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ASPECTS OF FRACTURE MECHANICS

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Aspects of Fracture Mechanics

We shall first discuss "linear elastic fracture mechanics" and later see how this fits into a proper elastic-plastic framework. To start things off, we study a simple two-dimensional problem of plane strain, for a crack in an infinite body.

1. Plane strain analysis

We start from the equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \quad , i = 1, 2\ ; j = 1, 2 \text{ (summed)} \qquad (1.1)$$

and the stress-strain relations

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} \quad ; \quad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad (1.2)$$

with $u_3 \equiv 0$. Equations (1.1) can be satisfied by taking

$$\sigma_{11} = \frac{\partial^2 \chi}{\partial x_2^2} \ , \ \sigma_{22} = \frac{\partial^2 \chi}{\partial x_1^2} \ , \ \sigma_{12} = -\frac{\partial^2 \chi}{\partial x_1 \partial x_2} \qquad (1.3)$$

for some $\chi(x_1, x_2)$, and it is well-known that, for compatibility with a displacement field, $\nabla^4 \chi = 0$. Hence, we can represent χ in the form

$$\chi = \operatorname{Re}\left\{ \bar{z}\phi(z) + \psi(z) \right\} ; \qquad z = x_1 + i x_2 \ , \qquad (1.4)$$

where ϕ and ψ are analytic. We have now, for example,

$$\sigma_{12} + i\sigma_{22} = -\frac{\partial^2 \chi}{\partial x_1 \partial x_2} + i\frac{\partial^2 \chi}{\partial x_1^2} = -\frac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial x_1}\right)\left\{ \bar{z}\phi(z) + z\bar{\phi}(\bar{z}) + \psi(z) + \bar{\psi}(\bar{z}) \right\},$$

where $\bar{\phi}(\bar{z}) = \overline{\phi(z)}$ so that $\bar{\phi}(z) = \overline{\phi(\bar{z})}$. Performing the differentiations gives

$$\sigma_{12} + i\sigma_{22} = i\left\{ \phi'(z) + \bar{\phi}'(\bar{z}) + z\bar{\phi}''(\bar{z}) + \bar{\psi}''(\bar{z}) \right\}. \qquad (1.5)$$

An expression for σ_{11} could be found similarly; in fact,

$$\sigma_{11} + \sigma_{22} = \nabla^2 \chi = 2\left\{ \phi'(z) + \bar{\phi}'(\bar{z}) \right\}.$$

Suppose now that a crack occupies $x_2 = 0,\ -a < x_1 < a$, and that tractions are applied "at infinity" that would produce the stresses σ_{ij}^{∞}, if the crack were not present. Let σ_{ij} (and associated $\phi(z)$, $\psi(z)$) represent the additional field induced by the crack. Then, if the faces of the crack are free of traction, when $x_2 = \pm 0,\ -a < x_1 < a$

$$\sigma_{12} + i\sigma_{22} = -\left(\sigma_{12}^{\infty} + i\sigma_{22}^{\infty}\right) = T, \text{ say.} \qquad (1.6)$$

Therefore, using (1.5), when $-a < x_1 < a$,

$$\left.\begin{aligned} \phi'(x_1+0i) + \bar\phi'(x_1-0i) + x_1\bar\phi''(x_1-0i) + \bar\psi''(x_1-0i) &= -iT, \\ \phi'(x_1-0i) + \bar\phi'(x_1+0i) + x_1\bar\phi''(x_1+0i) + \bar\psi''(x_1+0i) &= -iT, \end{aligned}\right\} \quad (1.7)$$

and ϕ and ψ are analytic elsewhere.

Adding eqns (1.7) gives

$$\left(\phi' + \bar\phi' + x_1\bar\phi'' + \bar\psi''\right)_+ + \left(\phi' + \bar\phi' + x_1\bar\phi'' + \bar\psi''\right)_- = -2iT, \qquad (1.8)$$

while subtracting them gives

$$\left(\phi' - \bar\phi' - x_1\bar\phi'' - \bar\psi''\right)_+ - \left(\phi' - \bar\phi' - x_1\bar\phi'' - \bar\psi''\right)_- = 0, \qquad (1.9)$$

using the notation $F_+ = F(x_1+0i),\ F_- = F(x_1-0i)$.

Equation (1.9) shows that

$\phi'(z) - \bar\phi'(z) - z\bar\phi''(z) - \bar\psi''(z)$ is continuous across the cut, and so entire, and is zero because $\sigma_{ij} \to 0$ as $|z| \to \infty$.

Equation (1.8) is a Hilbert problem for

$$F(z) = \phi'(z) + \bar\phi'(z) + z\bar\phi''(z) + \bar\psi''(z)$$

$$F_+(x_1) + F_-(x_1) = -2iT(x_1), \quad -a < x_1 < a$$

whence, if we define $G(z) = (z^2-a^2)^{1/2}F(z)$,

$$G_+(x_1) - G_-(x_1) = -2i\left[(x_1+0i)^2 - a^2\right]^{1/2}T(x_1) = 2\left[a^2 - x_1^2\right]^{1/2}T(x_1).$$

Hence, $G(z) = \frac{1}{\ldots}\int^{a} (a^2-x_1^2)^{1/2}\,T(x_1)\,dx_1 \ldots$

and $$F(z) = \frac{(z^2-a^2)^{-1/2}}{\pi i}\int_{-a}^{a}\frac{(a^2-x_1^2)^{1/2}T(x_1)\,dx_1}{x_1-z}. \qquad (1.10)$$

Now when z is close to a,

$$F(z) \sim -\frac{(z^2-a^2)^{-1/2}}{\pi i}\int_{-a}^{a}\left(\frac{a+x_1}{a-x_1}\right)^{1/2}T(x_1)\,dx_1\,,$$

by putting $z=a$ in the integrand. Hence, using (1.5) and the definition of $F(z)$, when $x_2=0$ and $x_1=a+\varepsilon$ for some small $\varepsilon>0$,

$$\sigma_{12}+i\sigma_{22} \sim -\frac{(x_1-a)^{-1/2}}{(2a)^{1/2}\pi}\int_{-a}^{a}\left(\frac{a+x}{a-x}\right)^{1/2}T(x)\,dx\,. \qquad (1.11)$$

We thus see that the stresses are singular at the crack tip, with an "$r^{-1/2}$" singularity. All the stress components can be found, with $x_2 \neq 0$, with a bit more labour.

The relative displacement of the crack faces is also of interest. Inverting the stress-strain relations (1.2) and using (1.3) and (1.4) gives the result

$$2\mu(u_1+iu_2) = \kappa\phi(z) - z\bar{\phi}'(\bar{z}) - \bar{\psi}(\bar{z}) \quad ; \quad \kappa = \frac{\lambda+3\mu}{\lambda+\mu} \qquad (1.12)$$

[see Muskhelishvili, Singular Integral Equations, 1953].

