INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

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SMR/23 / 25

AUTUMN COURSE ON APPLICATIONS OF ANALYSIS TO MECHANICS

22 September - 26 November 1976

FUNCTIONAL ANALYTIC METHODS IN ELASTIC-PLASTIC TORSION OF CYLINDRICAL BARS

Part III

and Summary

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Functional analytic methods in elastic-plastic torsion of cylindrical bars.

H. Lanchon (Nancy - France)

Summary

III. 5 Behaviour of θ_α for $\alpha_0 < \alpha < \infty$ elastic-plastic solutions.

For $\alpha > \alpha_0$, it is clear that $f_\alpha \notin K$ so $\theta_\alpha \neq f_\alpha$; the solution is no longer completely elastic. Let introduce then:

(54) $E_\alpha = \{ x / x \in \Sigma$ such that $|\text{grad}\, \theta_\alpha| < g \}$.

which is a priori the elastic part of the cross section; E_α is an open set of Σ thanks to the property $\theta_\alpha \in C^{1,1}(\bar{\Sigma})$

(55) $P_\alpha = \{ x / x \in \Sigma$ such that $|\text{grad}\, \theta_\alpha| = g \}$ is the plastic part of Σ.

We can prove the following result:

<u>Proposition 3</u>: <u>for $\alpha_0 < \alpha \leq \infty$, $P_\alpha \neq \Phi$ (empty set) and we have.</u>

(56) <u>$\Delta \theta_\alpha + 2\mu\alpha = 0$ on E_α and $\theta_\alpha \in C^\infty(E_\alpha)$</u>

Proof: We introduce

$$\Sigma_{g_n} = \{ x / x \in \Sigma \text{ s.t } |\text{grad}\, \theta_\alpha(x)| \leq g - \tfrac{1}{n} \}.$$

and for $\psi \in \mathcal{D}(\Sigma_{g_n})$:

$$\varphi_1 = \begin{cases} \theta_\alpha - \dfrac{\psi}{n h} \text{ on } \Sigma_{g_n} \text{ with } h = \sup\limits_{x \in \Sigma_{g_n}} |\text{grad}\, \psi(x)| \\ \theta_\alpha \text{ on } \Sigma - \Sigma_{g_n}. \end{cases}$$

then φ_1 and $\varphi_2 = 2\theta_\alpha - \varphi_1 \in K$

By successive application of (48) at φ_1 and φ_2, we prove that.

$$\int_{\Sigma_{g_n}} \text{grad}(\theta_\alpha - f_\alpha)\, \text{grad}\, \psi \, dx = 0 \qquad \forall \psi \in \mathcal{D}(\Sigma_{g_n})$$

and taking the limit when $n \longrightarrow \infty$ we obtain.

$$\int_{E_\alpha} \text{grad}(\theta_\alpha - f_\alpha)\, \text{grad}\, \psi \, dx = 0 \qquad \forall \psi \in \mathcal{D}(\Sigma_\alpha)$$

Whence we conclude the proposition.

III. 6 Behaviour of θ_α when $\alpha \longrightarrow \infty$: completely plastic solution

small displacements. However the behaviour of θ_α when $\alpha \to \infty$ will be very useful for the mathematic study. We obtain then the following result.

<u>Proposition 4</u>: <u>When $\alpha \longrightarrow \infty$ then</u>

a) <u>θ_α converges uniformely toward $\theta_\infty \in K$</u>

b) <u>$\theta_\infty(x) = g\,\delta(x, \partial\Sigma)$ where $\delta(x,\partial\Sigma)$ is the distance from x to $\partial\Sigma$</u>

c) <u>$|\text{grad}\,\theta_\infty| = g$ almost everywhere</u>

Proof: (48) can be written

$$\frac{\langle\theta_\alpha, \theta_\alpha\rangle}{\mu\alpha} - \frac{\langle\theta_\alpha, \varphi\rangle}{\mu\alpha} - \langle\gamma, \theta_\alpha\rangle + \langle\gamma, \varphi\rangle \leq 0 \qquad \forall\, \varphi \in K$$

and when $\alpha \longrightarrow \infty$, θ_α describes an infinite subset of K which is uniformely compact (cf properties of K).

So, $\exists$ a subsequence θ_{α_i} of $\{\theta_\alpha\}$ which converges uniformely to an element $\theta_\infty \in K$; the above mentioned inequality becomes.

(57) $\qquad \langle\gamma, \varphi - \theta_\infty\rangle \leq 0 \qquad \forall\, \varphi \in K$

The uniqueness of θ_∞ come from the fact that, if $\exists$ two such functions $\theta_\infty^{(1)} \neq \theta_\infty^{(2)}$, then $\varphi = \sup\left[\theta_\infty^{(1)}, \theta_\infty^{(2)}\right] \in K$ (cf. properties of K) and satisfies $\langle\gamma, \varphi - \theta_\infty^{(i)}\rangle > 0$ whence the contradiction.

