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BASIC IDEAS IN APPLIED FUNCTIONAL ANALYSIS (continued)

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c) A is surjectif; that is to say $AV = V'$
We know already that A is an isomorphism from V to AV and then every closed subset of V is transformed by A in a closed subset of AV. Particularly, V being an Hilbert space is complete and therefore closed. Whence we know that AV is closed; that is to say

$$AV = \overline{AV}$$

If we prove now that AV is dense in V' (that is to say $\overline{AV} = V'$) that will imply $AV = V'$

Let us admit the following important result wich is a consequence of Hahn Banach Theorem

<u>Theoreme 2</u> : <u>Let E be a normed space</u>
<u>M a closed vector subspace of E</u>
<u>M_0 a vector subspace of M</u>
<u>Then M_0 is everywhere dense in M if and only if every continuous linear form on E vanishing identically in M_0 vanishes identically also in M.</u>

Here V is an Hilbert space, then all $u \in V$ can be considered as a continuous linear form on V' which is closed set.

Let $u \in V$ such that $(v', u) = 0 \quad \forall\ v' \in AV$

That is to say $(Av, u) = 0 \quad \forall\ v \in V$ and particularly.

$$0 = (Au, u) = a(u, u) \geqslant \alpha \|u\|^2 \implies u = 0$$

Whence the density of AV in V' and therefor the conclusion of the proposition.

<u>Remark 2</u>: i) The theorem of Lax Milgram shows the equivalence between the problems (6) and (8); moreover it guarantees under large hypothesis the existence and uniqueness of their common solution. We are now able to understand the relation between this theorem and the problem of Dirichlet wich with we started

ii) The theorem of Lax-Milgram could be given on a more general form with V, complex Hilbert space and a sesquilinear form a but the proof would have been more complicated and the

iii) Before going back to the example (2), it is useful to speak about two particular cases of Lax. Milgram theorem

I.2 Case of the inclusion of one Hilbert space in an other one

Let suppose that V Hilbert space is now included in an other one H and is a dense subset of H.

We shall denote here by

$((\ , \ ))$ and $\| \ \|$ respectively the scalar product and the norm on V

$(\ , \ )$ and $| \ |$ " " " " " " " on H.

Let i be the injection from V to H, that is to say the application

$$\underset{\| \ \|}{v \in V} \longmapsto \underset{| \ |}{i(v) = v \in H}$$

and suppose that i is continuous. Then we can identify H', dual of H with a subset of V'. H' being dense in V' with continuous injection (The details of the proof will not be given here)

At the end we can indentify. H with H' thank to the theorem. of Riesz that we just recall here:

Theorem 3: (Representation theorem of Riesz): if L is a linear continuous form on a Hilbert space H (that is to say an element of H') there exist one and only one element $\ell \in H$ such that

$$L(v) = (v, \ell) \qquad \forall\, v \in V.$$

so the application $L \in H' \longmapsto \ell \in H$ is a one to one correspondence between H and H' and it is easy to remark that

$$\|L\|_{H'} = |\ell|_H .$$

Whence the possibility of identification between H and H'.

Now, we are in the following situation:

$$(9) \quad V \subset H \equiv H' \subset V'$$

the differents injections being continuous and each space being dense in the following.

if $f \in H$ then f can be identified to an element of V'.

The theorem 1 of Lax milgram implies then the existence and uniqueness of the solution of.

We call domain of A the set.

$$D(A) = \{u \,/\, u \in V \text{ such that } Au \in H\}$$

We can prove that $D(A)$ is dense in V' also that $D(A)$ is an Hilbert space for the scalar product.

$$(u, v) + (Au, Av)$$

A that A is an isomorphism of $D(A)$ onto H.

I.3 The case where $a(u, v)$ is symetric.