It is now easy to deduce that, for z close to a,

$$\phi(z) \sim \frac{K(z-a)^{1/2}}{(2\pi)^{1/2}i} \sim z\bar{\phi}'(z)+\bar{\psi}'(z),$$

where

$$K = -(\pi a)^{-1/2}\int_{-a}^{a}\left(\frac{a+x}{a-x}\right)^{1/2}T(x)\,dx\,, \qquad (1.13)$$

and hence that

$$\Big[(u_1+iu_2)\Big]_{x_2=-0}^{x_2=+0} \sim \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{\lambda+2\mu}{\lambda+\mu}\right)\frac{K}{\mu}(a-x_1)^{1/2} \qquad (x_1<a). \qquad (1.14)$$

Note for comparison that (1.11) states

$$\sigma_{12}+i\sigma_{22} \sim K\big/\big[2\pi(x_1-a)\big]^{1/2} \qquad (x_1>a) \qquad (1.15)$$

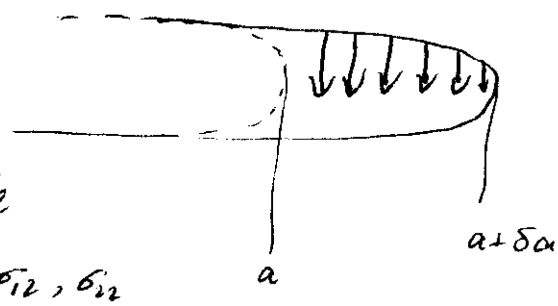
Equations (1.14) and (1.15) are universal, in that they describe the stress and relative displacement at any crack tip under conditions of plane strain, but K is usually not so easy to find, being given by (1.13) <u>only</u> for the present geometry.

2. Energy considerations in fracture.

The stress field given in the preceding section violates the condition that was assumed in its derivation that the strains should be small. Therefore, plainly, it should be <u>modified</u>, perhaps everywhere, but definitely in the vicinity of the crack tip, to allow for finite deformation plasticity or whatever behaviour the material is supposed to have. For the moment, however, we shall ignore this and ask how much energy would be released, on the basis of the preceding analysis, if the crack end moved from $x_1 = a$ to $x_1 = a + \delta a$, where δa is small. Now, after the extension, the relative displacements near $x_1 = a$ take the asymptotic form

$$[u_1 + i u_2] \sim \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{\lambda + 2\mu}{\lambda + \mu}\right) \frac{K}{\mu} \left(a + \delta a - x_1\right)^{1/2}, \qquad (2.1)$$

with K in effect unchanged, to lowest order in δa. To see how much energy has been released, we can "sew up" the crack again, so that its end is at "a":



and calculate the work that is required to do this. In the "sewing up" process, the tractions σ_{12}, σ_{22} increase from zero, when the displacement is given by (2.1), to the values given by (1.15), when the relative displacement between a and $a + \delta a$ is zero. Hence, since the body is linear the work done is

$$\delta W = +\frac{1}{2}\int_a^{a+\delta a}\left\{\sigma_{12}(a)\,u_1(a+\delta a) + \sigma_{22}(a)\,u_2(a+\delta a)\right\}dx_1 ,$$

where dependence on the location of the end of the crack has been admitted explicitly. Thus,

$$\delta W = +\frac{1}{2}\,\mathrm{Re}\int_a^{a+\delta a}\left(\sigma_{12}+i\sigma_{22}\right)_a\left(u_1-iu_2\right)_{a+\delta a}\,dx_1$$

$$\approx +\frac{1}{2}\,\mathrm{Re}\int_a^{a+\delta a}\frac{K}{\left[2\pi(x_1-a)\right]^{1/2}}\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{\lambda+2\mu}{\lambda+\mu}\right)\frac{\bar{K}}{\mu}\left(a+\delta a-x_1\right)^{1/2}dx_1 .$$

Performing the integration now gives

$$\delta W = +\frac{(\lambda+2\mu)}{4(\lambda+\mu)}\frac{|K|^2}{\mu}\,\delta a , \qquad (2.2)$$

which motivates defining the "strain energy release rate"

$$\mathcal{G} = \frac{\lambda+2\mu}{4(\lambda+\mu)}\frac{|K|^2}{\mu} . \qquad (2.3)$$

This idea was originated by G. R. Irwin; an early account is contained in Handbuch der Physik Vol VI, Irwin's article. It may be noted that tensile loading ($\sigma_{22}^{\infty} \neq 0$) is usually called "Mode I", while shear loading ($\sigma_{12}^{\infty} \neq 0$) is called "Mode II". We usually write

$$K = K_{II} + iK_{I} , \qquad (2.4)$$

correspondingly.

The classical fracture criterion due to Griffith states that a crack will extend whenever it can release energy in doing so. It postulated that work is done in forming new crack surfaces, however, so that, in extending from a to $a+\delta a$, an amount of the condition for crack extension is

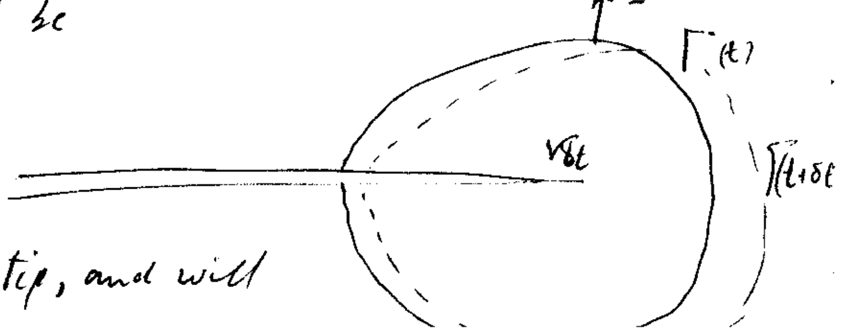
$$\mathcal{G} \geq 2\gamma . \qquad (2.5)$$

Griffith proposed, in 1921, a global energy statement, equivalent to (2.5), in the context of a particular problem; (2.5) is due to Irwin.

The drawback of the Griffith model is its assumption that only surface energy γ is important. It has been modified, in interpretation though not in mathematical form, by including in γ plastic dissipation, but this is not obviously a material constant, independent of the particular experiment. Still, the criterion for crack stability, that $\mathcal{G}$ should be less than a critical value, gives useful results. An explanation will be given later. First, though, we express the energy release rate $\mathcal{G}$ in a different way, which is capable of generalization.

3. The J-integral.

An integral, exploited extensively by J. D. Eshelby in dislocation theory, and independently by J. R. Rice in fracture, will now be discussed. We shall obtain it in the quickest possible way, by considering a crack in an infinite body. The crack moves with speed V and the loads follow it, so that the stress and displacement fields depend upon (x_1, x_2, t) in the combination $(x_1 - Vt, x_2)$. The crack faces will be supposed traction-free.