In the other hand,

$g\,[\delta(x,\partial\Sigma)] = \psi(x)$, upper envelopp of K [intuitively obvious)

and we can easily show that:

$\psi \in K$ and satisfies $\quad \langle\gamma, \varphi - \psi\rangle \leq 0 \quad \forall\, \varphi \in K$

Therefore: $\quad \theta_\infty = \psi = g\,\delta \qquad$ q.e.d.

c) is very easy to prove from b)

So <u>θ_∞ can be called the completely plastic solution</u>

III.1 <u>General behaviour of θ_α as a function of α</u>

<u>Proposition 5</u> <u>If. $\alpha < \alpha'$ then $\theta_\alpha < \theta_{\alpha'}$ on a subset of non zero measure and in particular</u>

Proof : i) $\theta_\alpha(x) \le \theta_{\alpha'}(x)$ if $\alpha < \alpha'$. In fact

Let us define.

$$\Sigma_0 = \{ x \,/\, x \in \Sigma \ \text{ s.t } \ \theta_\alpha(x) > \theta_{\alpha'}(x) \}$$
$$\Sigma_1 = \{ x \,/\, x \in \Sigma \ \text{ s.t } \ \theta_\alpha(x) \le \theta_{\alpha'}(x) \}$$

and let us apply the fundamental inequality (48) successively to φ_1 and φ_2 defined by.

$$\varphi_1(x) = \max\{\theta_\alpha(x), \theta_{\alpha'}(x)\} \quad \text{and} \quad \varphi_2(x) = \min\{\theta_\alpha(x), \theta_{\alpha'}(x)\}$$

We obtain by combination

$$(\alpha' - \alpha) \int_{\Sigma_0} \{\theta_{\alpha'}(x) - \theta_\alpha(x)\}\, dx \geqslant 0 \quad \text{which implies } \Sigma_0 = \phi$$

q.e.d

ii) if $\theta_\alpha(x) = \theta_{\alpha'}(x)$ everywhere on Σ then $\theta_\alpha = \theta_\infty$

Let us suppose in fact $\theta_\alpha = \theta_{\alpha'} \neq \theta_\infty$, then $\exists\, \Sigma_\alpha \subset \Sigma$ of non zero measure such that

$$|\operatorname{grad} \theta_\alpha| < g;$$

then on E_α, $\Delta\theta_\alpha = \Delta\theta_{\alpha'} = -2\mu\alpha = -2\mu\alpha'$ on E_α; that is impossible because $\alpha < \alpha'$

iii) For a finite α, $\theta_\alpha \neq \theta_\infty$

Let us suppose $\theta_\alpha = \theta_\infty$, then $\langle \theta_\infty - \varphi, \theta_\infty - f_\alpha \rangle \le 0 \quad \forall \varphi \in K$.

Let us introduce $M_\infty = \sup\limits_{x \in \Sigma} \theta_\infty(x)$ and for $0 < h < M_\infty$. The function

$$\varphi = \begin{cases} h & \text{on } \Sigma_h = \{x \,/\, x \in \Sigma \ ; \ \theta_\infty(x) \geqslant h\}. \\ \theta_\infty & \text{elsewhere.} \end{cases}$$

$\varphi \in K$; Let apply the fundamental inequality for this particular φ, we obtain $M_\infty - h \geqslant \dfrac{g^2}{2\mu\alpha} \quad \forall\, h \in]0, M_\infty[$ and that is trivially false. Whence the contradiction.

III.8. New definition of the elastic and plastic regions.

Proposition 6 : if we introduce

(58) $E'_\alpha = \{ x \,/\, x \in \Sigma,\ \theta_\alpha(x) < \theta_\infty(x) \}$.

then $E_\alpha = E'_\alpha$ and Therefore

(59) $P_\alpha = \{ x \,/\, x \in \Sigma,\ \theta_\alpha(x) = \theta_\infty(x) \}$

(60) $K' = \{ \varphi / \varphi \in H_0^1(\Sigma)$ s.t. $|\varphi| \leq \theta_\infty$ a.e on $\Sigma \}$; $K \subset K'$

H. Brezis and M Sibony (cf references) proved that the two problems

(48) $\theta_\alpha \in K$; $\langle \theta_\alpha - \varphi, \theta_\alpha - f_\alpha \rangle \leq 0 \quad \forall \varphi \in K$. and

(61) $\theta'_\alpha \in K'$; $\langle \theta'_\alpha - \varphi, \theta'_\alpha - f_\alpha \rangle \leq 0 \quad \forall \varphi \in K'$

have the same solution $\theta_\alpha = \theta'_\alpha$.

We use then the same methode as in the proposition 3 but in replacing Σg_n by.

$$\Sigma_n = \{ x / x \in \Sigma, \ \theta_\alpha(x) \leqq \theta_\infty(x) - \tfrac{1}{n} \}$$

to show that $\theta_\alpha \in C^\infty(E'_\alpha)$ and satisfied.

$$\Delta \theta_\alpha + 2\mu\alpha = 0 \quad \text{on } E'_\alpha$$

The maximum principle shows then : $|\operatorname{grad} \theta_\alpha| < g$ on E'_α hence. $E'_\alpha \subset E_\alpha$; that finishes the proove because.

$$\theta_\alpha(x) = \theta_\infty(x) \quad \text{on} \quad P'_\alpha = \complement_\Sigma E'_\alpha$$

now it is obvious that $P'_\alpha \subset P_\alpha$ and therefore.

$$E'_\alpha = \complement_\Sigma P'_\alpha \supset \complement_\Sigma P_\alpha = E_\alpha. \quad \Rightarrow E'_\alpha = E_\alpha \text{ and } P'_\alpha = P_\alpha$$

<u>Remark 5</u>: These two definitions of plastic and elastic regions are going to be very useful thanks to the simplicity of the function.

$$\theta_\infty(x) = g\,\delta(x, \partial\Sigma)$$

very well known for all shape of cross section.

