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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS TO MECHANICS

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EVOLUTION EQUATIONS

(Part IV)

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4. Variational Method

4-1. Basic results in functional Analysis

Let X a separable Banach space ; We denote by $|\ |_X$ the norm in X.

Let $T>0$; dt denotes the Lebesgue measure on $]0,T[$

$L^p(0T;X)$ (or $L^p(X)$), $1 \leq p \leq +\infty$, is the space of (classes of) functions $t \to f(t) : (0,T) \to X$ which are measurable and such that

$$(1) \quad \int_0^T |f(t)|_X^p \, dt < +\infty \quad , \quad 1 \leq p < +\infty$$

Equipped with the norm :

$$(2) \quad f \to \|f\|_{L^p(X)} = \left(\int_0^T |f(t)|_X^p \, dt\right)^{1/p}, \quad 1 \leq p < +\infty$$

$L^p(X)$ is a Banach space. (an Hilbert space if $p=2$)

We denote by $L^\infty(0T;X)$ (or $L^\infty(X)$) the space of (classes of) functions f which are measurable and which satisfies

$$(1)' \quad f \text{ is bounded a.e. on } (0,T)$$

and we put

$$(2)' \quad \|f\|_{L^\infty(X)} = \inf_{|f(t)|_X \leq M \text{ a.e.}} (M)$$

$L^\infty(X)$ is a Banach space

If X' denotes the dual space of X and $\langle\ ,\ \rangle$ the bracket in this duality then

$$(3)\begin{cases} \text{if } u \in L^1(0,T;X)\ ,\ f \in X' \\ \text{then } \langle f, \int_0^T u(t)\,dt \rangle = \int_0^T \langle f, u(t) \rangle\,dt \end{cases}$$

Since $L^p(X) \hookrightarrow L^1(X)$, $p \geq 1$, (3) is also valid if $u \in L^p$ (with $p \geq 1$).

We now recall the notion of Distribution on $]0,T[$ with values in X. A such Distribution is an element of $\mathcal{L}(\mathcal{D}(]0,T[);X)$ where $\mathcal{D}(]0,T[;X)$ is the space of C^∞ functions with compact support in $]0,T[$.

Thus

$$(4)\quad \mathcal{D}'(]0,T[;X) = \mathcal{L}(\mathcal{D}(]0,T[;X)$$

If $u \in L^1(0,T;X)$ (resp. $L^p(0,T;X)$) then the mapping

$$\varphi \to \int_0^T \varphi(t)\,u(t)\,dt \qquad (\varphi \in \mathcal{D}(]0,T[;\mathbb{C}) = \mathcal{D}(]0,T[)$$

defines one element $T_u \in \mathcal{D}'(]0,T[;X)$ which is identified with u.

Now if $T \in \mathcal{D}'(]0,T[;X)$, $m \in \mathbb{N}^*$ $\varphi \to (-1)^m T\left(\frac{d^m\varphi}{dt^m}\right)$

$\varphi \in \mathcal{D}(]0,T[)$ is a Distribution denoted by $\frac{d^mT}{dt^m}$.

thus we put :

$$(\ )\quad \frac{d^mT}{dt^m}(\varphi) = (-1)^m T\left(\frac{d^m\varphi}{\ }\right)$$

If X, Y are Banach spaces with $X \hookrightarrow Y$

then (6) $\mathcal{D}'(]0,T[; X) \hookrightarrow \mathcal{D}'(]0,T[; Y)$

and (7) $L^2(]0,T[; X) \hookrightarrow L^2(]0,T[; Y)$

Now let $u \in L^2(0,T; X)$; then $\forall \varphi \in \mathcal{D}(]0,T[)$

(8) $\frac{du}{dt}(\varphi) = -\int_0^T u(t)\varphi(t)\,dt$

We shall said that $u' = \frac{du}{dt} \in L^2(0T; Y)$ if there exists $v \in L^2(0T; Y)$ such that $\forall \varphi \in \mathcal{D}(]0,T[)$ $v(\varphi) = -u(\varphi')$

i.e.

(9) $\int_0^T v(t)\,\varphi(t)\,dt = -\int_0^T u(t)\,\varphi'(t)\,dt$.