We consider a curve Γ that moves rigidly with the crack tip, and will

In a time interval of length δt, the change of energy in the (moving) region bounded by Γ is, since the fields depend on $(x_1 - Vt, x_2)$,

$$\oint_\Gamma W V \delta t \, dx_2 + \oint_\Gamma \sigma_{ij} \frac{\partial u_i}{\partial t} \delta t n_j ds ,$$

where W is the stored energy: the first term comes through the motion of Γ while the second is from the rate of working of the tractions ($\underline{n}$ is the normal to Γ). Hence, we have

$$\dot{U} = V\left\{\oint_\Gamma \left(W dx_2 - \sigma_{ij} \frac{\partial u_i}{\partial x_1} n_j ds\right)\right\} \qquad (3.1)$$

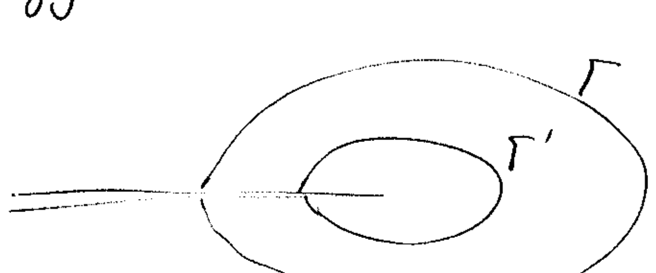
so that the flux of energy across Γ per unit of crack extension ("$V\delta t$") is

$$J = \oint_\Gamma \left(W dx_2 - \sigma_{ij} \frac{\partial u_i}{\partial x_1} n_j ds\right). \qquad (3.2)$$

Remark: In the above, we have disregarded kinetic energy, so we are dealing with small V.
Note that the above argument holds even if $W\left(\frac{\partial u_i}{\partial x_j}\right)$ corresponds to nonlinear elasticity, so long as we have also

$$\sigma_{ij} = \partial W / \partial(\partial u_i / \partial x_j) \qquad (3.3)$$

That is, (3.2) holds for nonlinear elasticity if σ_{ij} is interpreted as the first Piola-Kirchhoff (or "nominal") stress, introduced in earlier lectures by R. J. Knops. The integral J is plainly path-independent, for the energy contained between Γ and Γ' (see the picture) is independent of time, so that the flux across Γ equals the flux across Γ'.



the relation $\qquad \mathcal{G} = J \, . \qquad\qquad (3.4)$

Of course, this derivation is for a particular crack motion with the crack loaded in a special way. However, as $V \to 0$, the integral J defined by (3.2) remains path-independent and can be shown directly to retain its meaning as an energy flux.

[Comment. If we included inertia, simply replace W by $\left(W + \frac{1}{2}\rho V^2 \frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_1}\right)$. The derivation exploiting "$x_1 - Vt$" could also be extended to thermoelasticity by allowing W to be the internal energy function, depending also on entropy (or temperature); allowance for heat flux across Γ would then also have to be made.]

The above discussion now makes it possible to evaluate $\mathcal{G}$ (or J) by calculating the integral J around any convenient contour Γ. Plainly, the value of J depends upon K only, since Γ can be taken close to the crack tip, and it can be evaluated directly, from the field given in §1, as

$$J = \frac{\lambda+2\mu}{4(\lambda+\mu)} \frac{|K|^2}{\mu},$$

consistently with (3.4).

In fact, (in my opinion!) the "energy flux" interpretation of J is a red herring as regards its usefulness: it is really useful just because it is path-independent (even for a nonlinearly elastic body) and (in the linear elastic case) is in 1-1 correspondence with the value of $|K|$.

For a simple application of the J-integral, we consider a plastic body obeying the "deformation theory" constitutive law

$$\sigma'_{ij}/\sigma = \varepsilon_{ij}/\varepsilon \qquad (3.5)$$

where σ'_{ij} is the stress deviator: $\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}$, ε_{ij} is the strain tensor (previously written e_{ij}), $\varepsilon_{kk} = 0$ and $\varepsilon = (\varepsilon_{ij}\varepsilon_{ij})^{1/2}$, $\sigma = (\sigma'_{ij}\sigma'_{ij})^{1/2}$. The constitutive law (3.5) states that the stress and strain are coaxial; it is completed by relating σ and ε, for which we take the power-law form

$$\frac{\sigma}{\sigma_0} = \left(\frac{\varepsilon}{\varepsilon_0}\right)^{1/N} \qquad (3.6)$$

When $N = 1$, we have (incompressible) linear elasticity, while for $N \to \infty$, we get perfect plasticity ($\sigma = \sigma_0$). Of course, "deformation theory" is generally not so well regarded as incremental theory, but it provides a tolerable approximation so long as, in any particular solution (with monotonic loading, usually), the principal axes of strain remain the same, roughly, as the load increases.

Associated with the constitutive law (3.5) and (3.6), we have the energy function

$$W(\varepsilon_{ij}) = \int_0^{\varepsilon_{ij}} \sigma_{ij}\, d\varepsilon_{ij} = \int_0^{\varepsilon_{ij}} \sigma'_{ij}\, d\varepsilon_{ij}$$

$$= \int_0^{\varepsilon_{ij}} \sigma_0 \frac{\varepsilon_{ij}}{\varepsilon}\left(\frac{\varepsilon}{\varepsilon_0}\right)^{1/N} d\varepsilon_{ij} = \sigma_0 \int_0^{\varepsilon} \frac{\varepsilon^{1/N}}{\varepsilon_0^{1/N}}\, d\varepsilon ,$$

i.e. $$W(\varepsilon_{ij}) = \frac{N}{N+1}(\sigma_0\varepsilon_0)\left(\frac{\varepsilon}{\varepsilon_0}\right)^{\frac{N+1}{N}} . \qquad (3.7)$$

view of the path-independence of J, it is plausible that, when Γ

that $W(\mathcal{E}_{ij}) = (1/r) \times$ a fn of ϑ, so that (1

$$\left.\begin{aligned} \left(\frac{\mathcal{E}}{\mathcal{E}_0}\right) &= r^{-(N/N+1)} \times \text{ a fn of } \vartheta, \\ \text{and } \left(\frac{\sigma}{\sigma_0}\right) &= r^{-1/(N+1)} \times \text{ a fn of } \vartheta \end{aligned}\right\} \quad (3.8)$$

Thus, we might expect a strain singularity of order $r^{-(N/N+1)}$, and a stress singularity of order $r^{-1/(N+1)}$.

For linear elasticity, $N=1$ and both stress and strain are of order $r^{-1/2}$. For perfect plasticity, on the other hand, the strain singularity is of order r^{-1}, while the stress remains $O(1)$, as it should. [In perfect plasticity, the strains of order r^{-1} are consistent with a finite "crack opening displacement," right at the crack tip].

Detailed calculations of this model have been made by Rice and Rosengren, and Hutchinson, both in J. Mech. Phys. Solids, 1968. If the singularities (3.8) are admitted, the equilibrium equations reduce to nonlinear ODE's for the functions of ϑ. Hutchinson, in fact, assumed $(\mathcal{E}/\mathcal{E}_0) = r^{-\alpha} \times$ a fn. of ϑ and obtained thereby a nonlinear eigenvalue problem for α, which verified $\alpha = N/(N+1)$ numerically to high accuracy.

