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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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FUNCTIONAL ANALYTIC METHODS IN ELASTIC-PLASTIC TORSION OF CYLINDRICAL BARS

(Part II)

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II. A general boundary value problem with Hencky's law

II.1 Recapitulation

We want to resolve several problems about elasto-plastic torsion of cylindrical bars:

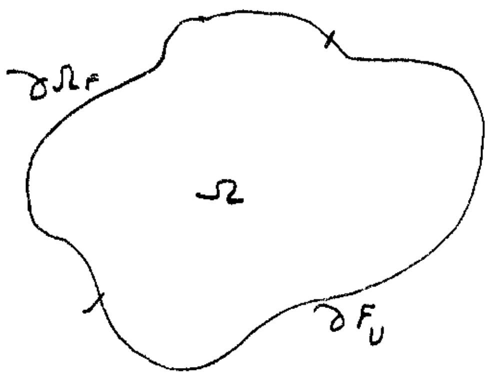
Homogeneous bars with simply connected cross sections
multiply connected cross sections.

Fibred reinforced bars with simply connected cross sections

Let us consider these problems like some particular cases of a more general one that we are going to study in this Chapter.

For reasons of simplicity we shall adopt the Hencky's constitutive law — That will be justified later on by the fact that for increasing loads (exclusively) we shall be able to prove that, in the case of the torsion problem, the solutions obtained by Hencky's and Prandtl Reuss's laws are the same.

II.2 Description of the problem (A)



Ω is here an arbitrary bounded open set of R^3 with a "regular" boundary $\partial\Omega$
(In fact we can allow $\partial\Omega$ to be piece-wise continuously differentiable)

The data are:

$$\begin{cases} f, & \text{body forces in } \Omega. \\ \overline{F_k} & \text{on } \partial\Omega_{F_k} \subset \partial\Omega. \\ U_\ell & \text{on } \partial\Omega_{U_\ell} \subset \partial\Omega. \end{cases}$$

(f, F_k, $\partial\Omega_{F_k}$ and $\partial\Omega_{U_\ell}$ being such that the problem is well posed)

We shall call $\partial\Omega_F$ the union of the parts of $\partial\Omega$ on which some F_k is given and $\partial\Omega_U$ the union of the parts of $\partial\Omega$ on which some U_ℓ is given.

Then we have to resolve the following problem:

Problème (A). :

To find the couple (σ, u) such that:

(15) $\sigma_{ij} = \sigma_{ji}$ in Ω

(16) $\sigma_{ij,j} + f_i = 0$ in Ω } (equilibrium)

(17). $\sigma_{kj} n_j = \bar{F}_k$ on $\partial\Omega_k$

(18) $u_\ell = U_\ell$ on $\partial\Omega_\ell$ } Boundary condition.

(19) $\mathcal{F}(\sigma) \leq 0$

(20) $\varepsilon_{ij}(u) = A_{ijhk}\,\sigma_{hk} + \lambda_{ij}$

(21) $\lambda_{ij}(\tau_{ij} - \sigma_{ij}) \leq 0 \quad \forall\ \tau \in R^9$ s.t. $\tau_{ij} = \tau_{ji}$ and $\mathcal{F}(\tau) \leq 0$

} Hencky's constitutive law

with:

(22) $\mathcal{F} : R^9 \longrightarrow R$, convex and continuous (Elasticity criterium)

(23) $A_{ijhk} : \Omega \longrightarrow R$, bounded on Ω

(24) $A_{ijhk} = A_{jihk} = A_{hkij}$.

(25) $\exists\ \beta > 0$ constant such that $A_{ijhk} X_{ij} X_{hk} \geqslant \beta\, |X|^2$. $\forall\ X \in R^9$ with $X_{ij} = X_{ji}$

} components of the elasticity tensor.

(26) $\varepsilon_{ij}(u) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \varepsilon_{ji}(u)$, linearized strain tensor.

II. 3. Principle of Haar Karman and variational formulation for (A)

It is very easy to prove the following statement.