We denote by $W(X,Y)$ the space:

(10) $W(X,Y) = \{u \mid u \in L^2(X),\ u' \in L^2(Y)\}$

Equipped with the norm $u \to \|u\|_W = \left(\|u\|^2_{L^2(X)} + \|u'\|^2_{L^2(Y)}\right)^{1/2}$

$W(X,Y)$ is a B- Space.

Let V, H are two separable Hilbert spaces. We denote by $\|\ \|$ the norm in V and by $(\ ,\ )$ the scalar product and by $|\cdot|$ the norm in H.

We suppose $V \hookrightarrow H$ ($\hookrightarrow$ Continuous injective mapping)

Then if we identify H and its dual we have

(11) $V \hookrightarrow H \hookrightarrow V'$

if V' denotes the dual of V with a norm denoted by $\| \, \|_*$

(all spaces are here real spaces)

Let $W(V)$ be the Hilbert space:

(12) $W(V) = \{ u \mid u \in L^2(V), \; u' \in L^2(V') \}$

$\left(L^2(X) \equiv L^2(0T; X) \right)$

Then (cf. Lions-Magenes's Book) we have

(13) $W(V) \hookrightarrow \mathcal{C}^0([0,T]; H)$

(14) $\mathcal{D}([0T]; V)$ is dense in $W(V)$

(15) $\begin{cases} u, v \in W(V) \\ \int_0^T \langle u', v \rangle dt + \int_0^T \langle v', u \rangle dt = (u(T), v(T)) - (u(0), v(0)) \end{cases}$

(Green-formula)

As a consequence of (15) we have

(16) $\begin{cases} u \in W(V), \quad v \in V \\ \langle u'(\cdot), v \rangle = \frac{d}{dt} (u(\cdot), v) \quad \text{in } \mathcal{D}'(]0T[) \end{cases}$

Here $\langle , \rangle$ mean the Duality (V', V). Because of the identification of H with H' often we shall denote

$\langle , \rangle$ by $(,)$.

4-2- Galerkin's approximation of V (with variable basis).

We consider V_m a family of (finite dimensional spaces) subspaces of a separable Banach space V which satisfies the following axioms

$$(17) \begin{cases} (i) & V_m \subset V \quad (\dim V_m < +\infty) \\ (ii) & \text{"} V_m \to V \text{"} \text{ when } m \to \infty \text{ in the following sense.} \\ & \text{There exists a dense subspace } \mathcal{V} \text{ of } V \text{ such that} \\ & \forall v \in \mathcal{V}, \text{ there exist a sequence } v_m \in V_m, v_m \to v \text{ in } V. \end{cases}$$

Examples.

1- $w_1, \ldots, w_n, \ldots$ is a family of orthogonal eigenfunctions of a self adjoint operator A.

V_m is spanned by $\{w_1, w_2, \ldots, w_m\}$

then Galerkin's method is the same as Fourier's method if $\{w_j\}$ are the eigenfunctions of the operator A which occurs in the equation.

2- V_m an increasing family of spaces corresponding to a *fixed* "basis" $w_1, \ldots, w_n, \ldots$ of V, whose elements $w_1, \ldots, w_m, \ldots$ are *not* eigenvalues.

It is the usual Galerkin method.

3- General case (17) correspond to ~~...~~ finite elements

Suppose that $V = H_0^1(\Omega)$.

Ω open bounded set of $\mathbb{R}^n$

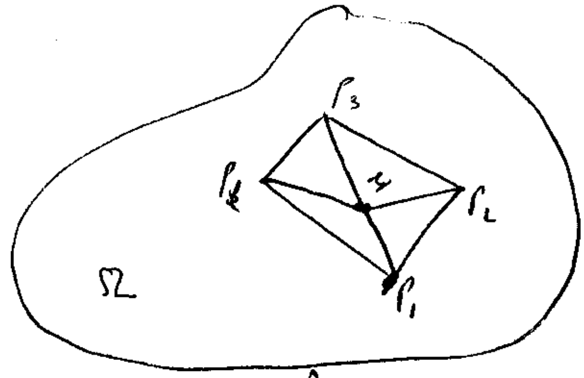

fig 1

We consider a triangulation of Ω : $\mathcal{T}_h$.

$\mathcal{T}_h$ is a family of triangles with the following properties.

