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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS TO MECHANICS

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EVOLUTION EQUATIONS

MONOTONIC PARABOLIC PROBLEM

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II - Monotonic Parabolic problems

1- Monotonic parabolic equations

1-1. Hypothesis

Let V be a reflexive separable Banach space with norm denoted by $\| \ \|$.

Let H be an Hilbert space; we denote by $(\, , \,)$ the scalar product and by $| \ |$, the norm of H.

We assume that $V \hookrightarrow H$ with continuous injection and density.

Then if we identify H and its dual we have

$$(1) \quad V \hookrightarrow H \hookrightarrow V'$$

where V' is the dual space of V with norm $\| \ \|_*$.

On the other hand let A be an operator (non linear operator)

$$A : V \mapsto V'$$

with the following properties:

(2) A is a <u>monotone</u> operator on V

(i.e. $^{**}(Au - Av, u - v) \geq 0 \quad \forall u, v \in V$)

(3) A is a <u>Bounded</u> operator : i.e. A maps Bounded Sets of V in Bounded Set in V' ; $\exists \, c, c'$ st > 0 $\quad \|Av\|_* \leq c\|v\| + c' \quad \forall v \in V$

(4) A possesses the following coerciveness property:

$\exists \alpha > 0$, $\lambda > 0$ c_1 such that

$$(Av, v) \geq \alpha \|v\|^2 - \lambda |v|^2 - c_1$$

i.e. $\dfrac{(Av,v)}{\|v\|} \to +\infty$ when $\|v\| \to +\infty$.

1-2. An existence and uniqueness Theorem.

Before to state the theorem, we introduce the space: ($T > 0$ given)

$$(5)\quad \mathcal{W}(V) = \{u;\ u \in L^2(0T; V),\ u' \in L^2(0T; V')\}$$

equipped with the norm

$$u \to \|u\|_{\mathcal{W}} = \left(\|u\|^2_{L^2(V)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(V')}\right)^{1/2}$$

$\mathcal{W}(V)$ is a Banach space and we have:

$$(6)\quad \mathcal{W}(V) \hookrightarrow \mathcal{C}^0[[0T]; H].$$

Then:

$\boxed{\text{Theorem 1-1}}$

Let (7) $\begin{cases} - & u_0 \in H \\ - & f \in L^2(0T; V') \end{cases}$

Then there exists one $u \in \mathcal{W}(V)$ which satisfies:

$$(8)\quad \frac{\partial u}{\partial t}(t) + Au(t) = f(t) \qquad \forall t \in]0,T[,\ \text{a.e. (equality in } V')$$

Proof. of Theorem 1.1.

a) Uniqueness.

If u_1 and u_2 are two solutions of (8), (9) and if we put $w = u_1 - u_2$, we have (by Green's formula of Part I which remains valid here)

(10) $\frac{1}{2}\frac{d}{dt}|w(t)|^2 + (Au_1 - Au_2, u_1 - u_2) = 0 \Rightarrow \frac{1}{2}\frac{d|w(t)|^2}{dt} \leq 0$

Since $w(0) = 0$, we obtain $w(t) = 0$. $t \in [0,T]$.

b) Existence of a solution:

We use Galerkin's method with a fixed basis for simplicity.

1) Approximate problem

Let $\{w_1, w_2, \ldots, w_m, \ldots\}$ be a "basis" of V.

Let V_m be spanned by $\{w_1, \ldots, w_m\}$.

We look for $u_m(t)$

(11) $u_m(t) = \sum_{i=1}^{m} g_{im}(t)\, w_i$

which satisfies

(12) $\begin{cases} (u_m'(t), w_j) + (A(u_m(t)), w_j) = (f(t), w_j), & 1 \leq j \leq m \\ u_m(0) = u_{0m} \in V_m & \text{where } u_{0m} \to u_0 \text{ in } H \text{ when } m \to \infty \end{cases}$

by general theory of differential equation system, there exists a local solution $u_m(t)$ for (12) at least in the interval $[0, t_m[$, $t_m > 0$

iii) A priori estimates

Take the product by $g_{jm}(t)$ in (12) and sum from 1 to m, we have

$$(13) \quad (u_m'(t), u_m(t)) + (A(u_m(t), u_m(t)) = (f(t), u_m(t))$$

Integrate on $(0,t)$, $0 < t < t_m$ we obtain

$$-\frac{1}{2}|u_{0m}|^2 + \frac{1}{2}|u_m(t)|^2 + \int_0^t (A u_m(\sigma), u_m(\sigma))\, d\sigma = \int_0^t (f(\sigma), u_m(\sigma))\, d\sigma$$