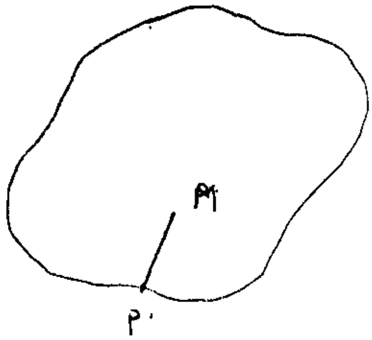
III.3 <u>Evolution of the plastic regions.</u>

<u>Remark 6</u> : For $\alpha \leq \alpha_0$, we saw that the whole cross section stay elastic. and that the maximum of $|\operatorname{grad} \theta_\alpha|$ is reached on $\partial\Sigma$. That leads us to think that the plasticity will appear first on a point of $\partial\Sigma$ and then is propagated by degrees therefrom. We are able to obtain that result rigorously.

In effect, these new definitions of E_α and P_α are essential to establish the following concrete properties which are now easy to prove.

i) <u>Contiguity of plastic regions with the boundary $\partial\Sigma$.</u>

We can show that; if $M \in P_\alpha$ and if $\varphi \in \partial\Sigma$ such that. $|M\varphi| = \delta(M, \partial\Sigma)$ then $M\varphi \subset P_\alpha$



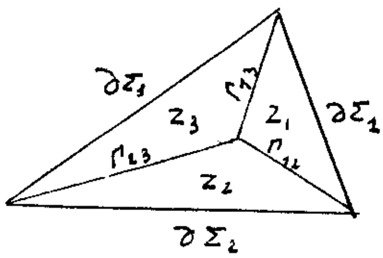
ii) <u>Partition of the cross section in zones of influence of the different parts of the boundary</u>

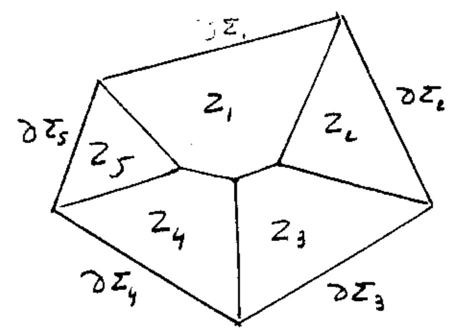
Let Γ be the set of point of Σ which are equidistant of at least two points of $\partial\Sigma$.

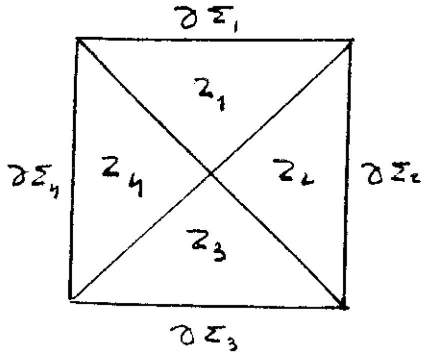
It is easy to prove that P_α cannot cross Γ (reasons of regularity) so, the different arc of Γ share the cross section in several parts, each of them being exclusively influenced by one determined part of $\partial\Sigma$; that is to say, if $z_i \in \Sigma$ is the zone of influence of $\partial z_i \subset \partial\Sigma$; $\forall\, M \in z_i$, $\exists$ an unique $\varphi \in \partial\Sigma$ such that $\delta(M, \partial z) = |M\varphi|$ and $\varphi \in \partial\Sigma$

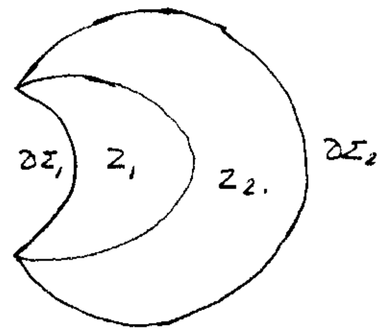
iii) <u>conclusion</u>: the plasticity begin to appear at $\alpha = \alpha_0$ at one of several points of $\partial\Sigma$ and is propagated by degrees therefrom; each connected components staying enclosed in one zone of influence.

