of the material points but do not affect the stress distribution.

b. <u>Velocity profile</u>. When a shear stress $\tau(k)$ is given the velocity profile can be obtained by integration from $(29)_2$

$\tau(r\omega'(r)) = c$

(34) gives: $\omega(r) = \Omega_1 + \int_{R_1} \frac{1}{r} \tau^{-1}\left(\frac{c}{r^2}\right) dr$

where c must be deduced by the condition $\omega(R_2) = \Omega_2$.

<u>Remark</u>: At least in the neighborhood of $k=0$, the invertibility of τ is a constitutive requirement coming from the Clausius-Duhem inequality.

c. <u>Moment</u> <u>per unit length</u> "M".

To maintain the steady motion a couple M must be applied to the cylinder that is given by the equilibrium equation

(35) $$M = \int_0^{2\pi} T\langle r\theta\rangle\, r^2\, d\varphi = c\, 2\pi \Rightarrow c = \frac{M}{2\pi}$$

<u>Remark</u>. For the Navier-Stokes fluid the shear stress function $\tau(k) = \eta_0 k$ and the explicit solution of this pb is given by

(36) $$\omega(r) = \Omega_1 + \int_{R_1}^{r} \frac{1}{r}\left(\frac{c}{r^2}\right)\frac{1}{\eta_0}\, dr = \Omega_1 - \frac{c}{2\eta_0}\left(\frac{1}{r^2} - \frac{1}{R_1^2}\right)$$

(36) suggests an experimental measure of the viscosity:

(37) $$\eta_0 = \frac{M}{4\pi}\ \frac{R_1^2 - R_2^2}{(\Omega_1 - \Omega_2)}\ \frac{1}{R_1^2 R_2^2}$$

7. APPENDIX -

<u>Vectors</u>: Let $u(x,y)$ be the translation of the Euclidean space onto itself bringing the point x into y. If a composition between translations: $u(x,y)\circ v(y,z) = w(x,z)$ and scalar multiples: $\alpha u(x,y) = w(x,z)$, where z is aligned with x and y and the distance $|xz| = \alpha\,|xy|$, are defined, it is possible to show that the family of translations on the Euclidean space has the algebraic structure of a vector space. It is useful to think of the vectors as translations. Here I adopt the notations

$$(A.1) \qquad \underset{\sim}{u} = \underset{\sim}{y} - \underset{\sim}{x} \quad , \quad \underset{\sim}{u} + \underset{\sim}{v} \quad \text{and } \alpha \underset{\sim}{u}$$

for $u(x,y)$, $u(x,y)\circ v(y,z)$ and $\alpha\, u(x,y)$.

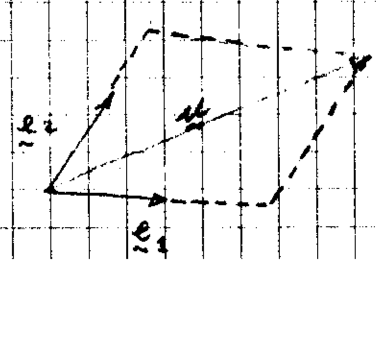
<u>Tensors</u>: A tensor $\underset{\sim}{T}$ is a <u>linear application</u> from the vectors into the vectors. I write for $\underset{\sim}{T}$, for the composition of tensors and their scalar multiples :

$$(A.2) \qquad \underset{\sim}{v} = \underset{\sim}{T}\,\underset{\sim}{u} \quad , \quad (\underset{\sim}{T}\,\underset{\sim}{S})\,\underset{\sim}{u} = \underset{\sim}{T}\,(\underset{\sim}{S}\,\underset{\sim}{u}) \quad , \quad (\alpha\underset{\sim}{T})\,\underset{\sim}{u} = \alpha\,(\underset{\sim}{T}\,\underset{\sim}{u})$$

where α is a scalar quantity.

<u>Components</u>: From Linear Algebra any vector $\underset{\sim}{u}$ can be represented as a linear combination of whatever three non-coplanar vectors $\{\underset{\sim}{e}_i\}$ (<u>base</u>)

$$(A.3) \qquad \underset{\sim}{u} = u^i\, \underset{\sim}{e}_i \qquad (\text{summation } i)$$



so that a bijective correspondence is given between the vectors $\underset{\sim}{u}$ and the triplets of scalars $(u^1\ u^2\ u^3)$

between tensors and matrices:

(A.4) $\quad \underset{\sim}{v} = \underset{\sim}{T}\,\underset{\sim}{u} \quad \Longrightarrow \quad v^i = T^i{}_j\, v^j \qquad$ (summation on j).