Using now (4) and hypothesis on f we have:

$$(14) \left\{ \begin{aligned} \frac{1}{2}|u_m(t)|^2 + \alpha \int_0^t \|u_m(\sigma)\|^2 d\sigma \le \frac{1}{2}|u_{0m}|^2 + C_1 t + \lambda \int_0^t |u_m(\sigma)|^2 d\sigma + \\ + \int_0^t \|f(\sigma)\|_* . \|u_m(\sigma)\|\, d\sigma . \end{aligned} \right.$$

But

$$(15) \quad \int_0^t \|f(\sigma)\|_* . \|u_m(\sigma)\|\, d\sigma \le \frac{\alpha}{2}\int_0^t \|u_m(\sigma)\|^2 d\sigma + \frac{1}{2\alpha}\int_0^t \|f(\sigma)\|^2 d\sigma$$

and moreover $|u_{0m}|$ is bounded.

Then

$$(15) \left\{ \begin{aligned} & \frac{1}{2}|u_m(t)|^2 + \frac{\alpha}{2}\int_0^t \|u_m(\sigma)\|^2 d\sigma \le \lambda \int_0^t |u_m(\sigma)|^2 d\sigma + \Gamma_1 \\ & (\text{When } \Gamma_1 \text{ is a constant which depend upon } u_0, T, f, C_1) \end{aligned} \right.$$

By Gronwall's inequality we deduce from (15)

$$(16) \quad |u_m(t)|^2 \le 2C_1 e^{2\lambda t} \le 2C_1 e^{2\lambda T} \qquad 0 \le t \le t_m .$$

apply the result on Differential system with u_{0m} as an initial data. (5)

Now from (16), (15) gives

(17) $\int_0^T \|u_m(\sigma)\|^2 d\sigma \leq \Gamma_2$ (= cte)

We have obtained:

(18) $u_m \in$ Bounded set of $L^2(V) \cap L^\infty(H)$

Since A is a bounded operator we have also

(19) $Au_m \in$ Bounded set of $L^2(V')$

Also from equation (12) we can deduce

(20) $\frac{\partial u_m}{\partial t} \in$ Bounded set of $L^2(V')$)

iii) $\underline{m \to \infty}$.

From (18) – (15) it is possible to extract of the sequence $\{u_m\}$ a sub-sequence $u_{m'}$, such that:

(21) $\begin{cases} u_m \to u & \text{in } L^2(V) \text{ equipped with Weak topology} \\ u_m \to u & \text{in } L^\infty(H) \quad " \quad " \quad \text{Weak-* topology} \\ Au_m \to \chi & \text{in } L^2(V') \quad " \quad " \quad \text{Weak topology} \\ & \qquad\qquad\qquad\qquad\qquad " \quad " \\ u_m(T) \to \xi & \text{in } H \quad " \quad " \end{cases}$

It remains to show that u is the required solution of the problem

iv) u is solution of (8) – (9)

a) Let $v \in V_{m_0}$ and $\varphi \in \mathcal{D}([0T])$

[$\mathcal{D}([0T])$ is the space of restrictions on $[0,T]$ of functions of $\mathcal{D}(\mathbb{R})$]

$\forall m \geq m_0$, we have

$$\int_0^T (u_m'(t), \varphi(t)v)\,dt + \int_0^T (Au_m(t), \varphi(t)v)\,dt = \int_0^T (f(t), \varphi(t)v)\,dt$$

If we integrate by part in the first integral, we obtain

$$(22)\quad (u_m(T), \varphi(T)v) - (u_m(0), v)\varphi(0) - \int_0^T (u_m, v)\varphi'(t)\,dt + \int_0^T (Au_m, v\varphi)\,dt = \int_0^T (f, \varphi v)\,dt$$