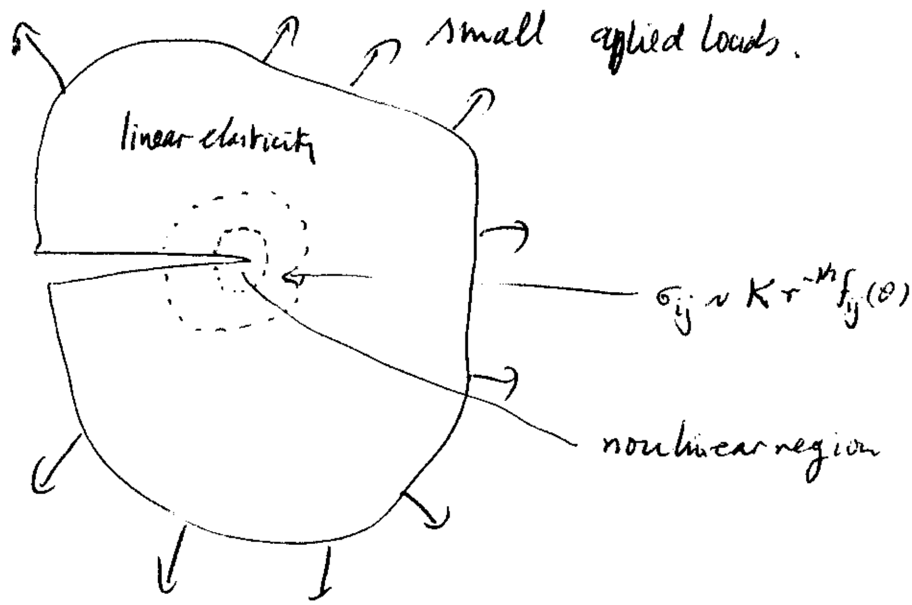
Of course, the above results are only asymptotic, as $r \to 0$. However, the strength of the singularity must be in 1-1 correspondence with the value of J, and this is really the key to the success of linear fracture mechanics. Essentially, we can regard a linear elastic description as providing an "outer" solution in a "boundary layer" formulation. If the applied stresses are not too high, then the

the crack tip, will have no effect on the elastic field. This field has the form $r^{-1/2}$ as the crack tip is approached, but the singularity is never reached because this form does not survive as $r \to 0$. Instead, there is an "inner" region, in which other phenomena take place (plastic flow being just one possibility). This inner region may be so small that it can be studied by considering a semi-infinite crack in an infinite body, subjected to the asymptotic boundary conditions

$$\sigma_{ij} \to K r^{-1/2} f_{ij}(\theta) \quad \text{as } r \to \infty, \qquad (3.9)$$

the functions $f_{ij}(\theta)$ being known (e.g. from §1).

Pictorially:

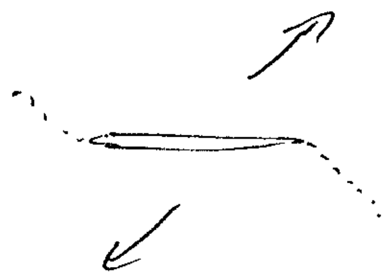


When this description is correct, there is _no need_ to solve the nonlinear problem (except for interest's sake) because, _whatever_ goes on in the vicinity of the crack tip _has_ to be governed by the value of K alone and a crack is _bound_ to extend at a critical value of K. If, for instance, the nonlinear law (3.5) and (3.1) applied near the crack tip, the _local_ solution would be that of Rice/Rosengren/Hutchinson, with J taking the value $\frac{\lambda+2\mu}{4(\lambda+\mu)}\frac{|K|^2}{\mu}$, since it could be evaluated in the "inner" part

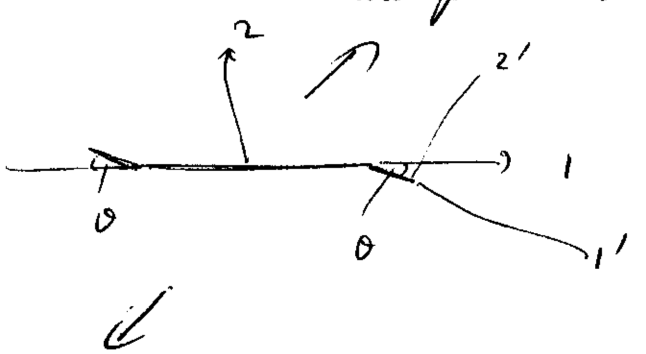
Note, however, that all of the above applies *only* for small applied loads *and*, furthermore, depends upon *some* prescribed fracture process being the only one possible. Also, the criterion of K_{crit} will predict *only* when the crack is *about to extend*. It could extend stably for some distance which the criterion by itself *cannot predict*. The "energy" theory, on the other hand, actually applies to extension, during which the energy is balanced. It is more complete, therefore, *except* that "γ" is not a material constant when plastic dissipation occurs, so "energy theories" are correspondingly empirical, in practice.

4. Out of plane extension of a crack.

Suppose a pre-existing crack is located obliquely:



It is plausible that it will not extend in its line, but along some curves, shown dashed. The energy balance, *and* the J-integral / critical K theories are all inadequate to predict when the crack will extend, because they are *scalar* statements and we have the additional unknown, namely the *angle* of propagation. Some other criterion is needed, and a variety have been proposed. The *easiest* is to evaluate $\lim_{r\to 0}\left(r^{1/2}\sigma_{\theta\theta}\right)$ and to find the angle θ for which this is maximal. Another is to solve the problem:

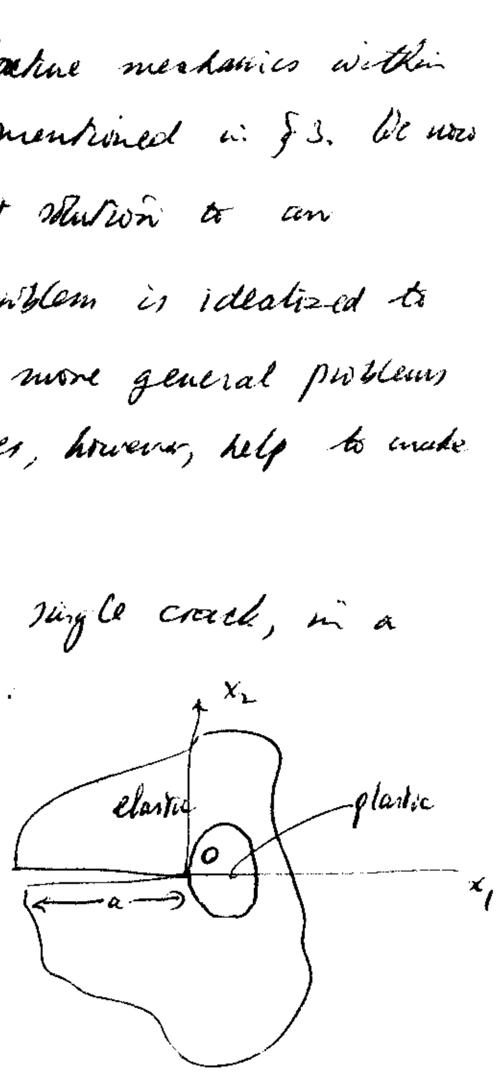


K at the ends. A postulate (of which I do not particularly approve) is that the correct angle θ is that for which K gives no singular shear component ahead of the crack ($\sigma_{1'2'} = 0$ in the diagram). Another possibility would be to study the angle θ that maximizes the strain energy release rate. This would involve a calculation like the one given in §2, but for "sewing up" an <u>angled</u> extension. This has been done (Willis, Int. J. Fracture, 1975) for the easy case of Mode III (longitudinal shear) loading (defined in detail below) but the mixed Mode I/II problem remains unresolved. Elastic-plastic (and all other nonlinear) problems are correspondingly further from resolution, except in one case, to be mentioned later.

5. Matched expansions in nonlinear fracture mechanics.

The possibility of interpreting linear elastic fracture mechanics within the framework of a 'boundary layer' formalism was mentioned in §3. We now enlarge upon this notion by giving an explicit solution to an elastic-plastic fracture problem. The problem is idealized to allow a closed-form solution to be found, and more general problems would require numerical work. The closed solution does, however, help to make the general structure of the arguments comprehensible.