Proposition 2: The hypotheses are the same as those of the theorem 1. We suppose moreover that $a(u, v)$ is symetric. Then $u \in V$ is solution of (2) if and only if u realize the minimum on V of the functional.

$$(10) \quad J(v) = a(v, v) - 2\,(f, v)$$

Therefore, there exists one and only one $u \in V$ such that

$$(11) \quad J(u) = \inf_{v \in V} J(v)$$

Proof. if u is solution of (2), the inequality

$$a(u - v, u - v) \geq \alpha \|v - u\|^2 \geq 0.$$

$$\Rightarrow J(u) - J(v) = a(u, u) - 2\,(f, u) - a(v, v) + 2\,(f, v)$$

$$= a(u, u) - 2\,a(u, u) - a(v, v) + 2\,a(u, v)$$

$$= -\,a(u - v, u - v) \leq 0 \quad \forall\, v \in V$$

Reciprocally: let suppose that u satisfies (11) then.

$$J(u) \leq J(u + \lambda v) \quad \forall\, \lambda \in R \text{ and } v \in V.$$

that is to say

$$\lambda^2 a(v, v) + 2\lambda\, a(u, v) - 2\lambda\,(f, v) \geq 0. \quad \forall\, \lambda \in R \text{ and } v \in V.$$

Considering successively the cases $\lambda > 0$ and $\lambda < 0$, dividing by λ and making λ to tend toward 0 we obtain.

$$a(u, v) = (f, v) \quad \forall\, v \in V. \qquad \text{q.e.d}$$

II Introduction of the Useful functional spaces.

II.1. Recapitulation

We are interested in the resolution of the probleme:

$$(2)\quad \begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega, \quad \lambda > 0. \\ \gamma_0 u = 0 \text{ on } \Gamma \end{cases}$$

u is supposed to belong to a functional space that we have not yet specified and f is given.

If u is a strong solution of (2) then we have proved that u is also solution of

$$(3)\quad a(u,v) = (f,v) \qquad \forall\, v \text{ sufficiently regular such that } \gamma_0 v = 0 \text{ on } \Gamma.$$

where $(12)\ a(u,v) = \int_\Omega (\text{grad}\, u\, \text{grad}\, v + \lambda\, u.v)\, dx.$

and $(13)\ (f.v) = \int_\Omega f \cdot v\, dx.$

Until now, the frame work of these two problems is still very vague - In particular, we are unable to go back from (3) to (2) in saying that u solution of (3) is also solution of (2) - Beside, if we demand for u a too strong regularity (for instance if we look for u in $\mathcal{C}^2(\Omega)$, space of functions twice differentiable) it is clear that the two formulations cannot be equivalent.

However, after having exposed the problem (3) in an abstract but very precise mathematic frame, we have obtained:

(i) an equivalence between the problem

$$(6)\quad u \in V, \quad a(u,v) = (f,v) \qquad \forall\, v \in V$$

and the problem

$$(8)\quad u \in V, \qquad Au = f$$

where V is a Hilbert space, f a given element of V' et A defined by $a(u,v) = (Au, v) \quad \forall\, v \in V$

(ii) existence a unicity of the common solution of (6) and (8)

We have now to find a convenient frame work for (3) in order that (2) and (3) should correspond respectively to (8) and (6)

<u>What are the ideas which can guide us to find this convenient frame work?</u>

We have first to give a meaning to (13) and (12):
For (13) it is sufficient to suppose that f and v belong to $L^2(\Omega)$, vector space defined on R by

$$(14) \quad L^2(\Omega) = \{ u \,/\, \underset{\Omega}{x} \longmapsto \underset{R}{u(x)},\ u \text{ measurable},\ \int_\Omega |u(x)|^2 dx < \infty \}$$

in fact if f and $v \in L^2(\Omega)$, then by the Minkowski inequality we have:

$$\left| \int_\Omega f(x) v(x)\, dx \right|^2 \leq \left(\int_\Omega |f(x)|^2 dx \right) \left(\int_\Omega |v(x)|^2 dx \right) < \infty$$

(f, v) defined by (13) is then nothing else that the scalar product on $L^2(\Omega)$ which is an Hilbert space for the associated norm. that we shall denote by.

$$(15) \qquad |u| = \left(\int_\Omega |u(x)|^2 dx \right)^{1/2}$$

If we want now give a meaning to (12) we have to suppose that of course u and $v \in L^2(\Omega)$ but also $u_{,i}$ and $v_{,i}$ $\forall\, i = 1, 2, \ldots, n$ $\left(u_{,i} = \frac{\partial u}{\partial x_i} \right)$ in order that:

$$|a(u, v)| \leq \left| \int_\Omega u_{,i} v_{,i}\, dx \right| + \lambda \left| \int_\Omega u v\, dx \right| < \infty$$