Since $u_m(0) \to u_0$ in H in the strong topology and $u_m(T) \to \xi$ in H in the weak topology from (22) we obtain:

$$(23)\quad \begin{cases} (\xi, v)\varphi(T) - (u_0, v)\varphi(0) - \int_0^T (u(t), v)\varphi'(t)\,dt + \int_0^T (\chi - f, v)\varphi\,dt = 0 \\ \forall v \in V_{m_0}. \end{cases}$$

But $\{\bigcup_{m_0} V_{m_0}\}$ is dense in V, thus (23) is valid $\forall v \in V$

b) If we choose $\varphi \in \mathcal{D}(]0,T[)$.

(23) becomes

$$(24)\quad \begin{cases} u' + \chi = f \\ \text{equality in } \mathcal{D}'(]0T[; V') \end{cases}$$

Now since f and $\chi \in L^2(V')$ by (24) $u' \in L^2(V')$ and thus for all $\varphi \in \mathcal{D}([0,T])$ we have

$$\int_0^T u'(t)\varphi(t)\,dt = -\int_0^T u(t)\varphi'(t)\,dt - u(0)\varphi(0) + u(T)\varphi(T)$$

$$(25)\ \begin{cases} (u(T), v)\varphi(T) - (u(0), v)\varphi(0) - \int_0^T (u(t), v)\varphi'(t)\,dt + \int_0^T (\chi - f, v)\varphi(t)\,dt = 0 \\ \forall v \in V,\ \varphi \in \mathcal{D}([0,T]). \end{cases}$$

$$(23) + (25) \Rightarrow \quad \begin{cases} u(T)\varphi(T) - u(0)\varphi(0) = \xi\varphi(T) - u_0\varphi(0) \\ \forall \varphi \in \mathcal{D}([0,T]) \end{cases}$$

(We choose φ with $\varphi(0)=0$, and next φ with $\varphi(T)=0$)

and (26) $\underline{u(0) = u_0}$, $\underline{\xi = u(T)}$;

It remains to show $\chi = Au$.

c)- $\underline{\chi = Au}$ $\underline{\text{Minty's method}}$

Let $v \in L^2(V)$, we have

$$(27)\quad \int_0^T (Au_m - Av, u_m - v)\,dt \geq 0$$

We use Eq. (13) to compute the term $\int_0^T (Au_m, u_m)\,dt$ and we obtain

$$\int_0^T (f, u_m)\,dt - \int_0^T (u_m'(t), u_m(t))\,dt + \int_0^T (Av, v - u_m)\,dt - \int_0^T (Au_m, v)\,dt \geq 0$$

or

$$(28)\quad \frac{1}{2}|u_m(T)|^2 - \frac{1}{2}|u_{0m}|^2 \leq \int_0^T [(f, u_m) - (Au_m, v) - (Av, u_m - v)]\,dt$$

Now we take inferior limit in inequality (28), because of the lower semi continuity of the norm in a B-space, all obtain

$$\frac{1}{2}|u(T)|^2 - \frac{1}{2}|u_0|^2 \leq \int_0^T ((f, u) - (\chi, v) - (Av, u - v))\,dt$$

(29) $\int_0^T (\chi - Av, u-v)\, dt \geq 0$.

Then let $w \in L^2(V)$ and $\lambda > 0$. We choose $v = u - \lambda w$ and put it in (29), we deduce from (29):

(30) $\int_0^T (\chi - A(u-\lambda w), w)\, dt \geq 0$

Because of the hemicontinuity of A, when $\lambda \to 0$ we obtain

(31) $\int_0^T (\chi - Au, w)\, dt \geq 0 \quad \forall w \in L^2(V)$

(31) implies

$$\chi = Au \quad \text{in } L^2(V')$$

Theorem 1-1. follows

1-3. Continuity with respect to the data.

Theorem 1.2 We give us V, H and A as in Section 1.1.

Let $u_0 \in H$, $u_0^* \in H$, $f \in L^2(V')$, $f^* \in L^2(V')$ are given; let u and u^* be the ~~solutions~~ corresponding solutions with data (u_0, f), (u_0^*, f^*).