Principle of Haar Karman: if the problème (A) has a sufficiently regular solution (σ^0, u^0) then σ^0 minimizes the functional

(27) $$J(\sigma) = \frac{1}{2}\int_\Omega A_{ijhk}\,\sigma_{ij}\,\sigma_{hk}\,dx - \int_{\partial\Omega_u} \sigma_{\ell j}\, n_j\, U_\ell\, d\Gamma$$

To prove that, we have to write down the difference $J(\sigma) - J(\sigma_o)$, using the properties (15) (24) (25) and also the fact that

$$\int_\Omega \varepsilon_{ij}(u_o)[\sigma_{ij} - \sigma_{ij}^o]\,dx = \int_\Omega u^o_{i,j}(\sigma_{ij} - \sigma_{ij}^o)\,dx = \int_{\partial\Omega} u_i^o(\sigma_{ij} - \sigma_{ij}^o)n_j\,d\Gamma - \int_\Omega u_i^o(\sigma_{ij,j} - \sigma^o_{ij,j})\,dx$$

$$= \int_{\partial\Omega_U} U_\ell(\sigma_{\ell j} - \sigma_{\ell j}^o)n_j\,d\Gamma$$

because the properties (18) for u_ℓ^o and (15)(16)(17)(19) for σ and σ^o. We have also to take account of the fact that u_o and σ_o, solution of (A) are related by (20) and (21)

<u>We have now to choose a functional framework in order to give a meaning to $J(\sigma)$</u>

- (23) implies that $A_{ijkh} \in L^\infty(\Omega)$ therefore,
- $\int_\Omega A_{ijkh}\sigma_{ij}\sigma_{kh}\,dx$ will have a meaning if $\sigma_{ij} \in L^2(\Omega)$ $\forall i,j = 1,2,3$
- Then, (20) suggests to look for $\varepsilon_{ij}(u) \in L^2(\Omega)$ and also $\lambda_{ij} \in L^2(\Omega)$.
- Hence, u_i can be reasonably supposed in $H^1(\Omega)$* and then $\gamma_o u_i \in H^{1/2}(\partial\Omega)$*.
- The second integrale of $J(\sigma)$ will have now a meaning if and only if $\sigma_{\ell j}n_j|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$. But we assumed $\sigma_{ij} \in L^2(\Omega)$ and σ is supposed to satisfy in particular (16) that is to say

$$\sigma_{ij,j} = -f_i$$

Now, we know from the theorem 7 (lecture notes H. Lanchon p 20) that if $\sigma_{ij} \in L^2(\Omega)$ and $\sigma_{ij,j} \in L^2(\Omega)$, then $\sigma_{ij}n_j|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$. It is therefore sufficient to make the following assumptions on the data:

(28) $f_i \in L^2(\Omega)$

(29) $\bar{F}_\ell \in H^{-1/2}(\partial\Omega)$

(30) $U_\ell \in H^{1/2}(\partial\Omega)$,

<u>Recalls (cf lectures notes H. Lanchon p14 and following)</u>

$$\left.\begin{array}{l} H^1(\Omega) = \{v \,/\, v \in L^2(\Omega),\ v_{,j} \in L^2(\Omega) \text{ for } j = 1,2,3\} \\ H^{1/2}(\partial\Omega) = \text{space of restrictions at } \partial\Omega \text{ of all the elements of } H^1(\Omega) \end{array}\right\} \text{Sobolev Spaces}$$

to obtain a convenient functional scope for the problem (A)

Let us introduce some more definitions

(31) $\mathcal{K} = \{ \sigma / \sigma \in [L^2(\Omega)]^9$ verifying (15)(16)(17)(19)$\}$ is the set of allowed stresses for the problem (A)

(32) $a(\sigma, \tau) = \int_\Omega A_{ijkh} \sigma_{ij} \tau_{kh} \, da$ is a bilinear form over $[L^2(\Omega)]^9$

We are now able to write down clearly a variational principle for the stress field.

Variational principle: If (σ^0, u^0) is a solution of (A) Then σ_0 is solution of a new problem that we shall call "(A_1)"

(33) $\sigma^0 \in \mathcal{K}$ and $J(\sigma^0) \leq J(\sigma) \quad \forall \sigma \in \mathcal{K}$ with

(34) $J(\sigma) = a(\sigma, \sigma) - 2\int_{\partial\Omega_U} \sigma_{ej} n_j U_e \, d\Gamma$

II.4 Results and conclusions about the problem (A_1)