In effect if u and v are continuously differentiable on $\bar{\Omega}$, as the nature of the probleme (3) can let us hope, the hypothesis $u, v, u_{,i}, v_{,i} \in L^2(\Omega)$ are satisfied but the difficulty is that the space $C^1(\bar{\Omega})$ is not a complete one except for the norme

$$\|u\| = \sup_\Omega |u(x)| + \sum_{i=1}^{n} \sup_\Omega |u_{,i}(x)|$$

wich does not give to it a structure of Hilbert space.
The idea then consists in providing $C^1(\bar{\Omega})$ with the norm:

$$(16) \quad \|u\| = \left(\int_\Omega |u(x)|^2 dx \right)^{1/2} + \left(\int_\Omega |\operatorname{grad} u|^2 dx \right)^{1/2}.$$

associated to the scalar product

(17) $((u,v)) = \int u\,v\,dx + \int \operatorname{grad} u \cdot \operatorname{grad} v\,dx$.

and to complete $C^1(\bar{\Omega})$ with all the limits of the Cauchy sequences (defined with this norm) wich are not necessarily differentiable we obtain by this way, a Hilbert space $H^1(\Omega)$ wich is exactly one of the Sobolev spaces. (c.f textbooks about normed spaces for "completion")

The new difficulty then is to give a meaning to (17) for the ~~new~~ elements added to $C^1(\bar{\Omega})$ wich are not derivable, hence the necessity to introduce the space of distributions.

II.2. notions on the distribution

Definition 3 : Let Ω be an open subset of R^n. We call $\mathcal{D}(\Omega)$ the set of functions, indefinitely derivable, with compact support in Ω

Definition 4 : a sequence $\{\varphi_m\} \in \mathcal{D}(\Omega)$ is said to converge toward 0 in $\mathcal{D}(\Omega)$ if :

i) the elements φ_m have their support in a fixed compact $K \subset \Omega$

ii) if all the partial derivativatives $D^\alpha \varphi_m$ converge toward 0 uniformly in Ω

Definition 5 : a distribution L on Ω is a continuous linear form on $\mathcal{D}(\Omega)$ in the sens of

i) $L : \underset{\mathcal{D}(\Omega)}{\varphi} \longrightarrow \underset{R\ (\text{ou } \mathcal{C})}{L(\varphi)}$

ii) if $\varphi_m \to 0$ in Ω, $L(\varphi_m) \to 0$ in R (or $\mathcal{C}$)

The set of all the distributions on Ω is denoted by $\mathcal{D}'(\Omega)$, it is a vector space.

examples 2 : i) Let $f \in L^2(\Omega)$

then $L_f(\varphi) = \int_\Omega f(x)\,\varphi(x)\,dx$. for $\varphi \in \mathcal{D}(\Omega)$

defines a continuous linear form on $\mathcal{D}(\Omega)$, therefore $L_f \in \mathcal{D}'(\Omega)$

it is usual to identify L_f and f and to say that f is a distribution $L^2(\Omega)$ is therefore a space of distributions

ii) let $a \in \Omega$, the mapping:

$$\delta_a : \quad \varphi \in \mathcal{D}(\Omega) \longrightarrow \delta_a(\varphi) = \varphi(a) \in \mathbb{R} \ (\text{or } \mathbb{C})$$

is a distribution called Dirac (or delta) distribution in a.

we shall write $\varphi(a) = \int_\Omega \delta(x-a)\,\varphi(x)\,dx$.

Let α be a multiinteger : $\alpha = (\alpha_1, \alpha_2, \dots \alpha_n) \in N^n$ with $\sum_{i=1}^{n} \alpha_i = m$ we shall denote by $D^\alpha \varphi$ the m^{th} order partial derivative

$$D^\alpha \varphi = \frac{\partial^{\alpha_1+\alpha_2+\dots\alpha_n}}{\partial x_1^{\alpha_1} \dots\dots \partial x_n^{\alpha_n}}$$

Definition 6 : Let $f \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. the derivative of f in the sense of distributions, the distribution defined by

(17) $\quad \langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} (f, D^\alpha \varphi) \qquad \forall \varphi \in \mathcal{D}(\Omega)$

where $\quad |\alpha| = \sum_{i=1}^{n} \alpha_i$

this definition is a generalization of the green formula. obtained for regular fonctions.