Moreover we assume that A satisfie:

$\ldots \mid \exists \alpha_1 > 0,\ \lambda_1 \geq 0$, $(Aw - Av, w-v) \geq \alpha_1 \|w-v\|^2 - \lambda_1 |w-v|^2$

Then (11)

$$(33)\quad \begin{cases} \|u^*-u\|^2_{L^\infty(H)} \leq e^{2\lambda T}\left(\frac{1}{\alpha_1}\|f^*-f\|^2_{L^2(V')} + |u_0^*-u_0|^2\right) \\ \|u^*-u\|^2_{L^2(V)} \leq e^{\frac{2\lambda T}{\alpha_1}}\left(\frac{1}{\alpha_1}\|f^*-f\|^2_{L^2(V')} + |u_0^*-u_0|^2\right) \end{cases}$$

Remark: This result implies strong convergence in Galerkin's method.

Proof: Put $W(t) = u^*(t) - u(t)$, $u_0^* - u_0 = W(0)$, $f^* - f = g$

we have

$$(W', W) + (Au^* - Au, W) = (g, W)$$

As in eq. (13) we deduce

$$\frac{1}{2}|W(t)|^2 - \frac{1}{2}|W(0)|^2 + \alpha_1\int_0^t \|W(\sigma)\|^2 d\sigma \leq \frac{1}{2\alpha_1}\int_0^t \|g\|_*^2 d\sigma + \lambda\int_0^t |W(\sigma)|^2 d\sigma + \frac{\alpha_1}{2}\int_0^t \|W(\sigma)\|^2 d\sigma$$

thus

$$(34)\quad |W(t)|^2 + \alpha_1\int_0^t \|W(\sigma)\|^2 d\sigma \leq 2\lambda\int_0^t |W(\sigma)|^2 d\sigma + \frac{1}{\alpha_1}\int_0^T \|g\|_*^2 d\sigma + |W(0)|$$

by Gronwall's Inequality first we obtain

$$|W(t)|^2 \leq \frac{1}{\alpha_1}\left(\int_0^T \|g\|_*^2 d\sigma + |W(0)|^2\right)e^{2\lambda t}$$

and next we have

$$(35)\quad \alpha\int_0^t \|W(\sigma)\|^2 d\sigma \leq \frac{1}{\alpha_1}\left(\|g_*\|^2_{L^2(V')} + |W(0)|^2\right)e^{2\lambda t}$$

2- Monotonic Parabolic Variational Inequalities.

2-1 - Statement of the result

Let V, H, V' are as in section 1 and let A be a monotone, bounded, hemicontinuous, coercive operator : $V \to V'$.

We also assume:

$$(1) \begin{cases} \exists \alpha > 0, \lambda \geq 0 \text{ with} \\ (Aw - Av, w - v) \geq \alpha_1 \|w - v\|^2 - \lambda |w - v|^2. \end{cases}$$

Let K be a closed convex set in V, with $0 \in K$.

We suppose that

(2) $u_0 \in K$ (u_0 given)

and

f given with

(3) (i) $f, f' \in L^2(0T; V') = L^2(V')$, and (3)(ii) $Au_0 - f(0) \in H$

Theorem 2-1

There exists one $u \in \mathcal{W}(V)$ with $u' \in L^2(V) \cap L^\infty(H)$

which satisfies

$$(4) \begin{cases} (i) \ (u', v - u) + (Au, v - u) \geq (f, v - u) \quad \forall v \in K, \ \forall t \in (0, T) \\ (ii) \ u(t) \in K \quad t \in [0, T] \\ (iii) \ u(0) = u_0. \end{cases}$$

A. Uniqueness

Let u_1, u_2 are two solutions of (4). Then we have

$$(5)\begin{cases} (u_i', v-u_i) + (Au_i, v-u_i) \geq (f, v-u_i) & i=1,2 \\ u_i(t) \in K & \forall v \in K. \\ u_i(0) = u_0 \end{cases}$$

Takes $v = u_2$ in eq. (5) with $i=1$, and $v = u_1$ in eq. (5) with $i=2$.