We have already studied the general problem of minimizing a functional and this preceding one is already prepared to be interpreted as a Varional Inequalities (or "Inequation"?) - We have just to check now the properties of $\mathcal{K}$, $a(\, ,\,)$ and $\int_{\partial\Omega_U} \sigma_{ej} n_j U_e \, d\Gamma$ to be able to apply the theorem 15 (lecture notes of H. Lanchon p41)

i) $\mathcal{K}$ is a closed convex set of $[L^2(\Omega)]^9$, the norm of this space being defined by the scalar product

$$(\varphi, \psi) = \int_\Omega \varphi_{ij} \psi_{ij} \, da$$

In fact, Let $\sigma^{(m)}$ be a cauchy sequence in $\mathcal{K}$ then

• $\sigma_{ij,j}^{(n)}$ converges then toward $\sigma_{ij,j} \in \mathcal{D}'(\Omega)$ because the continuity of the operator $\frac{\partial}{\partial x_j}$ in $\mathcal{D}'(\Omega)$ (cf lecture notes H. L p 13)

moreover: $\sigma_{ij,j}^{(n)} = - f_i$ implies $\lim \sigma_{ij,j}^{(n)} = \sigma_{ij,j} = - f_i \in L^2(\Omega)$

• now, we already saw above that $\sigma_{ij} \in L^2(\Omega)$ and $\sigma_{ij,j} \in L^2(\Omega)$ implies $\sigma_{ij} n_j \in H^{-1/2}(\partial\Omega)$ - We have now to check that:

$$\sigma_{hj} n_j = \bar{F}_h \text{ over } \partial\Omega_{F_h}$$

For that, let ϕ be an element of $H^{1/2}(\partial\Omega)$; we know that $\exists$ $\varphi \in H^1(\Omega)$ unique such that:

$$\begin{cases} \Delta\varphi = 0 \text{ in } \Omega \\ \varphi\big|_{\partial\Omega} = \phi \end{cases} \qquad \text{(cf lecture notes H. L p 28)}$$

and then $\displaystyle\int_{\partial\Omega} (\sigma_{ij}^{(n)} n_j - \sigma_{ij} n_j)\, \phi \, d\Gamma = \int_\Omega (\sigma_{ij,j}^{(n)} - \sigma_{ij,j})\, \varphi \, dx + \int_\Omega (\sigma_{ij}^{(n)} - \sigma_{ij})\, \varphi_{,j} \, dx$

Whence $\displaystyle\lim_{n \to \infty} \int_{\partial\Omega} (\sigma_{ij}^{(n)} n_j - \sigma_{ij} n_j)\, \phi \, d\Gamma = 0 \quad \forall \; \phi \in H^{1/2}(\partial\Omega).$

and Therefore $\displaystyle\lim_{n \to \infty} \sigma_{ij}^{(n)} n_j = \sigma_{ij} n_j$ almost every-where over $\partial\Omega$.

$$\Rightarrow \quad \sigma_{hj} n_j = \bar{F}_h \quad \text{over } \partial\Omega_{F_h}.$$

• The convergence of $\sigma_{ij}^{(n)}$ toward σ_{ij} in $L^2(\Omega)$ implies the convergence almost everywhere in Ω of a subsequence $\sigma_{ij}^{(p)}$ toward σ_{ij}.

then $\displaystyle\lim_{p \to \infty} \mathcal{F}(\sigma^{(p)}) = \mathcal{F}(\sigma) \leq 0.$ almost everywhere in Ω.

• The convexity of $\mathcal{K}$ is obvious

ii) $a(\sigma, \tau)$ is a continuous, coercive symetric bilinear form over the subspace of symetric element in $(L^2(\Omega))^9$

That is obvious. from (23) (24) (25)

iii) $\displaystyle\int_{\partial\Omega_u} \sigma_{\ell j} n_j u_\ell \, d\Gamma$ is a continuous linear form over $\mathcal{K}$

The linearity is obvious. Let U be a fixed element of $(H^1(\Omega))^3$ satisfying (18)

Therefore $\int_{\partial\Omega_U} \sigma_{ej} n_j U_e \, d\Gamma = (\sigma, \varepsilon(U)) + \text{constant} \quad \forall \sigma \in K$

hence the continuity expected.

All the hypothesis of the theorem 15 (mentioned above) are satisfi
and we can state:

Theorem 1: Let us suppose that $K \neq \phi$, then.
$\exists \; \sigma_0 \in K$ unique such that
(34) $J(\sigma_0) \leq J(\sigma) \quad \forall \sigma \in K$.
and σ° is characterized by the variational inequality.