$$\int_\Omega (D^\alpha f)\,\varphi\,dx = (-1)^{|\alpha|} \int_\Omega f(x)\, D^\alpha \varphi\, dx.$$

in such a way that when f is derivable, the two notions are coinciding

Example 3. : Let $\Omega = [0,1]$. and $f = \begin{cases} 0 & \text{on } [0,\frac{1}{2}[\\ 1 & \text{on }]\frac{1}{2},1] \end{cases}$; $f \in L^2[0,1]$.

$$\langle \frac{\partial f}{dx}, \varphi \rangle = -\int_0^1 f(x)\,\frac{d\varphi}{dx}\,dx = -\int_{\frac{1}{2}}^{1} \frac{d\varphi}{dx}\,dx = -\varphi(1) + \varphi(\tfrac{1}{2}) = \varphi(\tfrac{1}{2})$$

hence. $\quad \dfrac{df}{dx} = \delta_{\frac{1}{2}}$

Definition 7 : Let $\{f_n\}$ a sequence in $\mathcal{D}'(\Omega)$; One say that $f_n \xrightarrow[n\to\infty]{} f$ in $\mathcal{D}'(\Omega)$ if $\langle f_n, \varphi \rangle \xrightarrow[n\to\infty]{} \langle f, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega)$

Example 4 : if $f_n \to f$ in $L^2(\Omega)$ then $f_n \to f$ in $\mathcal{D}'(\Omega)$

in fact

$$\left| \int_\Omega (f_n - f)\,\varphi\,dx \right| \leq \left| \int_K (f_n - f)\,\varphi\,dx \right| \leq \left(\int_K |f_n - f|^2 dx \right)^{1/2} \cdot \left(\int_K |\varphi|^2 dx \right)^{1/2}$$

Theorem 4 : if $f_m \to f$ in $\mathcal{D}'(\Omega)$, then $D^\alpha f_m \to D^\alpha f$ in $\mathcal{D}'(\Omega)$ $\forall \alpha$; that is to say that the derivation is a continuous operation

in fact, $\langle D^\alpha f_m, \varphi \rangle = (-1)^{|\alpha|} (f_m, D^\alpha \varphi) \longrightarrow (-1)^{|\alpha|} (f, D^\alpha\varphi)$
$= (D^\alpha f, \varphi) \quad \forall \varphi \in \mathcal{D}(\Omega)$.

II.3. notions on the Sobolev spaces

Definition 8 : $L^p(\Omega)$, for p positif integer, is the space of class of functions f, measurable such that

$$(18) \qquad \|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p\,dx\right)^{1/p} < \infty \qquad p \neq \infty$$

$$(19) \qquad \|f\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} \operatorname{ess} |f(x)| < \infty$$

provided with the norm (18) or (19), $L^p(\Omega)$ is a Banach space. if $p = 2$, we already saw that $L^2(\Omega)$ is a Hilbert space. $L^p(\Omega)$ is still a space of distributions.

Definition 9 : We call Sobolev space of m^{th} order on $L^p(\Omega)$ and we denote by $W^{m,p}(\Omega)$ the space defined by.

$$(20) \quad W^{m,p}(\Omega) = \{v \,/\, v \in L^p(\Omega),\ D^\alpha v \in L^p(\Omega),\ |\alpha| \leq m\}$$

and provided with the norm

$$(21) \qquad \|v\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^\alpha v\|^p_{L^p(\Omega)}\right)^{1/p}$$

the case $p = 2$ is fundamental. To simplify writing we usually denote by $H^m(\Omega)$ the spaces $W^{m,2}(\Omega)$. Provided with the scalar products

$$\langle u, v\rangle_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \langle D^\alpha u, D^\alpha v\rangle_{L^2(\Omega)}$$

these spaces are Hilbert ones.

Remark 3 : The space $H^1(\Omega)$ defined above by.

$$(22) \quad H^1(\Omega) = \left\{ v \,/\, v \in L^2(\Omega),\ \frac{\partial v}{\partial x_1}, \ldots \frac{\partial v}{\partial x_n} \in L^2(\Omega)\right\}.$$

is the same as this obtained by completion of $C^1(\Omega)$ in.