1) The family is finite

2) if $T, T' \in \mathcal{T}_h$ and $T \neq T'$ then

or $T \cap T' = \phi$ or $T \cap T' = \{\text{common side}\}$, $T \cap T' = \{\text{common vertex}\}$

3) $\begin{cases} T \subset \Omega, \ \forall T \in \mathcal{T}_h \\ \text{measure}(T) \leq \varphi(h), \ \forall T \in \mathcal{T}_h \end{cases}$ with $\varphi(h) \to 0$ when $h \to 0$

(4) If $\Omega_h = \bigcup_{T \in \mathcal{T}_h} T$, then

$\begin{cases} \text{"}\Omega_h \to \Omega\text{" in the following sense : } \forall \text{ all compact } K \subset \Omega \\ \exists\, h_k \text{ such that } K \subset \Omega_h \quad \forall h \leq h_k \end{cases}$

<u>Space $\hat{V}_h$.</u>

Let M be an "interior" vertex for triangles of $\mathcal{T}_h$; we denote by W_M the linear piecewise function defined on every triangle with vertex M, which satisfies:

$$(18) \begin{cases} W_M(M) = 1 \\ W_M(P_i) = 0, \quad i = 1, \dots k \end{cases} \quad (\text{cf fig 1})$$

... to show that

then we put

$\hat{V}_h$ be the space spanned by W_M where M is an "interior point" for the triangulation $\mathcal{T}_h$.

We have $\dim \hat{V}_h = \mathrm{Card}\{M\}$

If $h = \frac{1}{m}$, $m \in \mathbb{N}^*$, we put $V_m = \hat{V}_h$.

The family V_m satisfies axiom (17) [(i), (ii)], with $\mathcal{V} = \mathcal{D}(\Omega)$.

In fact if $v \in \mathcal{V}$, $v_h = \sum_M v(M) W_{hM}$ and it is possible to show (as a exercise) that $v_h \to v$ in $H_0^1(\Omega)$.

4-3- Linear parabolic Problem-

Le V, H, V' are separable Hilbert spaces which satisfy (11).

$\forall t \in [0,T]$, $a(t; u, v)$ is a bilinear form which continuous on $V \times V$.

We suppose

$$(20)\begin{cases} (i)\ t \to a(t; u, v) \text{ is measurable on } (0,T)\ \forall u, v \in V \\ (ii)\ \exists M \text{ (independent } t\text{) with} \\ \quad |a(t; u, v)| \leq M . \|u\| . \|v\| \quad \forall u, v \in V. \end{cases}$$

$$(21)\begin{cases} \exists \alpha > 0 \text{ and } \lambda \geq 0 \text{ with:} \\ a(t; u, u) \geq \alpha \|u\|^2 - \lambda |u|^2 \quad \forall u \in V \end{cases}$$

Now

(22) $u_0 \in H$ is given

(23) $f \in L^2(V')$ is given.

We state the following result:

Theorem 4-1

There exist one function u, $u \in W(V) \subset \mathcal{C}^0([0,T];H)$

With:

$$(24)\begin{cases}(i)\ \left(\frac{du}{dt},v\right)+a(t;u,v)=(f(t),v) & \forall v\in V\\ (ii)\ u(0)=u_0 .\end{cases}$$

Proof. of the theorem

A- Uniqueness.

If u_1 and u_2 are two solutions of problem (24), $w=u_1-u_2$ satisfies (by Green formula):

$$(25)\begin{cases}\frac{1}{2}\frac{d|w(t)|^2}{dt}+a(t,w(t),w(t))=0\\ W(0)=0\end{cases}$$

Using (21) we have

$$(26)\begin{cases}\frac{1}{2}\frac{d|w(t)|^2}{dt}+\alpha\|w(t)\|^2\leq\lambda|w(t)|^2\\ W(0)=0\end{cases}$$

From (26) we deduce

$$(27)\quad \begin{cases} |W(t)|^2 \leq 2\lambda \int_0^t |W(\sigma)|^2 d\sigma \\ W(0)=0 \end{cases}$$

and the <u>uniqueness</u> follows by Gronwall's inequality.

B. <u>Existence</u>.

a) We use <u>Galerkin's method with variable basis</u>.

Let V_m be a finite dimensional space which satisfies (17). Let $\mu_m = \dim V_m$ and let $\{w_{1,m}, \ldots, w_{\mu_m,m}\}$ a basis of V_m.