(35) $$\sigma_0 \in K \qquad a(\sigma^\circ, \sigma - \sigma^\circ) \geq \int_{\partial\Omega_U} (\sigma_{ij} - \sigma^\circ_{ij}) n_j U_i \, d\Gamma, \quad \forall \sigma \in K.$$

II.5. Recapitulation about the problem (A)

What have we got now?

i) If σ_0 is solution of (A), then σ_0 is the unique solution of (A_1); whence the uniqueness of the stress field solution of (A)

ii) We shall be able to obtain the existence of a solution for (A) if we can show that K is not empty and also that $\exists \; u^\circ \in H^1(\Omega)$ satisfying (18) and associated to σ° by (20) and (21)

Remark 1: We obtained also a variational principle for the displacement field but, this one is much more complicated and we have just obtained from this principle a very weak solution. It is therefor better now to particularize the problem (A) in order to go ahead.

II.6 Bibliography.

About Sobolev space.

J.L. Lions and E Magenes
" Non homogeneous boundary value problems"
Springer Verlag 1972

III Elastic-plastic torsion of cylindrical bars.

III 1. Application of the theorem 1 for the torsion of an homogeneous bar.

Problem (B)

Ω is a cylinder with a cross section Σ and a higher $h=1$. (cf p. 2)

(16) $\mathcal{K} = \{\sigma / \sigma \in [L^2(\Omega)]^9 ;\ \sigma_{ij} = \sigma_{ji} ;\ \sigma_{ij,j} = 0 ;\ \mathcal{F}(\sigma) \leq 0$ a.e in Ω and
$\sigma_{ij} n_j = 0$ a.e over $\partial\Omega_L$; $\sigma_{33} = 0$ a.e over $\partial\Omega_0$ and $\partial\Omega_1 \}$

(17) $\mathcal{F}(\sigma) = \frac{1}{2}\sigma_{ij}\sigma_{ij} - \frac{1}{6}(\sigma_{ii})^2 - g^2 = \frac{1}{2}\sigma_{ij}^D \sigma_{ij}^D - g^2$ with $\sigma^D = \sigma - \frac{\text{trace}\,\sigma}{3} \mathbb{1}$

(18) $J(\sigma) = \frac{1}{2}\int_\Omega \left\{ \frac{1+\nu}{E}\sigma_{ij}\sigma_{ij} - \frac{\nu}{E}\sigma_{ii}^2 \right\} dx - \alpha \int_{\partial\Omega_1} (x_1 \sigma_{23} - x_2 \sigma_{13})\, dx_1\, dx_2 .$

(19) $A_{ijkh} x_{ij} x_{kh} > \frac{1-2\nu}{E} |x|^2$; $0 \in \mathcal{K} \Rightarrow \mathcal{K} \neq \phi$. Then $\exists\ \sigma_0 \in \mathcal{K}$ unique s.t

$$J(\sigma_0) \leq J(\sigma) \quad \forall \sigma \in K .$$

σ^0 is the only possible stress tensor field solution for the problem B

Proposition 1 : σ^0 is such that.

1/ $\sigma_{ij}^0 = 0$ a.e in Ω except $\sigma_{13}^0 = \sigma_{31}^0$ and $\sigma_{23}^0 = \sigma_{32}^0$.

2/ $\sigma_{ij,3}^0 = 0$ a.e in Ω.

Ideas of the proof : Suppose the proposition is not true - then let us introduce $\bar{\sigma}$:

$\bar{\sigma}_{ij} = 0$ except $\bar{\sigma}_{p3} = \bar{\sigma}_{3p}$ $(p = 1, 2)$ defined as distribution on Σ by.