The problem to be considered is that of a single crack, in a body of any geometry, as shown in the Figure.



The idealization, though, is that only a special loading will be considered so that, on the boundary, $\sigma_{1j} n_j = \sigma_{2j} n_j = 0$,
$\sigma_{3j} n_j = \tau(x_1, x_2)$; $\underline{n} = (n_1, n_2, 0)$.

This defines a state of "anti-plane strain" or Mode III loading will be the same elastic-perfectly plastic one selected by H. Landon in her analysis of the torsion problem. For the loading considered, there are only two non-zero components of stress, σ_{13} and σ_{23}, so that the yield criterion reduces to the form

$$\sigma_{13}^2 + \sigma_{23}^2 = k^2 \tag{5.1}$$

and the associated flow rule gives

$$\dot{\gamma}_{i3} = \frac{\partial \dot{u}_3}{\partial x_i} = \dot{\lambda}\sigma_{i3} \quad , \quad i = 1, 2, \tag{5.2}$$

the only non-zero displacement being $u_3(x_1, x_2)$.

In the region of elastic deformation, the stress-strain law

$$\sigma_{i3} = G\frac{\partial u_3}{\partial x_i} \quad , \quad i = 1, 2 \tag{5.3}$$

and the equilibrium equation

$$\frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{23}}{\partial x_2} = 0 \tag{5.4}$$

together imply that u_3 is a harmonic function of (x_1, x_2), which is conveniently expressed in the form

$$u_3 = \frac{ak}{G}\,\mathrm{Im}\{g(z)\}, \tag{5.5}$$

where $z = \dfrac{x_1 + i x_2}{a}$ and $g(z)$ is analytic.

[It should, perhaps, be noted that the notation employed here conforms to that used in some work of Rice, with which results are later compared]

(a) Solution of the elastic problem in Mode III.

Suppose the greatest value of τ on the boundary is much smaller than k, the yield stress. Then the approach of linear elastic fracture problem can be converted into the problem of finding the analytic function $g(z)$. The boundary conditions are that the normal component of stress, $\sigma_{13} n_1 + \sigma_{23} n_2$, or equivalently $G\, \partial u_3/\partial n$, is given. Hence, using the Cauchy-Riemann equations,

$$\frac{\partial}{\partial s}\{Re(g(z))\} \text{ is given on the boundary, and so,}$$

by integration, $Re\{g(z)\}$ is known on the boundary of the domain. By choosing the constant appropriately, since the crack faces are traction-free, we can demand, in particular, that $Re\{g(z)\} = 0$ on the crack faces, $z = x_1 \pm 0i$; $x_1 < 0$. One particular $g(z)$ that satisfies this condition (but not conditions on other boundaries) is $g(z) = 2Kz^{1/2}$, with K real; this gives $(\sigma_{23} + i\sigma_{13})/k = Kz^{-1/2}$, which is the Mode III analogue of the asymptotic form of the stresses near the crack tip. Other candidates are $g(z) = iA$, iBz, $Cz^{3/2}$, and also $K'z^{-1/2}$, $K''z^{-3/2}$, A'/z, iA''/z^2, and so on. The non-singular ones could well appear in the asymptotic expansion of $g(z)$ near the crack tip, but not the singular ones, because they would imply infinite displacements.

However, in the elastic-plastic problem, the elastic solution does not apply as $z \to 0$, and we shall see later how these singular forms have their use.

In the elastic-plastic problem, the elastic field is only fully specified once some boundary conditions are applied at the elastic-plastic boundary, which itself has to be found as part of the solution. The fortunate aspect of the Mode III problem, however, is that the plastic region can be described very simply; this we now do.

(b) Solution for the plastic region.

In the plastic region, we have

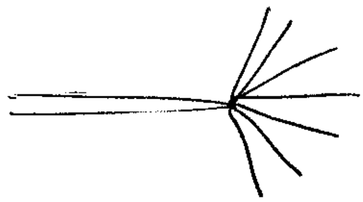
$$\sigma_{13}^2 + \sigma_{23}^2 = k^2$$

and $\dfrac{\partial\sigma_{13}}{\partial x_1} + \dfrac{\partial\sigma_{23}}{\partial x_2} = 0$. Hence, if we set

$\sigma_{13} = -k\sin\phi$, $\sigma_{23} = k\cos\phi$, there results the first order d.e.

$$\cos\phi \frac{\partial\phi}{\partial x_1} + \sin\phi \frac{\partial\phi}{\partial x_2} = 0$$

for ϕ, which implies that ϕ = constant along lines for which $\dfrac{dx_2}{dx_1} = \tan\phi$. It proves sufficient to assume that the characteristics form a centred fan:



so that, relative to polar coordinates based on the crack tip, $\sigma_{r3} = 0$, $\sigma_{\theta 3} = k$, by identifying ϕ with the polar angle θ. The flow rule (5.2) now implies $\gamma_{r3} = \dfrac{\partial u_3}{\partial r} = 0$, so that u_3 is a function of θ only. Hence,

$$\gamma_{\theta 3} = \frac{1}{r}\frac{\partial u_3}{\partial \theta} = \frac{k}{G}\frac{R(\theta)}{r}$$

for some function $R(\theta)$. This choice of representation is made, however, because at the elastic-plastic boundary, stresses are continuous and so are strains, so that $\gamma_{\theta 3} = k/G$. Thus, the elastic-plastic boundary is described in polar coordinates by the equation

$$r = R(\theta),$$

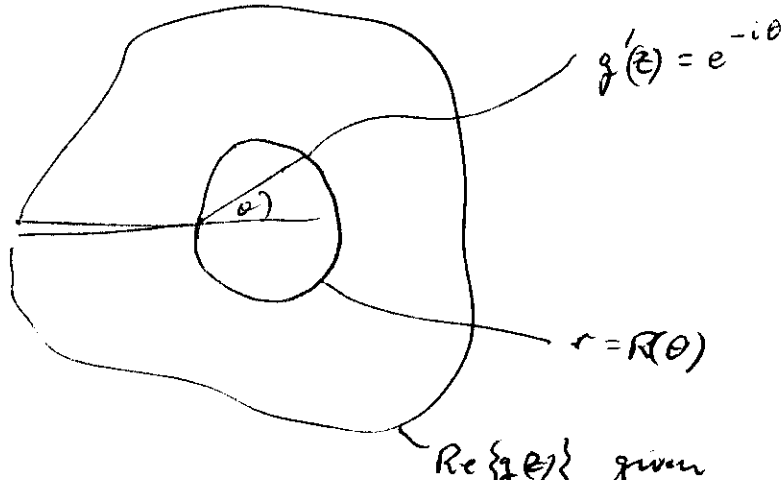
where $R(\theta)$ is so far unknown.

boundary, for we have

$$\sigma_{23} - i\sigma_{13} = k e^{i\theta}$$

anywhere in the plastic region, and so particularly on $r = R(\theta)$.