When a base is chosen, all the operations defined above can be performed by means of components, for instance:

$$(\underset{\sim}{u}+\underset{\sim}{v})^i = u^i + v^i \quad , \quad (\alpha\, \underset{\sim}{u})^i = \alpha\, u^i$$

(A.5)

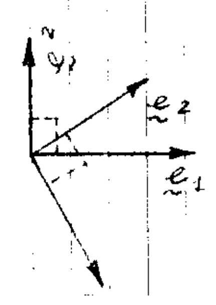
$$(\underset{\sim}{T}\,\underset{\sim}{S})^i{}_{\cdot j} = T^i{}_{\cdot k}\, S^k{}_{\cdot j} \qquad \text{(summation on } k\text{)}.$$

<u>Remark</u>: Sometimes it is convenient to use different bases to represent the argument $\underset{\sim}{u}$ and the vector $\underset{\sim}{T}\,\underset{\sim}{u}$. Thus for a fixed pair of bases a new correspondence is given between tensors and matrices.

<u>Remark</u>: As the correspondence with components is subordinate to the chosen basis, the <u>same</u> vector may have <u>different</u> components with respect to two different bases. If $\{\underset{\sim}{e}_\alpha\}$ and $\{\underset{\sim}{e}_i\}$ are distinct bases and $\underset{\sim}{e}_i = c^\alpha{}_i\, \underset{\sim}{e}_\alpha$, then the components of $\underset{\sim}{u}$ change according to

(A.6) $\quad \underset{\sim}{u} = u^i \underset{\sim}{e}_i = u^i c^\alpha{}_i\, \underset{\sim}{e}_\alpha \Rightarrow u^\alpha = c^\alpha{}_i\, u^i$.

When the scalar product between the vectors is considered, for any base $\{\underset{\sim}{e}_i\}$ a <u>dual base</u> $\{\underset{\sim}{e}^i\}$ is defined by the conditions



$$\underset{\sim}{e}_i \cdot \underset{\sim}{e}^j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

For vectors we have

and from (A.3), (A.3)' and (A.4)

$$(A.7) \qquad v^i = v \cdot e^i \;, \quad v_i = v \cdot e_i \;, \quad T^i{}_j = e^i \cdot T e_j$$

It is also helpful to introduce the physical components, i.e. the components w.r. to an orthonormal base. If $\{e_i\}$ is an orthogonal base they are given by

$$(A.8) \qquad v\langle i\rangle = v \cdot e_i \frac{1}{|e_i|} \;, \quad T\langle ik\rangle = e_i \cdot T e_k \frac{1}{|e_i||e_k|}$$

<u>Def.</u> The <u>transpose</u> of T is the tensor associated to the matrix

$$(A.9) \qquad e^i \cdot T^T e_j = (T^T)^i{}_j = T_j{}^i = e_j \cdot T e^i$$

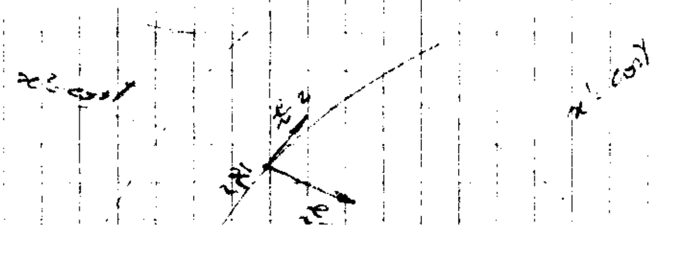
<u>Def.</u> T is <u>symmetric</u> (<u>skew symmetric</u>) if $T = T^T$ ($T = -T^T$).

<u>Def.</u> Q is an orthogonal tensor if $QQ^T u = u$ for any u.

<u>Exercise</u>: Prove that an orthogonal tensor preserves the scalar product between vectors:

$$(Qu)\cdot(Qv) = u \cdot v \qquad \text{for any } u, v.$$

<u>Coordinate systems</u>: A coordinate system is a labelling of the points in the Euclidean space by means of triplets of parameters $x = x(x^1, x^2, x^3)$. For a coordinate system at any point $\bar{x}$ a base is given by



$$e_i(\bar{x}) = \frac{\partial x}{\partial x^i}(\bar{x}_1, \bar{x}_2, \bar{x}_3).$$

Exercise: Find the base and the dual base in a spherical coordinates system. Find also the components of the metric tensor $g_{ij} = \underset{\sim}{e}_i \cdot \underset{\sim}{e}_j$.