(19) $\langle \varphi, \bar{\sigma}_{p3} \rangle = \int_\Omega \varphi(x_1, x_2)\, \sigma_{p3}^0 (x_1, x_2, x_3)\, dx \qquad \forall \varphi \in \mathcal{D}(\Sigma)$

$\bar{\sigma} \in K$; in fact.

i/ $|\langle \varphi, \bar{\sigma}_{p3} \rangle| \leq \| \sigma_{p3}^0 \|_{L^2(\Omega)} \| \varphi \|_{L^2(\Omega)} = \| \sigma_{p3}^0 \|_{L^2(\Omega)} \| \varphi \|_{L^2(\Sigma)}$

$\Rightarrow \quad \bar{\sigma}_{p3} \in$ dual $L^2(\Sigma) \equiv L^2(\Sigma)$

ii) $\sigma_{kj,j} = 0$ in Ω because:

for $k = p = 1$ or 2.

$$\langle \varphi, \bar{\sigma}_{pj,j} \rangle = - \langle \varphi_{,3}, \bar{\sigma}_{p3} \rangle = 0 \quad \forall \varphi \in \mathcal{D}(\Sigma)$$

for $k = 3$

$$\langle \varphi, \bar{\sigma}_{3j,j} \rangle = - \int_{\Omega} \varphi_{,p} \sigma^{0}_{3p} \, dx = - \int_{\partial\Omega_1} \varphi \sigma^{0}_{33} \, dx_1 dx_2 + \int_{\partial\Omega_0} \varphi \sigma^{0}_{33} \, dx_1 dx_2 .$$

$$- \int_{\partial\Omega_L} \varphi (\sigma^{0}_{3p} n_p) \, dx = 0 \qquad \forall \varphi \in \mathcal{D}(\Sigma)$$

$\Longrightarrow \quad \sigma_{kj,j} = 0 \quad$ for $k = 1, 2, 3$ a.e on Ω.

iii) $\bar{\sigma}_{3p} n_p = 0$ on $\partial\Omega_L$ because.

$\bar{\sigma}_{kj} \in L^2(\Omega)$ and $\sigma_{kj,j} = 0 \in L^2(\Omega) \Longrightarrow \sigma_{kj} n_j \in H^{-1/2}(\partial\Omega)$

but also (40) $\bar{\sigma}_{p3} = \int_0^1 \sigma^{0}_{p3} \, dx_3$ by definition.

$$\Rightarrow \quad \bar{\sigma}_{31} n_1 + \bar{\sigma}_{32} n_2 = \int_0^1 (\sigma^{0}_{13} n_1 + \sigma^{0}_{23} n_1) \, dx_3 = 0 .$$

IV. $\quad \mathcal{F}(\bar{\sigma}) = \bar{\sigma}_{13}^2 + \bar{\sigma}_{23}^2 - g^2 \leq 0$ because.

$$\bar{\sigma}_{13}^2 + \bar{\sigma}_{23}^2 = \int_0^1 (\sigma^{0}_{13} \bar{\sigma}_{13} + \sigma^{0}_{23} \bar{\sigma}_{23}) \, dx_3$$

$$\leq (\bar{\sigma}_{13}^2 + \bar{\sigma}_{23}^2)^{1/2} \left[\int_0^1 (\sigma^{0\,2}_{13} + \sigma^{0\,2}_{23}) \, dx_3 \right]^{1/2}$$

$$\Rightarrow \qquad \sigma_{13}^2 + \sigma_{23}^2 \leq g^2 .$$

- $\underline{J(\bar{\sigma}) \leq J(\sigma^0)}$; in fact:

$$J(\sigma) = \frac{1}{2} a(\sigma, \sigma) - \alpha \int_{\partial\Omega_1} (x_1 \sigma_{23} - x_2 \sigma_{13}) \, dx_1 dx_2$$

we are going to prove that $\int_{\partial\Omega_1} (x_1 \bar{\sigma}_{23} - x_2 \bar{\sigma}_{13}) \, dx_1 dx_2 = \int_{\partial\Omega_1} (x_1 \sigma^{0}_{23} - x_2 \sigma^{0}_{13}) \, dx_1 dx_2$

and also that $a(\bar{\sigma}, \bar{\sigma}) \leq a(\sigma^0, \sigma^0)$.

(41) $$\int_{\partial\Omega_1} (x_1 \bar{\sigma}_{23} - x_2 \bar{\sigma}_{13}) \, dx_1 dx_2 = \int_{\Omega} (x_1 \sigma^0_{23} - x_2 \sigma^0_{13}) \, dx .$$

now $\forall \sigma \in \mathcal{K}$ and $u \in (H^1(\Omega))^3$ satisfying $u_1 = u_2 = 0$ over $\partial\Omega_0$ and $u_1 = -\alpha x_2$, $u_2 = \alpha x_1$ over $\partial\Omega_1$, we have.