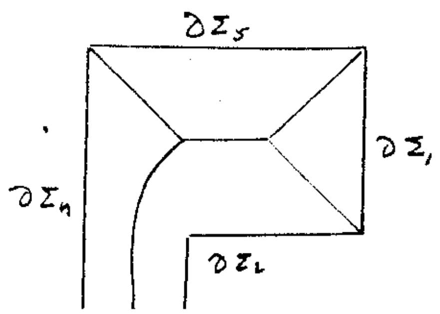
iv <u>Examples</u>

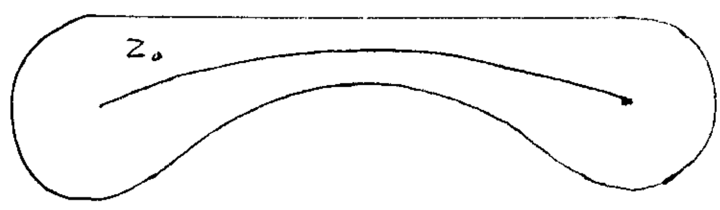












III. 10 Determination of the displacements field.

(we can now lighten the notations in denoting simply the stress tensor solution of (43)(44) by σ and more generally in suppressing for each symbole the index α)

- We are looking for $\lambda \in L^2(\Omega)$ such that

$$(62) \begin{cases} \lambda(x) = 0 & \text{if } \mathcal{F}[\sigma(x)] < 0, \text{ that is to say on } E \times]0,1[\\ \lambda(x) \geq 0 & \text{if } \mathcal{F}(\sigma(x)) = 0, \quad \text{" " " " on } P \times]0,1[\\ \text{and } \exists u \in \mathcal{U}_{ad} & \text{such that } \varepsilon_{ij}(u) = d_{ij} = A_{ijkh}\sigma_{kh} + \lambda \frac{\partial \mathcal{F}}{\partial \sigma_{ij}} \end{cases}$$

Where

$$(63) \quad \mathcal{U}_{ad} = \{ v / v \in [H^1(\Omega)]^3, \; v_1 = v_2 = 0 \text{ on } \partial\Omega_0, \; v_1 = -\alpha x_2, \; v_2 = \alpha x_1 \text{ on } \partial\Omega_1 \}$$

__Remark 7__ : It is easy to see that if $\mathcal{F} \in C^1(R^9)$, then :
$\lambda_{ij}(\tau_{ij} - \sigma_{ij}) \leq 0$ in Ω $\forall \tau \in R^9$ such that $\tau_{ij} = \tau_{ji}$ and $\mathcal{F}(\tau) \leq 0$ is equivalent to $\lambda_{ij} = \lambda \dfrac{\partial \mathcal{F}}{\partial \sigma_{ij}}$ with λ sastifying (62)

- $\sigma_{ij} = 0$ except $\sigma_{13} = \dfrac{\partial \theta}{\partial x_2}$ and $\sigma_{23} = -\dfrac{\partial \theta}{\partial x_1}$ $\Longrightarrow$.

$$d_{11} = d_{22} = d_{33} = d_{12} = 0; \quad d_{13} = \left(\frac{1}{2\mu} + \lambda\right)\frac{\partial \theta}{\partial x_2}; \quad d_{23} = -\left(\frac{1}{2\mu} + \lambda\right)\frac{\partial \theta}{\partial x_1}.$$

so, if the u of (62) exists, it is such that

$$(64) \begin{cases} u_1 = -\alpha x_2 x_3 \\ u_2 = \alpha x_1 x_3 \\ u_3 = \alpha \eta(x_1, x_2) \end{cases}$$

with $\eta \in H^1(\Omega)$ and $\lambda \in L^2(\Omega)$ s.t.

$$(65) \begin{cases} \frac{\alpha}{2}[-x_2 + \eta_{,1}] = \left(\frac{1}{2\mu} + \lambda\right)\frac{\partial \theta}{\partial x_2} \\ \frac{\alpha}{2}[x_1 + \eta_{,2}] = -\left(\frac{1}{2\mu} + \lambda\right)\frac{\partial \theta}{\partial x_1} \end{cases};$$

λ is then determined by the integrability condition of (65). So we have to resolve the following problem :

Problem B_2: Find $\Lambda \in L^2(\Sigma)$, $(\Lambda = 2\mu\lambda)$ such that

s.t

$$\left.\begin{array}{l} \Lambda = 0 \quad \text{on } E \\ \Lambda \geq 0 \quad \text{on } P \end{array}\right\} \Longleftrightarrow \Lambda\left(g - |\operatorname{grad}\theta|\right) = 0 \quad \text{almost everywhere on } \Sigma$$

$$-\frac{\partial}{\partial x_p}\left[\Lambda \frac{\partial \theta}{\partial x_p}\right] = \Delta\theta + 2\mu\alpha \qquad (p = 1, 2) \qquad \text{a.e on } \Sigma$$

- Λ is a multiplier of Lagrange; H Brezis solved this problem (cf ref) using the following properties
  - $\Delta\theta + 2\mu\alpha = 0$ on E_α
  - $\Delta\theta + 2\mu\alpha \geq 0$ a.e on $P_\alpha \Longleftarrow$ variational inequality (48)
  - $\delta(x, \partial\Sigma)$ (and therefore θ_∞) $\in C^2(\bar{E} - \bar{\Gamma})$ result obtained by T.W. Ting
  - $\bar{\Gamma} \subset E_\alpha \quad \forall$ finite α
  - if $x \in P_\alpha$, $\exists\, y$ unique $\in \partial\Sigma$ such that $|x - y| = \delta(x, \partial\Sigma)$
  - The extension of $[y, x[$ cross the elastic plastic boundary at least in one point

- Conclusions of H. Brezis : Λ exists, is unique and belongs to $L^\infty(\Sigma)$ and then, $\exists$ a displacements field unique at a translation along ox_3

So the problem B is about completely solved at least in the two cases of regularity pointed out by H. Brezis and C. Stampacchia cf. paragraphe III. 3.2 p 20.