If we put $w = u_1 - u_2$ we obtain:

$$(6)\quad \frac{1}{2}\frac{d}{dt}|w(t)|^2 + \underbrace{(A(u_1) - A(u_2), u_1 - u_2)}_{\geq 0} \leq 0 \quad ; \quad w(0)=0$$

and it follows that $w(t) = 0$, $t \in (0,T)$.

B. Existence.

Let β be a penalty operator from V to V' i.e. a monotone hemicontinuous, bounded operator such that $\beta(v) = 0 \Leftrightarrow v \in K$ (as $0 \in K$ we have $\beta(0) = 0$ and $(\beta(0), v) \geq 0$ $\forall v \in V$) there exists such operators (Cf. Lions. "Quelques methodes ....")

Let ε be >0 and let u_ε be the solution (which exist by Th. 1-1) of the problem:

$$(7)\begin{cases} u_\varepsilon' + Au_\varepsilon + \frac{1}{\varepsilon}\beta u_\varepsilon = f, & t \in]0,T[. \\ u_\varepsilon(0) = u_0 \end{cases}$$

In order to pass at the limit when $\varepsilon \to 0$, we first obtain

A priori estimates (I).

In a same way as for eq. [(13) - Section 1] we obtain:

(8) $$\frac{1}{2}|u_\varepsilon(t)|^2 + \alpha\int_0^t \|u_\varepsilon(\sigma)\|^2 d\sigma \leq \lambda\int_0^t |u_\varepsilon(\sigma)|^2 d\sigma + C_1 + \int_0^t (f(\sigma), u_\varepsilon(\sigma))d\sigma + \frac{1}{2}|u_0|^2$$

(because of $(\beta(\varepsilon)u_\varepsilon, u_\varepsilon) \geq 0$)

We deduce from (8):

$$\frac{1}{2}|u_\varepsilon(t)|^2 + \frac{\alpha}{2}\int_0^t \|u_\varepsilon(\sigma)\|^2 d\sigma \leq \lambda\int_0^t |u_\varepsilon(\sigma)|^2 d\sigma + \frac{1}{2\alpha}\int_0^t \|f(\sigma)\|^2 d\sigma + C$$

and by Gronwall's inequality we obtain:

(9) $$\sup_{t\in[0,T]} |u_\varepsilon(t)|^2 \leq C \qquad (C \text{ cte independent on } \varepsilon)$$

(10) $$\int_0^T \|u_\varepsilon(t)\|^2 dt \leq C'$$

A priori estimates (II)

a) First we state a lemma.

$\boxed{\text{Lemma}}$

Let φ be a function : $[0,T] \to \mathbb{R}$ such that:

(11) $\varphi \in \mathcal{W}(0,T) = \{\varphi \mid \varphi \in L^2((0,T), \mathbb{R}),\ \varphi' \in L^2(0,T;\mathbb{R})\}$

Let $h>0$ and let φ_h be the function:

(12) $$\varphi_h : \begin{cases} \varphi_h(t) = \dfrac{\varphi(t+h)-\varphi(t)}{h} & \text{if } t \in [0, T-h] \\ \varphi_h(t) = 0 & \text{if } t \in]T-h, T] \end{cases}$$

Then $\|\varphi_h\|_{L^2} \leq \|\varphi'\|_{L^2}$

b) Now let $t \in [0,T[$ and $h > 0$ such that $t+h \in [0,T]$

let $u_{\varepsilon n}$ be Galerkin's approximation of order n for u_ε.

(Cf. Proof of th. 1.1)

We have

$$(14) \begin{cases} (u'_{\varepsilon n}(t+h), w_i) + (\hat{A} u_{\varepsilon n}(t+h), w_i) = (f(t+h), w_i) \\ (u'_{\varepsilon n}(t), w_i) + (\hat{A} u_{\varepsilon n}(t), w_i) = (f(t), w_i) \end{cases}$$

Where $\hat{A} = A + \frac{1}{\varepsilon}\beta$.

From (14) we obtain

$$(15) \quad (u'_{\varepsilon n}(t+h) - u'_{\varepsilon n}(t), w_i) + (\hat{A} u_{\varepsilon n}(t+h) - \hat{A} u_{\varepsilon n}(t), w_i) = (f(t+h) - f(t), w_i)$$

We can replace w_i by $u_{\varepsilon n}(t+h) - u_{\varepsilon n}(t)$.