$$\int_{\Omega} \sigma_{ij} \, \varepsilon_{ij}(u) \, dx = \alpha \int_{\partial\Omega_1} (\sigma_{23} x_1 - \sigma_{13} x_2) \, dx_1 dx_2 .$$

in particular for $u^* = (-\alpha x_2 x_3, \alpha x_1 x_3, 0)$

$$\int_{\Omega} \sigma_{ij} \, \varepsilon_{ij}(u^*) \, dx = \alpha \int_{\Omega} (\sigma_{23} x_1 - \sigma_{13} x_2) \, dx = \alpha \int_{\partial\Omega_1} (\sigma_{23} x_1 - \sigma_{13} x_2) \, dx_1 dx_2 .$$

then for $\sigma_{ij} = \sigma^0_{ij}$, $\int_{\Omega} (\sigma^0_{23} x_1 - \sigma^0_{13} x_2) \, dx = \int_{\partial\Omega_1} (\sigma^0_{23} x_1 - \sigma^0_{13} x_2) \, dx_1 dx_2 .$

Whence (41)

In an other end, from (40) and the inegality $\nu < \frac{1}{2}$, we obtain.

$$a(\sigma^0, \sigma^0) - a(\bar{\sigma}, \bar{\sigma}) \geq \frac{1+\nu}{2E} \int_{\Omega} \left(\sigma^{0D}_{ij} \sigma^{0D}_{ij} - \bar{\sigma}^D_{ij} \bar{\sigma}^D_{ij} \right) \geq 0$$

$\Rightarrow J(\bar{\sigma}) \leq J(\sigma^0) \Rightarrow \bar{\sigma} = \sigma^0$ because the uniqueness of σ^0 and then the Prop 1

Remark 2 : This proposition is still true in the two following cases;

a) Ω is a cylindar with some cylindrical cavities $\Omega_1, \dots \Omega_q$ with same direction of generatrix: we have then to apply the preceeding proof to the multiply connected domain

$$\Omega^* = \Big[\complement_{\Omega} \overbrace{\bigcup_{i=1,\dots,q} \Omega_i} \Big]$$

b) Ω is a cylindar reenforced by cylindrical fibers $\Omega_1, \dots \Omega_q$ with same direction of generatrix. In this case. ν and E are discontinuous.

$$\nu \text{ (and } E) = \begin{cases} \nu_{f} \text{ or } (E_f) & \text{on the fibers } \Omega_1, \dots \Omega_q . \\ \nu_m \text{ or } (E_m) & \text{on the matrix } \Omega^* \end{cases}$$

Remark 3 : The problem can be reduce now to a bidimentional one

III.2 New formulation of the problem (A_1) in the particular case of Torsion

$$\mathcal{K}_T = \{ \sigma \,/\, \sigma \in \mathcal{K},\ \sigma_{ij} = 0 \text{ except } \sigma_{p3} = \sigma_{3p};\ p = 1,2 \} \qquad (T \text{ or Torsion})$$

We know now that $\sigma^0 \in \mathcal{K}_T$ and that $J(\sigma^0) \le J(\sigma) \quad \forall \sigma \in \mathcal{K}_T \subset \mathcal{K}$

Now:

Lemma 1 : $\forall \sigma \in \mathcal{K}_T$, $\exists\ \theta \in H_0^1(\Sigma)$ unique such that.

i) $\theta_{,2} = \sigma_{13}$; $\theta_{,1} = -\sigma_{23}$ a.e on Σ (hom section of Ω)

ii) Furthermore $|\operatorname{grad}\theta| \le g$.

Proof: • $\sigma_{ij} \in L^2(\Omega)$; $\sigma_{ij,j} = 0$ and $\sigma_{33} = 0$ on $\partial\Omega_0$ and $\partial\Omega_1 \Longrightarrow$

$\exists\ \varphi \in H^1(\Sigma)$ such that $\varphi_{,2} = \sigma_{13}$ and $\varphi_{,1} = -\sigma_{23}$.

• $\sigma_{ij} n_j = 0$ on $\partial\Omega_L \Leftrightarrow \overrightarrow{\operatorname{grad}}\varphi \wedge \vec{n} = 0$ on $\partial\Sigma$

$\Longleftrightarrow \varphi = k = \text{constante on } \partial\Sigma$.