It would be interested now to think about the validity of Hencky law; this question will be mentioned in the following chapter

III. 11 Références.

H. Brezis and G. Stampacchia " Sur la regularité de la solution d'inequations elliptiques" Bull. Soc. Math. France Vol 96 (1968) p 153.180

H. Brezis and M. Sibony. " equivalente de deux inequations variationnelles et application " Arch. Rat. Mech. Anal Vol 41 (1971) p 254.265

T.W. Ting " Elastic plastic torsion of simply connected cylindrical bars (Indiana University Math. J Vol 20 n° 11 1971 p. 1047-1076

H. Brezis " multiplicateurs de Lagrange en torsion élastoplastique "

IV. Diverse generalisations.

IV.1 Case of a homogeneous cylindrical bar with multiply connected cross-section

we consider the same boundary problem than precedingly except that now the elastic plastic media occupyn only a multiply connected part of Ω : denote by Ω^* this multiply connected domain and by $\Omega_1, \Omega_2, \dots \Omega_q$ the cylindrical cavities - The lateral boundary of these cavities are a part of the lateral boundary of Ω^*

If we define K_T^* as in K_T in the paragraphe III.2 but in replacing Ω by Ω^* then the lemma 1 can be replace by the following one. where $\Sigma, \Sigma_i, \Sigma^*$ are the respective cross section of $\Omega, \Omega_i, \Omega^*$

Lemma 3 : $\forall\ \sigma \in K_T^*,\ \exists\ \theta \in H_0^1(\Sigma)$ unique such that

(66) $\theta_{,2} = \sigma_{13}$ and $\theta_{,1} = -\sigma_{23}$ p.p on Σ^*.

(67) $\theta = k_i$ on each Σ_i (k_i being a constante)

(68) $|\operatorname{grad}\theta| \leq g$ p.p on Σ

Whe obtain in fact as above the condition θ = constante on $\partial\Sigma^*$ but, becaus the multiply connecticity of Σ^*, one differente constante k_i is associated to each part $\partial\Sigma_i$ of $\partial\Sigma^*$ if we choose 0 over $\partial\Sigma$. The trick consists in extending θ over each Σ_i by the constant value k_i in order that the domain of definition for the functional space could be still Σ

We have then to solve the following problem

Probleme B_1^* : Find.

(69) $\theta^* \in K^*$ s.t. $\langle \theta^*_\alpha - \varphi, \theta^*_\alpha - f_\alpha \rangle \leq 0 \quad \forall\ \varphi \in K^*$.

Where.

(70) $K^* = K \cap E$; $E = \{\varphi \,/\, \varphi \in H_0^1(\Sigma),\ \varphi = \text{constante a.e on each } \Sigma_i\}$

The elastic solution is characterize by.

and satisfies:

$$(72)\quad \begin{cases} f_\alpha^* \in H_0^1(\Sigma) \cap C^\infty(\bar{\Sigma}^*) \\ \Delta f_\alpha^* + 2\mu\alpha = 0 \quad \text{in } \Sigma^* \\ f_\alpha^* = 0 \quad \text{on } \partial\Sigma \\ f_\alpha^* = h_{\alpha i} \quad \text{on } \Sigma_i \end{cases}$$

where $h_{i\alpha}$ for $i = 1, 2, \dots, q$ are some constants such that:

$$(73)\quad \int_{\partial\Sigma_i} \frac{d f_\alpha^*}{\partial\nu}\, ds = -2\pi\alpha A_i$$

(73) insuring the univocity of the displacement field over Σ^*.
ν being the outward unitary normal to $\partial\Sigma_i$ and A_i the area of Σ_i

We can then introduce as above:

$$(74)\quad M^* = \sup_{x \in \bar{\Sigma}^*} |\operatorname{grad} \gamma^*| \qquad \text{and} \quad \alpha_0^* = \frac{g}{\mu M^*}.$$

Because the maximum principle there is still working, but we don't know a priori on which part $\partial\Sigma$ or $\partial\Sigma_i$ of $\partial\Sigma^*$ the max of $\operatorname{grad} \gamma^*$ is reached.

We can again introduce the elastic and plastic regions

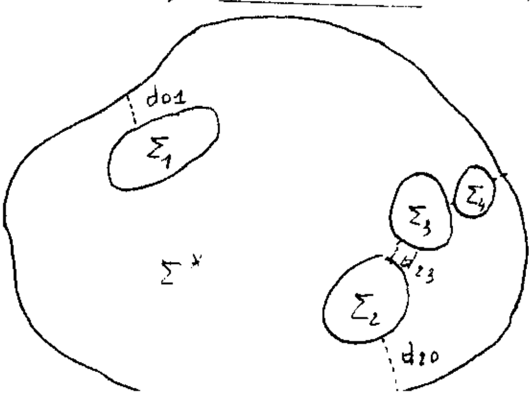
$$E_\alpha^* = \{ x \,/\, x \in \Sigma^* \text{ such that } |\operatorname{grad} \theta_\alpha^*| < g \text{ p.p.} \}.$$

$$P_\alpha^* = \{ x \,/\, x \in \Sigma^* \text{ such that } |\operatorname{grad} \theta_\alpha^*| = g \text{ p.p} \}.$$

and we have then the following results:

i) <u>for $\alpha \le \alpha_0^*$</u>; $f_\alpha^* \in K^*$ then $\theta_\alpha^* = f_\alpha^*$ and the whole section Σ^* is kept elastic.