Then in a same way that for eq (13) section §1 we have (using (1))

$$(16) \begin{cases} W_n = \dfrac{u_{\varepsilon n}(t+h) - u_{\varepsilon n}(t)}{h} \\ \dfrac{1}{2}\dfrac{d}{dt}|W_n|^2 + \alpha_1 \|W_n(t)\|^2 - \lambda |W_n(t)|^2 \leq \left(\dfrac{f(t+h) - f(t)}{h}, W_n(t)\right) \end{cases}$$

Now by integration on $(0,t)$

$$(17) \quad \frac{1}{2}|W_n(t)|^2 + \alpha_1 \int_0^t \|W_n(\tau)\|^2 d\tau - \lambda \int_0^t |W_n(\tau)|^2 d\tau \leq \int_0^t \left(\frac{f(\tau+h) - f(\tau)}{h}, W_n(\tau)\right) d\tau + \frac{1}{2}|W_n(0)|^2$$

In the Galerkin's approximation of u_ε we can choose $w_1 = u_0$

(17)

$$u_{\varepsilon n} = \sum_{j=1} g_{nj}(t)\, w_j$$

and

$$(18) \quad \sum_{j=1}^{n} g'_{nj}(t)(w_j, w_i) = (f - \hat{A} u_{\varepsilon n}, w_i)$$

From properties of $\hat{A}$ and f we have

$$(19) \qquad g_{nj} \in \mathcal{C}^1(0,T;\mathbb{R})$$

Then let

$$(20)\ X = \int_0^t \left(\frac{f(\sigma+h) - f(\sigma)}{h}, w_n(\sigma) \right) d\sigma = \sum_{j=1}^{n} \int_0^t \left(\frac{f(\sigma+h)-f(\sigma)}{h}, w_j \right)\left(\frac{g_{nj}(\sigma+h)-g_{nj}(\sigma)}{h} \right) d\sigma$$

Functions $\varphi_j(\sigma)$ given by $\varphi_j(\sigma) = (f(\sigma), w_j)$ satisfies the hypothesis of lemma and also g_{nj}. It follows that it is possible to take the limit in X to obtain

$$(21) \quad \lim_{h\to 0} X = \int_0^t (f'(\sigma), u'_{\varepsilon n}(\sigma))\, d\sigma$$

Since $W_n(\sigma) \in$ a finite dimensional space and because of (19) we have:

$$(22) \begin{cases} \lim\limits_{h\to 0} W_n(\sigma) = u'_{\varepsilon n}(\sigma) \\ \lim\limits_{h\to 0} |W_n(\sigma)| = |u'_{\varepsilon n}(\sigma)| \\ \lim\limits_{h\to 0} \|W_n(\sigma)\| = \|u'_{\varepsilon n}(\sigma)\| \end{cases}$$

and since W_n is bounded on $(0,t)$ with values in V or H

$$(23)\quad \begin{cases} \lim\limits_{\varepsilon\to 0} \int_0^t |W_n(\sigma)|^2\, d\sigma = \int_0^t |u'_{\varepsilon n}(\sigma)|^2\, d\sigma \\[2ex] \lim\limits_{\varepsilon\to 0} \int_0^t \|W_n(\sigma)\|^2\, d\sigma = \int_0^t \|u'_{\varepsilon n}(\sigma)\|^2\, d\sigma \end{cases}$$

Now it is possible to pass at the limit in eq. (17) to obtain

$$(24)\quad \begin{cases} \frac{1}{2}|u'_{\varepsilon n}(t)|^2 + \alpha_1 \int_0^t \|u'_{\varepsilon n}(\sigma)\|^2 d\sigma - \lambda \int_0^t |u'_{\varepsilon n}(\sigma)|^2 d\sigma \le \\ \qquad \le \int_0^t (f'(\sigma), u'_{\varepsilon n}(\sigma))\, d\sigma + \frac{1}{2}|u'_{\varepsilon n}(0)|^2 \end{cases}$$