• Let $\theta = \varphi - k$, then $\theta \in H_0^1(\Sigma)$

• $|\operatorname{grad}\theta|^2 = \sigma_{13}^2 + \sigma_{23}^2$ and $|||\theta|||^2_{H_0^1(\Sigma)} = \int_\Sigma |\operatorname{grad}\theta|^2 dx$.

implies the uniqueness of such a θ in $H_0^1(\Sigma)$

• $\mathcal{F}(\sigma) \le 0 \Longrightarrow |\operatorname{grad}\theta| \le g$.

It is easy to see now that $\exists$ a bijection between $\mathcal{K}_T$ and K.

(42) $$K = \{ \varphi \,/\, \varphi \in H_0^1(\Sigma) \text{ s.t } |\operatorname{grad}\varphi| \le g \};$$

for two corresponding elements σ and θ.

(42) $$2\mu\, J(\sigma) = J_\alpha(\theta) = \int_\Sigma |\operatorname{grad}\theta|^2 dx - 4\mu\alpha \int_\Sigma \theta\, dx \quad \text{with } \mu = \frac{E}{2(1+\nu)}$$

Therefore, in the particular case of the torsion of a homogeneous, elastic-plastique cylindrical bar, the problem (A_1) is replace by the equivalent problem (B_1) Find θ_α such that

(43) $$\theta_\alpha \in K\ ; \quad J_\alpha(\theta_\alpha) \le J_\alpha(\varphi) \quad \forall \varphi \in K$$

Lemma 2 : $\exists\ f_\alpha \in H_0^1(\Sigma)$, unique such that

(45) $\quad 2\mu\alpha \int_\Sigma \varphi\, dx = \langle \varphi, f_\alpha \rangle \quad \forall \varphi \in H_0^1(\Sigma)$

with (46) $\quad \langle \varphi, \psi \rangle = \int_\Sigma \operatorname{grad}\varphi \cdot \operatorname{grad}\psi\, dx$, scalar product on $H_0^1(\Sigma)$

This lemma is nothing else that a direct application of the Riesz theorem (cf Lecture notes H.L. p 7)

Then we got a new expression of the functional.

(47) $\quad J_\alpha(\varphi) = |||\varphi - f_\alpha|||^2 - |||f_\alpha|||^2$.

and because f_α is a fixed element of $H_0^1(\Sigma)$, we obtain a new form for the probleme B_1, because θ_α is the element which minimizes the distance $|||f_\alpha - \varphi|||$ of f_α to the closed convex K of $H_0^1(\Sigma)$. We obtain then the following conclusion (cf example 6 p 39 in lecture notes of H.L)

> The solution θ_α of the probleme (B_1) is exactly the projection on the closed convex K of the element f_α defined by (45). θ_α is characterized by the variational inequality
>
> (48) $\quad \theta_\alpha \in K \ ; \quad \langle \theta_\alpha - \varphi, \theta_\alpha - f_\alpha \rangle \leq 0 \quad \forall \varphi \in K$

III.3 First Properties of the stress function θ_α

To obtain this properties, it is natural to study first the general properties of K.

III.3.1. Properties of K (cf references given p 7)

- K is the set of Lipschitz functions of $H_0^1(\Sigma)$ with modulus g. that is to say.

(49) $\quad |\varphi(x) - \varphi(x')| \leq g\,|x - x'| \quad \forall \varphi \in K$ and $\forall x$ and $x' \in \Omega$

Therefore these functions are uniformly continuous on $\bar{\Sigma}$

- K is compact for the norm of uniform convergence.

Proof: That come from Ascoli theorem which says that: if X and Y are two compact metric spaces, if E is a set of equicontinuous mapping of X into Y, then the closure of E in $\mathcal{C}(X,Y)$, set of continuous mapping from X into Y, is compact

here: $X = \bar{\Sigma}$ (compact of R^2); $Y = \bigcup_{\varphi \in K} \varphi(\bar{\Sigma})$ (compact of R)

K is a set of equicontinuous mapping from X into Y.

$\Rightarrow$ if $\mathcal{C}(X,Y)$ is supplied with the norm of the uniform convergence then $\bar{K}$ is uniformely compact and it is easy to prove that with this topology $K = \bar{K}$

- if φ_1 and $\varphi_2 \in K$ then $\sup(\varphi_1, \varphi_2)$ and $\inf(\varphi_1, \varphi_2) \in K$

III 3.2 Regularity of θ_α

From H. Brezis and G Stampacchia (cf. reference) we have.