ii) <u>for $\alpha > \alpha_0^*$</u>, $f_\alpha^* \notin K^*$ then $\theta_\alpha^* \neq f_\alpha^*$; the plasticity becomes to appear in some part of Σ^* which have to be determined.



iii) <u>When $\alpha \to \infty$</u>; θ_∞^* converges uniformly toward

$$\theta_\infty^* = g\, \delta^*(x, \partial\Sigma^*),$$

where $\delta^*(x, \partial\Sigma^*)$ is a generalized distance to

$$\delta^*(\alpha, \partial\Sigma) = \inf_{i \in \{0,1,\dots q\}} \{ \delta(\alpha, \Sigma_i) + d_i \}$$

q being the number of cavities.

Σ_0 the outside of $\bar{\Sigma}$

d_i the shortest path between $\partial\Sigma_i$ and $\partial\Sigma \equiv \partial\Sigma_0$ among those wich consists in going as quick as possible from a domain Σ_i to another one and in counting for nothing each domain crossing.

In the example of the preceding figure:

$d_0 = 0$ (always true); $d_1 = d_{01}$; $d_2 = d_{23} + d_{34} + d_{40} < d_{20}$

$d_3 = d_{34} + d_{40} < d_{30} < d_{23} + d_{30}$.

$d_4 = d_{40}$.

The regularity of θ^*_α for an arbitrary α was given by Cl Gerhardt (cf reference): $\theta_\alpha^* \in C^{1,\alpha}(\bar{\Sigma}^*)$

We can wonder: what about the equivalence between the problem B_1^* and the following one.

$$(71)\quad \theta_\alpha'^* \in K'^* = \{ \varphi / \varphi \in E, \ |\varphi(\alpha)| \le \theta^*_\infty(\alpha) \ \text{a.e on } \Sigma \}$$

$$(72)\quad \langle \theta_\alpha'^* - \varphi, \ \theta_\alpha'^* - f_\alpha \rangle \le 0 \quad \forall \ \varphi \in K'^* \quad ?$$

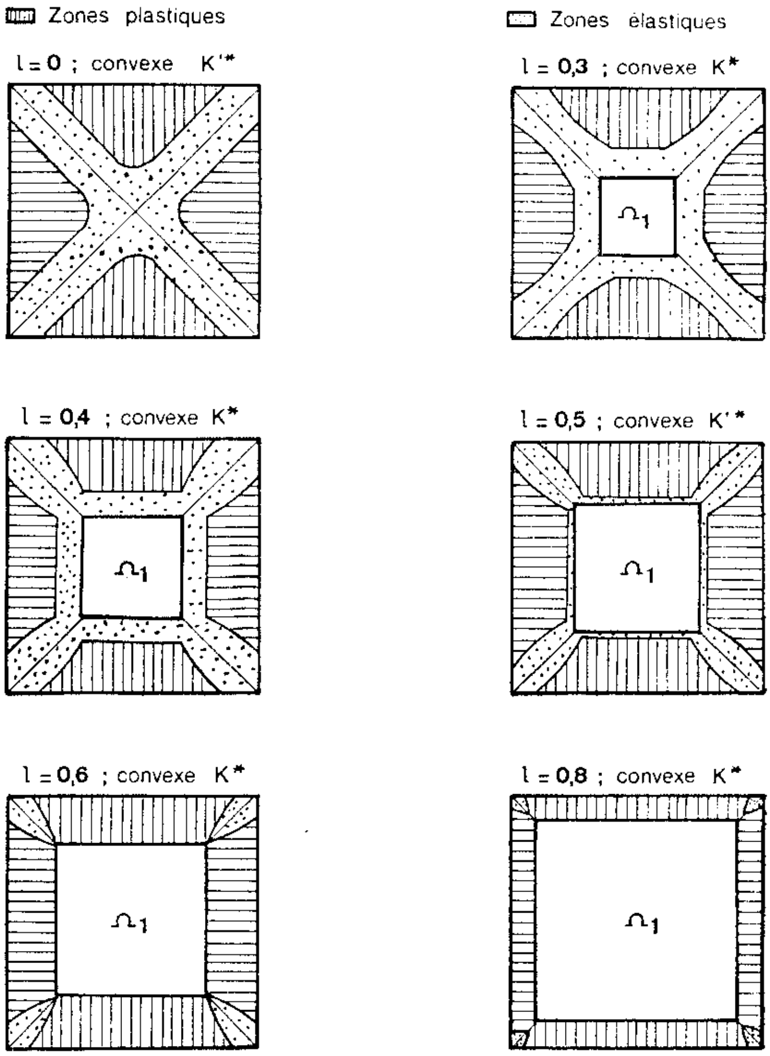
This very difficult question is still open; but, if we assume this equivalence; the cross section Σ^* can be, by analogy with the simply connected case, shared in zones of influence of the different parts of the boundary. $\partial\Sigma^*$ and the results about the propagation of plasticity follows. (cf References LANCHON - GLOWINSKY)

The figures below, illustrated the numerical results obtained in several cases of cross section.