We look for $u_m(t) = \sum_{j=1}^{\mu_m} g_{jm}(t)\, w_{j,m}$ as a solution of the finite linear system:

$$(28)\quad \begin{cases} (u_m'(t), w_{jm}) + a(t; u_m(t), w_{jm}) = (f_m, w_{jm}) \\ \qquad 1 \leq j \leq \mu_m \\ u_m(0) = u_{0m}, \quad u_{0m} \to u_0 \text{ in the strong topology of } H. \end{cases}$$

$$\left(f_m = \mathrm{Proj}_{V_m} f, \quad u_{0m} = \mathrm{Proj}_{V_m} u_0\right)$$

There exists $u_m \in \mathcal{C}^0([0,T]; V_m)$, $u_m' \in L^2(0,T; V_m)$ a unique solution of (28).

b) <u>A priori estimates</u>.

We take the product by $g_{jm}(t)$ in equation (28) and sum from 1 to m. Next we integrate between 0 and t, we have:

$$\ldots \frac{1}{2}|u_m|^2 + \int^t (f, u_m)\, d\sigma$$

We deduce from (21)

$$(30)\quad \frac{1}{2}|u_m(t)|^2 + \alpha\int_0^t \|u_m(\sigma)\|^2 d\sigma \le \frac{1}{2\alpha}\int_0^t \|f\|_*^2 d\sigma + \frac{\alpha}{2}\int_0^t \|u_m(\sigma)\|^2 d\sigma + \lambda\int_0^t |u_m(\sigma)|^2 d\sigma + \frac{1}{2}|u_{0m}|^2$$

that is:

$$(31)\quad \frac{1}{2}|u_m(t)|^2 + \frac{\alpha}{2}\int_0^t \|u_m(\sigma)\|^2 d\sigma \le \frac{1}{2}|u_0|^2 + \frac{1}{2\alpha}\int_0^T \|f\|_*^2 d\sigma + \lambda\int_0^t |u_m(\sigma)|$$

By Gronwall's inequality, we have:

$$(32)\quad \sup_{t\in[0,T]} |u_m(t)|^2 \le C \qquad (C = \text{cte indep. of } m \text{ and } t)$$

$$(33)\quad \int_0^t \|u_m(\sigma)\|^2 d\sigma \le C.$$

c) - $m \to +\infty$

By weak compactness of unit ball of an Hilbert space (also for the dual of a normed space), we can extract a subsequence $u_{m'}$ with the following properties:

$$(34)\quad \begin{cases} i)\ u_{m'} \to u \text{ in } L^\infty(H) \text{ in the weak-star topology} \\ ii)\ u_{m'} \to u \text{ in } L^2(V) \text{ in the weak topology} \end{cases}$$

then $A(t)u_{m'}$ ($A(t)$ defined by $a(t,u,v)$ see Lando's lectures) also tend weakly to $A(t)u$ in $L^2(V')$.

Now let $v \in \mathcal{V}$ and $v_{m'}$ the subsequence which satisfies (17)(2i).
Then we have:

$$(35)\quad \begin{cases} (u_{m'}'(t), v_{m'}) + a(t; u_{m'}(t), v_{m'}) = (f_{m'}(t), v_{m'}) \\ u_{m'}^{*}(0) = u_{0m}. \end{cases}$$

Let φ be an element of $\mathcal{D}(]0,T[)$, from (35) we deduce

$$(36)\quad -\int_0^T (u_{m'}(t), v_{m'})\varphi'(t)\,dt + \int_0^T a(t; u_{m'}(t), v_{m'})\varphi(t)\,dt = \int_0^T (f_{m'}(t), v_{m'})\varphi(t)\,dt$$

Now we can pass to the limit in (36) because of the fact that $v_{m'} \to v$ strongly in V and by (34):

We obtain:

$$(37)\quad \begin{cases} -\int_0^T (u(t), v)\varphi'(t)\,dt + \int_0^T a(t; u(t), v)\varphi(t)\,dt = \int_0^T (f(t), v)\varphi(t)\,dt \\ \forall v \in \mathcal{V} \text{ and } \forall \varphi \in \mathcal{D}(]0,T[). \end{cases}$$

This result gives us equ. [(24)-(i)].

We also have $u(0) = u_0$. (Exercise) - (Cf also Next Lecture)

and u is the solution of Problem (24).

Remark:. In fact we can prove that $u_m \to u$ strongly in $L^\infty(H)$ and in $L^2(V)$.

This result may be deduced from the theorem 1-2-

Continuity with respect to the data. (Next lectures)

Bibliography -- Part I.-

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