If we put $t=0$ in eq. (18) we obtain

$$(u'_{\varepsilon n}(0), w_i) = (f(0) - Au_0, w_i) \qquad \forall\, i = 1, \dots, n$$

Which implies because of ((3) - ii)

$$(25)\qquad |u'_{\varepsilon n}(0)| < |f(0) - Au_0|$$

then by (24) - (25) we deduce

$$(26)\quad \begin{cases} \frac{1}{2}|u'_{\varepsilon n}(t)|^2 + \frac{\alpha_1}{2}\int_0^t \|u'_{\varepsilon n}(\sigma)\|^2 d\sigma \le \lambda \int_0^t |u'_{\varepsilon n}(\sigma)|^2 d\sigma + \frac{1}{2\alpha_1}\int_0^t \|f'(\sigma)\|^2 d\sigma \\ \qquad + \frac{1}{2}|f(0) - Au_0|^2 . \end{cases}$$

Using Gronwall's Inequality we have:

$$(27)\quad |u'_{\varepsilon n}(t)|^2 \le \left[\frac{1}{\alpha_1}\int_0^T \|f'(\sigma)\|_*^2 d\sigma + |f(0) - Au_0|^2\right] e^{2\lambda t}$$

$$(28)\quad \int_0^t \|u'_{\varepsilon n}(\sigma)\|^2 d\sigma \le \left[\frac{1}{\dots}\int_0^T \|f'(\sigma)\|_*^2 d\sigma + |f(0) - Au_0|^2\right] e^{2\lambda t}$$

but when $\eta \to \infty$ we know that $u_{\varepsilon\eta} \to u_\varepsilon$ in the weak topology of $L^\infty(H)$ (w-*) and $L^2(V)$. It is the same for $u'_{\varepsilon\eta}$ and we have (16

(28) $$|u'_\varepsilon(t)|^2 \leq \left[\frac{1}{\alpha_1}\int_0^T \|f'(\sigma)\|_*^2\, d\sigma + |f(0) - Au_0|^2\right] e^{2\lambda T}$$

(28)' $$\alpha_1 \int_0^T \|u'_\varepsilon\|^2 dr \leq \left[\frac{1}{\alpha_1}\int_0^T \|f'(\sigma)\|_*^2\, d\sigma + |f(0) - Au_0|^2\right] e^{2\lambda T}$$

c) $\varepsilon \to 0$

We can extract of u_ε a subsequence still denoted by u_ε such that:

(29) $$\begin{cases} \text{(i)}\ u_\varepsilon \to u & \text{in } L^\infty(H) \text{ weak-star and } L^2(V) \text{ weak top.} \\ \text{(ii)}\ u'_\varepsilon \to u' & \text{" " " " " "} \\ \text{(iii)}\ u_\varepsilon(t) \to u(t) & \text{in } H \text{ weakly } \forall t \in [0,T] \end{cases}$$

moreover

(30) $$Au_\varepsilon \to \chi \quad \text{in } L^2(V') \text{ weakly}$$

Using eq. (27) we have

$$\left\|\frac{1}{\varepsilon}\beta(u_\varepsilon)\right\|_* \leq C \implies \beta(u_\varepsilon) \to 0 \text{ in strong topology of } L^2(V')$$

Since β is a monotone operator then we have

(31) $$u(t) \in K, \quad t \in [0,T].$$

Let $v \in W(V)$ with $v(t) \in K$, $t \in [0,T]$

we have

$$\int_0^T (\ldots)\, dr + \int_0^T (A\ldots, v - u_\varepsilon)\, dr + \frac{1}{\varepsilon}\int_0^T (\beta(u_\varepsilon), v - u_\varepsilon)\, dr = 0$$

But $\forall u \in K \Rightarrow \beta(v)=0$ and

$$\int_0^T (\beta(u_\varepsilon), v-u_\varepsilon)\,dt = \int_0^T (\beta(u_\varepsilon)-\beta(v), u_\varepsilon - v)\,dt \leq 0$$

Then

$$(33)\quad \int_0^T (u'_\varepsilon, v-u_\varepsilon)\,dt + \int_0^T (Au_\varepsilon - f, v-u_\varepsilon)\,dt \geq 0$$