We also give the curve representing the values of the torque couple with respect to α.

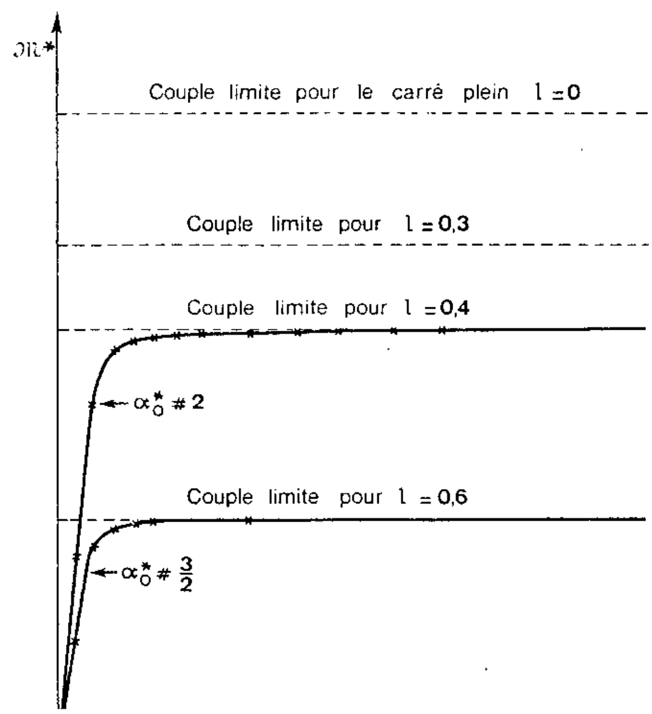
$$\mathcal{M}^*(\alpha) = 2 \int_{\Sigma} \theta_\alpha^* \, dx.$$

in the case of a square cross section with a square hole having same axis of symmetry. Different values of $\mathcal{M}^*(\infty)$: limit couple are given when the diameter of the hole changes.



Influence of the size of the cavity : ℓ is the relative diameter of the hole

Variation of the couple $\mathcal{M}^*$ with respect to α and ℓ



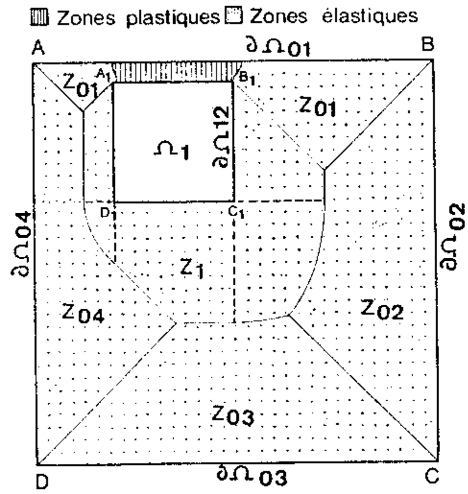


Fig. 7 *a*. — Zones d'influence des différents morceaux de $\partial\Omega^*$ et plastification pour $\alpha \neq 1$

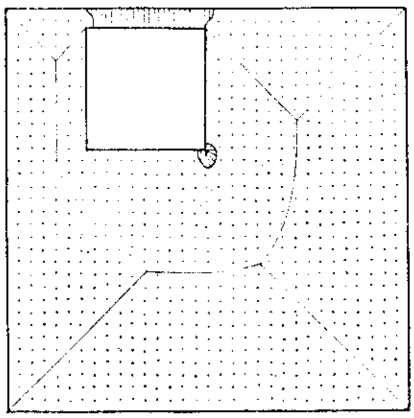


Fig. 7 *b*. — Plastification pour $\alpha = 1,5$.

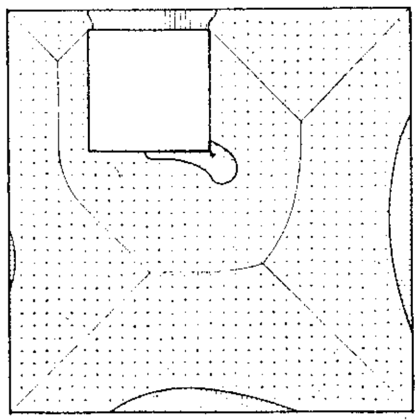


Fig. 7 *c*.— Plastification pour $\alpha = 2$.