(c) <u>Description of the problem for the elastic domain.</u>



It is convenient to define the complex stress variable

$$\zeta = (\sigma_{23} - i\sigma_{13})/k \quad , \qquad (5.6)$$

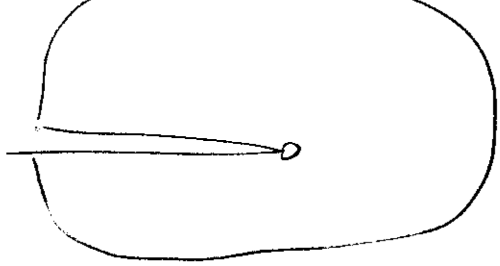
so that the stress-strain relations imply

$$\bar{\zeta} = g'(z) \qquad (5.7)$$

Hence, on the elastic-plastic boundary, we require $\bar{\zeta} = e^{-i\theta} = g'(z)$. The problem for the elastic domain is thus as shown in the picture above. One clever trick, employed by McClintock, and (later) Rice and (later still) Edmunds and Willis, is to solve the problem by inverting (5.7) to give a conformal mapping from the ζ-plane to the z-plane. It becomes extremely complicated except for simple geometries of the body, however.

We shall now define

(d) The outer problem.



When $\varepsilon \ll 1$, the plastic zone is small and we can reasonably expect, away from the crack tip, that

$$g(z) \sim \varepsilon g_1(z) + (\alpha_1 \varepsilon^3 + \alpha_2 \varepsilon^5 \ldots) g_2(z) + (\alpha_3 \varepsilon^5 + \ldots) g_3(z) + \ldots, \qquad (5.8)$$

where $\varepsilon g_1(z)$ is the linear elastic solution (proportional to ε because it is linear in τ), satisfying the boundary conditions on all the boundaries.

$g_2(z)$ is now taken so that $g_2(z) \sim -2z^{-1/2}$ as $z \to 0$ and $Re(g_2(z)) = 0$ on all boundaries, and

$g_3(z) \sim -\frac{2}{3} z^{-3/2}$ as $z \to 0$ and $Re(g_3(z)) = 0$ on all boundaries.

[Remember that the behaviour as $z \to 0$ is O.K. because the solution is not applicable as $z \to 0$; the conditions are just for mathematical convenience. The form (5.8) also simulates a sort of "multipole expansion" representation for the effect of the plasticity, far from the plastic zone].

We shall regard these elastic problems as solved; $g_2(z)$ and $g_3(z)$ can certainly be found for any domain for which $g_1(z)$ can be found. Having obtained the $g_i(z)$, we can find their small-argument expansions:

$$\left.\begin{aligned} g_1'(z) &\sim \gamma_{11} z^{-1/2} + \gamma_{12} z^{1/2} + \gamma_{13} z^{3/2} \ldots \\ g_2'(z) &\sim z^{-3/2} + \gamma_{22} z^{-1/2} + \gamma_{23} z^{1/2} \ldots \end{aligned}\right\} \qquad (5.9)$$

These will be used later. Remember that the γ_{ij} come from solving linear elastic problem alone.

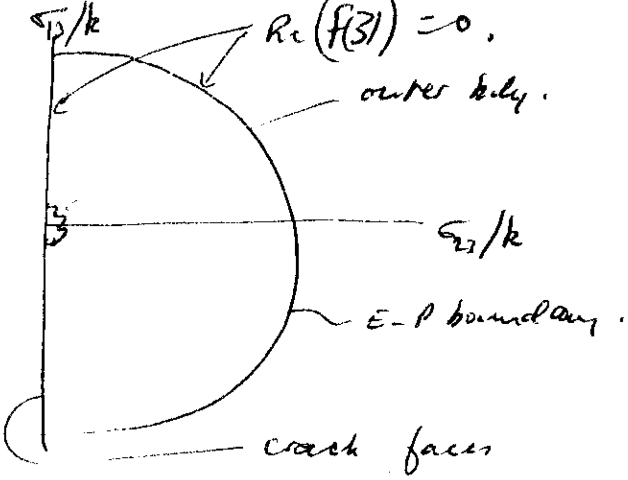
c) The inner problem.

Close to the crack tip, the crack appears semi-infinite and the body appears infinite.

Let us define the inverse to (5.7) as

$$\bar{z} = f'(\zeta) \qquad (5.10)$$

and develop a boundary value problem for $f(\zeta)$.



On the crack faces, $x_2 = 0$ so $Im(f'(\zeta)) = 0$. Also, on $x_2 = 0$, $\sigma_{23} = 0$. Hence, the crack faces map into the axis $\sigma_{23} = 0$ and, on this axis, $Im(f(\zeta)) = 0 \Leftrightarrow Re(f(\zeta)) = 0$

On the elastic-plastic boundary, the yield criterion is satisfied, so $|\zeta| = 1$.

At the outer boundary of the body, $\sigma_{i3}/k = O(\varepsilon)$, so the outer boundary maps onto some curve distant $O(\varepsilon)$ from the origin.

Finally, on the elastic-plastic boundary, we have

$\zeta = e^{i\theta}$ and $z = \frac{R(\theta)}{a} e^{i\theta}$. Hence,

$$a f'(e^{i\theta}) = R(\theta) e^{-i\theta}, \text{ or } a e^{i\theta} f'(e^{i\theta}) = R(\theta),$$

so $\frac{d}{d\theta} f(e^{i\theta})$ is imaginary, and we can take

$$\underline{Re(f(\zeta)) = 0}, \qquad (5.11)$$

both on the e-p. bdy $|\zeta| = 1$ and on the crack faces.

its effect by a singularity:

$$f(\zeta) \sim \left(B_1 \varepsilon^2 + B_2 \varepsilon^4 + B_3 \varepsilon^6 \ldots\right)\zeta^{-1} + \left(B_4 \varepsilon^4 + B_5 \varepsilon^6 \ldots\right)\zeta^{-3} + B_6 \varepsilon^6 \zeta^{-5} + \ldots \quad \text{as } \zeta \to 0. \qquad (5.12)$$

$f(z)$ now follows immediately as

$$f(z) = \left(\beta_{11} \varepsilon^2 + \beta_{12} \varepsilon^4 + \beta_{13} \varepsilon^6 \ldots\right)\left(\zeta - \zeta^{-1}\right) + \tfrac{1}{3}\left(\beta_{22} \varepsilon^4 + \beta_{23} \varepsilon^6 \ldots\right)\left(\zeta - \zeta^{-1}\right)^3 + \tfrac{1}{5}\, \beta_{33} \varepsilon^6 \left(\zeta - \zeta^{-1}\right)^5 \ldots, \qquad (5.13)$$

in which the β's are just as good unknown constants as the B's. We shall really need $\bar{\zeta}$ in terms of z; inverting (5.13) by iteration gives

$$\bar{\zeta} \sim \left(\frac{z}{\varepsilon^2 \beta_{11}} - 1\right)^{-1/2} + \text{higher-order terms (complicated, so not written out!)} \qquad (5.14)$$

(f) <u>Matching: first thoughts</u>

First, suppose we just retained the <u>leading terms</u> of (5.8) and (5.14). From (5.8), as z becomes small (near the crack tip),

$$\bar{\zeta} = g'(z) \sim \varepsilon \gamma_{11} z^{-1/2}, \qquad (5.15)$$

using (5.9), while, as $\left(\frac{z}{\varepsilon^2}\right)$ becomes large, (5.14) gives

$$\bar{\zeta} \sim \varepsilon \beta_{11}^{1/2} z^{-1/2} \qquad (5.16)$$

The forms (5.15) and (5.16) <u>agree</u> if

$$\beta_{11} = \gamma_{11}^2. \qquad (5.17)$$

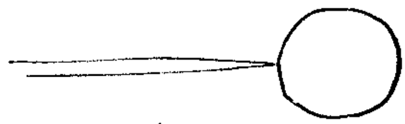
relation $\quad R(\theta) = a e^{i\theta} f'(e^{i\theta}) = a|f'(e^{i\theta})|$.