$\theta_\alpha \in C^{1,1}(\bar{\Sigma})$ (set of continuous functions with Lipschitz derivatives) in the two following cases

Σ, convex set of R^2 with Lipschitz boundary.

Σ, non convex set but with more regular boundary.

III.4 Behaviour of θ_α for α "small": elastic solution

Proposition 2: (50) Let $\gamma = \frac{f_\alpha}{2\mu\alpha}$ then:

a) γ is the solution of the Dirichet problem.

(51) $\Delta\gamma + 2 = 0$ on Σ and $\gamma = 0$ on $\partial\Sigma$

b) $\gamma \in W^{2,p}(\Sigma) \cap C^\infty(\Sigma) \cap C^{1,1}(\bar{\Sigma})$ $\forall\ 2 \le p < \infty$.

c) $\gamma > 0$ in Σ

d) $|\operatorname{grad}\gamma|$ is bounded on $\bar{\Sigma}$ and $\sup|\operatorname{grad}\gamma|$ must be reached only on the boundary $\partial\Sigma$

Remark 4 : the practical consequence of this result is the following:

(52) Let $M = \sup_{\bar{\Sigma}} |\operatorname{grad} \gamma|$ from b) we know that M is bounded

from d) we know that $|\operatorname{grad}\gamma| < M$ on Σ

Let us introduce then (53) $\alpha_0 = \dfrac{g}{\mu M}$ then for $0 \leq \alpha \leq \alpha_0$, $|\operatorname{grad} f_\alpha| \leq g$.

$\Rightarrow$ $f_\alpha \in K$ and then $\theta_\alpha = f_\alpha$

then from a)

$$\begin{cases} \Delta\theta_\alpha + 2\mu\alpha = 0 & \text{in } \Sigma. \\ \theta_\alpha = 0 & \text{on the boundary}. \end{cases}$$

$|\operatorname{grad}\theta_\alpha| < g$ on Σ.

We can recognize there the classical solution of elastic torsion for a homogeneous cylindrical bar

Moreover, from d) we know that the plasticity will appear first on the boundary.

Proof of the proposition 2 :

a) $\langle \varphi, \gamma \rangle = 2\int_\Sigma \varphi \, dx \quad \forall \varphi \in H_0^1(\Sigma)$ and in particular.

$\forall \varphi \in \mathcal{D}(\Omega)$ then

$$\int_\Sigma (\operatorname{grad}\varphi \cdot \operatorname{grad}\gamma - 2\varphi) \, dx = 0 \quad \forall \varphi \in \mathcal{D}(\Omega)$$

$$\int -(\Delta\gamma + 2)\varphi \, dx = 0 \quad \forall \varphi \in \mathcal{D}(\Omega).$$

$$\Rightarrow \left.\begin{array}{l} \Delta\gamma + 2 = 0 \quad \text{in } \Sigma. \\ \gamma = 0 \quad \text{on } \partial\Sigma. \end{array}\right\} \text{ homogeneous dirichlet problem. (Lecture notes H.2)}$$

$\Delta\gamma \in L^p \quad \forall p \geq 2$ and $\gamma \in H_0^1(\Sigma) \Rightarrow \gamma \in W^{2,p}(\Sigma) \cap H_0^1(\Sigma)$

$\Rightarrow \gamma \in C^{1,1}(\bar{\Sigma})$

(general results of regularity mentioned orally in the first lecture)

$\gamma \in C^\infty(\Sigma)$ is a consequence of the constant coefficients of (51)

b) c and d are direct consequences of the maximum principle:

In fact: $\Delta\gamma = -2 < 0$ shows that γ is superharmonic and therefore, γ can reach its minimum only on $\partial\Sigma$; $\gamma = 0$ on $\partial\Sigma$ implies then $\gamma > 0$ on Σ.

Otherwise: $\gamma \in C^{1,1}(\bar{\Sigma})$ implies that $|\operatorname{grad}\gamma|$ is bounded on $\bar{\Sigma}$

$$\Delta|\operatorname{grad}\gamma|^2 = 2 \operatorname{grad}\Delta\gamma \cdot \operatorname{grad}\gamma + 2\gamma_{,il}\gamma_{,il} = 2\gamma_{,il}\gamma_{,il} \geq 0 \text{ on } \Sigma$$