It is possible to write (33) in the form:

$$(34)\quad \int_0^T (Au_\varepsilon - f, v-u_\varepsilon)\,dt + \int_0^T (u'_\varepsilon, v)\,dt \geq \frac{1}{2}|u_\varepsilon(T)|^2 - \frac{1}{2}|u_0|^2$$

By lower semi-continuous property of the norm we have:

$$(35)\quad \inf.\lim \int_0^T (Au_\varepsilon - f, v-u_\varepsilon)\,dt + \int_0^T (u', v-u)\,dt \geq 0$$

We can choose $\underline{v=u}$ in (35) and this choice implies

$$(36)\quad \liminf \int_0^T (Au_\varepsilon, u-u_\varepsilon)\,dt \geq 0.$$

On the other hand by monotonicity of A we have

$$\int_0^T (Au_\varepsilon, u-u_\varepsilon)\,dt \leq \int_0^T (Au, u-u_\varepsilon)\,dt$$

and also

$$(37)\quad \cancel{\sup}\ \cancel{\lim_\varepsilon} \int_0^T (Au_\varepsilon, u-u_\varepsilon)\,dt \leq 0$$

Thus (36) and (37) imply

$$(38)\quad \lim_{\varepsilon\to 0} \int_0^T (Au_\varepsilon, u-u_\varepsilon)\,dt = 0$$

Then we have

(18)

$$(39)\quad \lim_{\varepsilon\to 0}\int_0^T (Au_\varepsilon, v-u_\varepsilon)\,dr = \lim_{\varepsilon\to 0}\int_0^T (Au_\varepsilon, v-u)\,dr + \lim_{\varepsilon\to 0}\int_0^T (Au_\varepsilon, u-u_\varepsilon)\,dr =$$

$$= \int_0^T (\chi, v-u)\,dr.$$

Now (35) implies

$$(40)\quad \int_0^T (\chi-f, v-u)\,dr + \int_0^T (u', v-u)\,dr \geq 0$$

Chooses $v_0 \in K$, $t_0 \in [0,T]$ and define $v \in W(V)$ by

$$(41)\quad \begin{cases} v(t) = 0 & \text{if } t \notin [t_0-\eta, t_0+\eta] \subset [0,T] \\ v(t) = v_0 & \text{if } t \in [t_0-\eta, t_0+\eta] \end{cases}$$

(40) implies

$$(42)\quad \frac{1}{2\eta}\int_{t_0-\eta}^{t_0+\eta} (u'+\chi-f, v_0-u)\,dr \geq 0$$

the theoreme of Lebesgue said us that if $\eta \to 0$

$(u'+\chi-f, v_0-u)(t_0) \geq 0$ except if $t_0 \in$ set of measure $= 0$

thus we have

$$(43)\quad (u'+\chi-f, v-u) \geq 0 \quad \forall v \in K, \quad t \in [0,T] \text{ a.e.}$$

To prove th. 2-1 it remains to prove $\underline{Au = \chi}$

Let $v \in L^2(V)$ we put

$$(44)\quad Y = \int_0^T (Av - Au_\varepsilon, v-u_\varepsilon)\,dt \geq 0$$

We have

$$\lim_{\varepsilon\to 0} Y = \lim_{\varepsilon\to 0} \int_0^T (Av - Au_\varepsilon, v-u)\,dt + \lim_{\varepsilon\to 0} \int_0^T (Av - Au_\varepsilon, u-u_\varepsilon)\,dt$$

by (38) it follows

$$(45) \qquad \lim_{\varepsilon\to 0} Y = \int_0^T (Av - \chi, v-u)\,dt \geq 0$$

By Minty's process we obtain as in section 1 - $Au = \chi$.

Since it is clear by [(29), iii) that $u(0) = u_0$.

The theorem is proved.

Bibliography (Part II).

1- J. L. LIONS. Quelques méthodes de résolution de problèmes aux limites non linéaires. Dunod Gauthier-Villars Paris 1969

2- J. L. LIONS. E. MAGENES. Vol 1 Pbs. aux limites non homogènes Dunod - Paris 1968

3- M. Artola lectures. Last Session 1973 -