Hence, $\quad R(\theta) \sim a\gamma_{11}^2 \varepsilon^2 \left(1+e^{-2i\theta}\right)e^{i\theta}$,

from (5.13), so

$$R(\theta) \sim 2a\varepsilon^2\gamma_{11}^2\cos\theta . \qquad (5.18)$$

To this order of approximation, therefore, the elastic-plastic boundary is a circle, radius $a\varepsilon^2\gamma_{11}^2$, touching the crack tip:



Note that the elastic stress "concentration factor" is

$K_{III} = (2\pi a)^{1/2} k\varepsilon\gamma_{11}$, so that the diameter of the plastic zone is actually

$$R(0) \sim \frac{K_{III}^2}{\pi k^2} \qquad (5.19)$$

Strains are singular at the crack tip but, integrating

$\gamma_{\theta 3} = \frac{1}{r}\frac{\partial u_3}{\partial\theta} = \frac{k}{G}\frac{R(\theta)}{r}$, we have the "crack opening displacement"

$$[u_3] = \frac{k}{G}\int_{-\pi/2}^{\pi/2} R(\theta)\,d\theta \sim \frac{4ak}{G}\varepsilon^2\gamma_{11}^2 . \qquad (5.20)$$

(g) Matching to three terms.

Let us now apply the method of matched expansions properly to find all of the parameters introduced in the preceding discussion. The outer expansion, (5.8), is supposed to represent an expansion of $g(z)$ in powers of ε, as $\varepsilon \to 0$ with z fixed. Thus, the right by application of the operator O_3 : $g(z) \mapsto$ rhs of (5.8) by performing power series expansion and truncating after ε^5.

Dually, the inner expansion was obtained by looking at the region when $\xi = O(1)$ and hence $z = O(\varepsilon^2)$ (see (5.18), for example). Formally, it could be obtained from $g(z)$, if this were known exactly by expanding $g(z)$ asymptotically as $\varepsilon \to 0$, while keeping z/ε^2 fixed. This may be associated with an inner operator, I_3 say.

We may now postulate that these two operators commute:

$$I_3(O_3 g) = O_3(I_3 g) \; ; \qquad (5.21)$$

eqn (5.17) came, in effect, from the corresponding postulate for one-term expansions. Therefore, if (5.21) can be believed, we must apply I_3 to the right side of (5.8), differentiated wrt z, and O_3 to the right side of (5.14), and equate the two expressions that result.

This gives, in fact,

$$\left.\begin{aligned}
\beta_{11} &= \gamma_{11}^{2}, \\
\beta_{12} &= \gamma_{22}\gamma_{11}^{4} + 3\gamma_{12}\gamma_{13}^{2}, \\
&\;\vdots \quad \text{etc!} \\
\alpha_1 &= \tfrac{1}{2}\gamma_{11}^{3}, \\
\alpha_2 &= \tfrac{3}{4}\gamma_{22}\gamma_{11}^{5} + \tfrac{5}{8}\gamma_{12}^{2}\gamma_{11}^{4} - \tfrac{3}{8}\gamma_{32}\gamma_{11}^{5}, \\
\alpha_3 &= \tfrac{3}{8}\gamma_{11}^{5},
\end{aligned}\right\} \qquad (5.22)$$

and 3-term inner and outer expansions are now known, being expressed entirely in terms of the parameters γ_{ij}, that come from solving linear elastic

L. E. Fraenkel, Proc. Camb. Phil. Soc., 1969. It is explained in several books, including Van Dyke, Perturbation Methods in Fluid Mechanics; the principle embodied in (5.21) was first assumed by Van Dyke, I believe.

(h) Properties of the solution

Once the constants α_i, β_j are known, the solution is complete. Properties in the plastic region are of particular interest and, because we have there

$$\gamma_{r3} = 0, \quad \gamma_{\theta 3} = \frac{1}{r}\frac{\partial u_3}{\partial \theta} = \frac{k}{G}\frac{R(\theta)}{r},$$

we just need to find $R(\theta)$. Performing the algebra, we have

$$R(\theta) = a\left|f(e^{i\theta})\right| = 2a\cos\theta\Big[\gamma_{11}^2\varepsilon^2 + (\delta_1 + \delta_2\sin^2\theta)\varepsilon^4 + (\delta_3 + \delta_4\sin^2\theta + \delta_5\sin^4\theta)\varepsilon^6 + O(\varepsilon^8)\Big], \qquad (5.23)$$

where

$$\left.\begin{aligned} \delta_1 &= \gamma_{11}^4\gamma_{22} + 3\gamma_{11}^3\gamma_{12}, \\ \delta_2 &= -8\gamma_{11}^3\gamma_{12} \\ &\text{and so on} \end{aligned}\right\} \qquad (5.24)$$

The plastic strain is most intense on the line $\theta = 0$, and its magnitude is measured by the value of $R(0)$, which may thus be regarded as a possible fracture parameter. From (5.23),

$$R(0) = 2a\Big[\gamma_{11}^2\varepsilon^2 + \delta_1\varepsilon^4 + \delta_3\varepsilon^6 + O(\varepsilon^8)\Big]. \qquad (5.25)$$

Other interesting parameters, that may be useful in predicting fracture, are the crack tip opening displacement δ_t, and the integral J; it is not hard to show that

$$\delta_t = \frac{2ak}{G}\Big[2\gamma_{11}^2\varepsilon^2 + \big(2\delta_1 + \tfrac{2}{3}\delta_2\big)\varepsilon^4 + \big(2\delta_3 + \tfrac{2}{3}\delta_4 + \tfrac{2}{5}\delta_5\big)\varepsilon^6 + O(\varepsilon^8)\Big], \qquad (5.26)$$

It may be noted that, to lowest order in ε, each of $R(0)$, δ_t and J depend only upon γ_{11}^2 or, equivalently, upon the stress concentration factor K_{III}. Thus if any of these quantities were assumed to govern fracture, in the sense that the crack would extend when the quantity reached a critical value, they would each correspond to the attainment of a critical value of K_{III}. Thus, the "linear elastic fracture mechanics" criterion, that K_{III} reaches a critical value for crack extension, is seen aphiatly to be reasonable, so long as $\varepsilon \ll 1$. In practice, linear elastic fracture mechanics should apply, so long as the extent of the plastic zone $R(0)$, as given by the lowest approximation (5.19), is a small fraction of the crack length "a". The ASTM criterion for Mode I loading is that $K_I < Y(2a/5)^{1/2}$, for example, where Y is the tensile yield strength. In Mode III, this would be

$$K_{III} < k(2a/5)^{1/2},$$

corresponding to $R(0) < \left(\frac{2}{5\pi}\right)a$.

It fracture occurs in the nonlinear regime, with large-scale plasticity, however, a fracture criterion is less easy to identify; δ_t, $R(0)$ and J are all possible candidates, but they all depend differently upon ε and so would lead to different predictions of the stress level at which the crack would extend. It may be noted, however, that J is path-independent, as the material behaves like an elastic one in Mode III and hence, if some other mechanical behaviour were going to occur at the crack tip and if this could the crack tip should be in correspondence with the value of J. For Mode I and Mode II deformations, J is not strictly path-independent but finite-element studies have verified that a J can be defined that is approximately so, except in the immediate vicinity of the crack tip. Perhaps, therefore, a criterion based upon J has some merit, though δ_t, the 'crack opening displacement', receives emphasis in UK, particularly.

One remark may be made concerning J, but not the other criteria. Because it can be evaluated from a path Γ in the 'outer' elastic region, and because the method of matched expansions can be viewed as an iterative development, the solution to order ε^3 (see eqn (5.8)) in the outer region can be obtained from knowledge of the inner solution to order ε^2 (i.e. the one-term solution) only. Hence, J can be calculated to order ε^4 (i.e. to two terms), from knowledge of the one-term inner solution embodied in (5.18) alone. A prescription for a "plastic zone corrected" estimate of J, or K_{III}, proposed by G.R. Irwin, is that the crack problem should be solved, elastically, with the crack extending to the centre of the plastic zone, that is, with the crack length 'a' replaced by '$a + r_y$', where $r_y = \frac{1}{2}R(0) = \frac{K_{III}^2}{2\pi k^2}$.

It can be shown that the new estimate of K_{III} that is generated in this way, if related to J through the formula

$$J = K_{III}^2/2G,$$

which is the 'linear elastic' result, produces an estimate of J that agrees with (5.27) to order ε^4, that is, the this way, and then applying the 'linear elastic fracture mechanics' criterion $K_{III} \le K_{III}(\text{critical})$ is correct, to order ε^4, provided that the true nonlinear criterion is $J \le J_{(\text{critical})}$, but not otherwise!

(i) Examples.

The solution can be used for any problem for which the δ_{ij} can be found.

Consider first a single crack, length $2a$, in an infinite body subjected to loading $\sigma_{23} = \tau$ at infinity; by images, this is equivalent to an edge crack of depth 'a'. It follows immediately that

$$\left.\begin{aligned} g_1'(z) &= (z+1)\left[(z+1)^2-1\right]^{-1/2}, \\ g_2'(z) &= 2^{3/2}(z+1)\left[(z+1)^2-1\right]^{-3/2}, \\ g_3'(z) &= 2^{5/2}(z+1)\left[(z+1)^2-1\right]^{-5/2}, \end{aligned}\right\} \quad (5.28)$$

from which the g's are easy to extract. It follows that

$$\left.\begin{aligned} \delta_t &= \frac{2ak}{G}\left[\varepsilon^2 + \tfrac{1}{4}\varepsilon^4 + \tfrac{1}{4}\varepsilon^6 + O(\varepsilon^8)\right], \\ R(0) &= a\left[\varepsilon^2 + \tfrac{5}{4}\varepsilon^4 + \tfrac{5}{4}\varepsilon^6 + O(\varepsilon^8)\right], \\ J &= \frac{\pi a k^2}{G}\left[\varepsilon^2 + \tfrac{1}{2}\varepsilon^4 + \tfrac{1}{4}\varepsilon^6 + O(\varepsilon^8)\right]. \end{aligned}\right\} \quad (5.29)$$

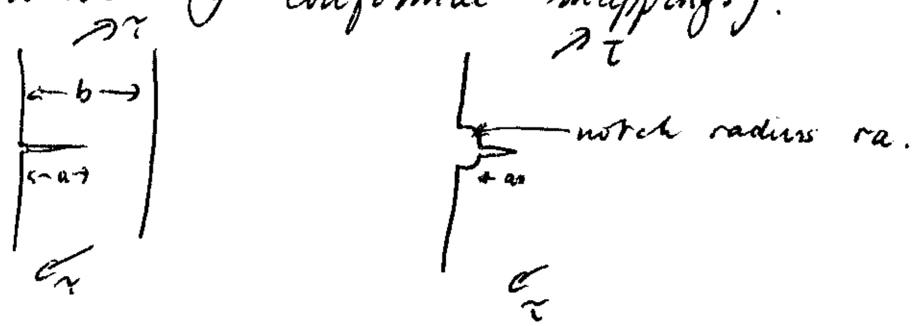
This simple problem can actually be solved exactly (Hult & McClintock, 1956; Rice, 1966), by a development of the method of section (e). Rice (1966) gave

$$\left.\begin{aligned} \delta_t &= \frac{2ak}{G}\left[\frac{2}{\pi}(1+\varepsilon^2)E_1(\varepsilon^2)-1\right], \\ R(0) &= a\left[\frac{2}{\pi}\left(\frac{1+\varepsilon^2}{1-\varepsilon^2}\right)E_2\left(\frac{2\varepsilon}{1+\varepsilon^2}\right)-1\right] \end{aligned}\right\} \quad (5.30)$$

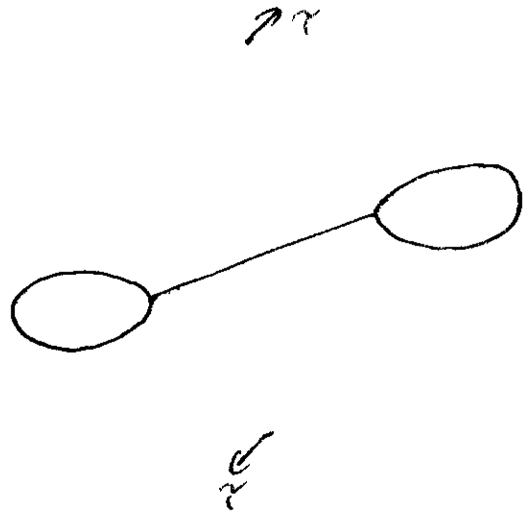
here E_1, E_2 are elliptic integrals. The first two of (5.29) agree with the asymptotic expansions of (5.30). Furthermore, the 3-term expansion of R01 agrees with the exact value of R01 rather closely, to $\varepsilon = .75$, compared with the 1-term expansion, which is good only up to about $\varepsilon = 0.3$. [The "ASTM limit", $R(0) < \frac{2}{5\pi} a$, corresponds to $\varepsilon^2 < (2/5\pi)$, or $\varepsilon < (2/5\pi)^{1/2} \sim 0.35$].

It may be noted, too, that $\varepsilon = 1$ corresponds to $\tau = k$, the yield stress, so that the whole body would flow. Thus, $\varepsilon = 0.75$ corresponds to loading up to 3/4 of "limit load."

More complicated examples can also be worked out. For instance, an edge crack in a strip of finite width, and cracks emanating from notches (the latter give problems for y's that are easy to solve by conformal mappings).



An account of this work will appear soon in J. Mech. Phys. Solids (Edmunds and Willis). We have also performed similar calculations for a power-law hardening material (this will appear in J Mech Phys. Solids as well). Here, the analysis of the plastic region is more involved than that given in Section (6). T.M. Edmunds has also done calculations for an asymmetrically loaded body; these will appear in Int. J. Solids & Structures. He has shown, for instance, plastic zones behaving as in the picture:



Thus, the plasticity tends to extend transverse to the applied shear. The line on which the plastic strains are maximal is correspondingly different from the line of the crack, and might imply that the crack would extend at an angle to its original line.

We are working on the Mode I / Mode II problems now! The plasticity problem, analogous to that of § (b), is now intractable analytically, and we are using a combination of 'matched expansion' analysis and finite element calculations.