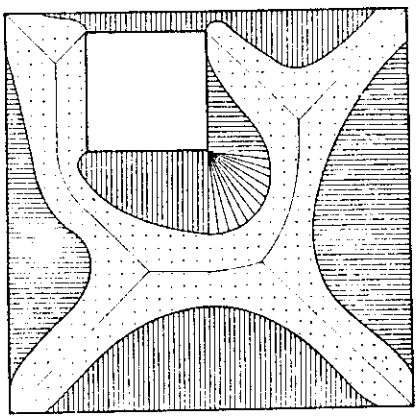
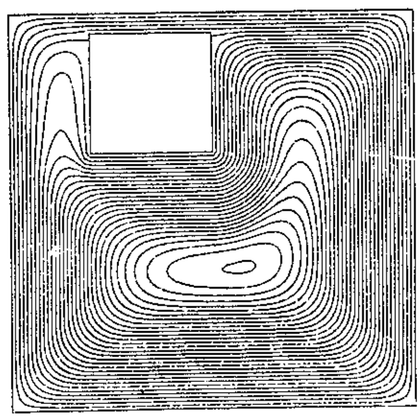
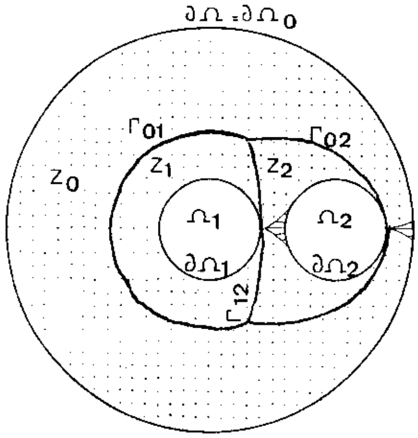


Fig. 7 *d*. — Plastification pour $\alpha = 5$.

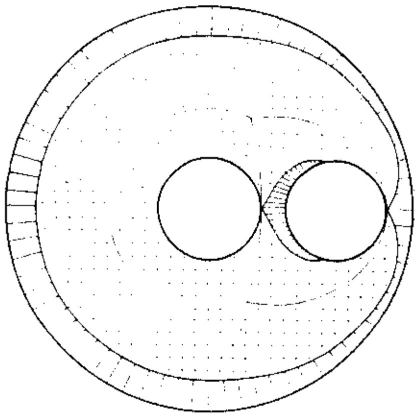


level lines of the solution θ_5^*

Zones of influence expected by the theory and propagation of the plasticity.



9 *a*



9 *b*

Fig. 9 *a*. — Zones d'influence et plastification pour $\alpha = 1$.

Fig. 9 *b*. — Plastification pour $\alpha = 1{,}5$.

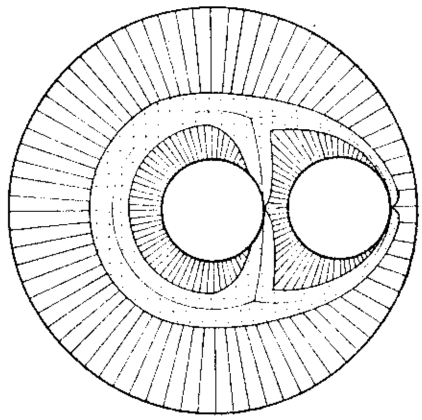
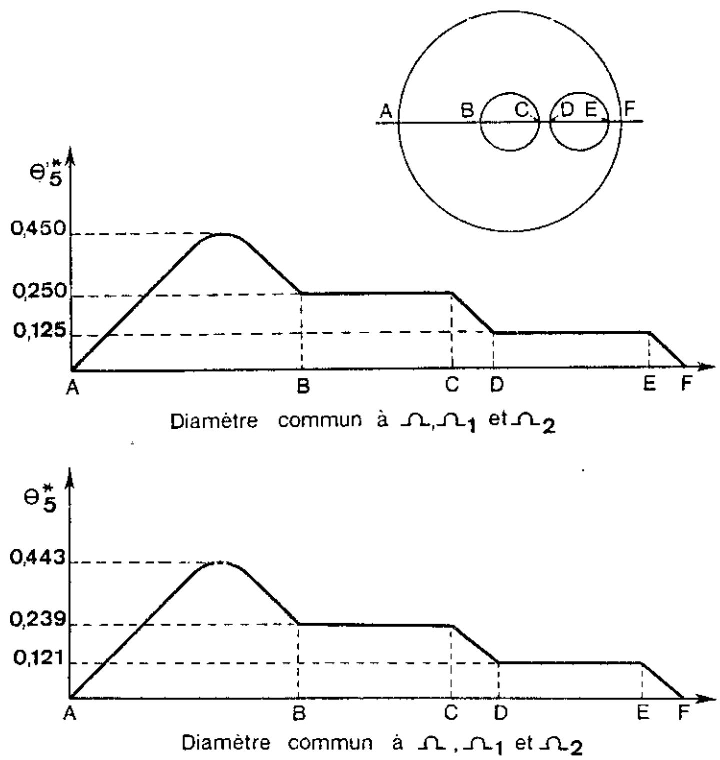


Fig. 9 *c*. — Plastification pour $\alpha = 5$.



IV. 2. Torsion for a homogeneous cylindrical bar with Prandtl Reuss Law

The formulation and results for a general boundary value problem with Prandtl Reuss' Law can be found in G. Duvaut - J.L. Lions (cf Reference). This general problem is formulated with the constitutive law defined by (5') and we need some initial conditions:

$$u(0) = u_0 \text{ on } \Omega.$$
$$\dot{u}(0) = u_1 \text{ on } \Omega.$$
$$\sigma(0) = \sigma_0 \text{ on } \Omega.$$

In the particular case of elastoplastic torsion, we assume still
$$F(\sigma) = \frac{1}{2}\sigma_{ij}\sigma_{ij} - \frac{1}{6}\sigma_{ii}^2 - g^2$$ an α depending slowly in time.
and the initial conditions are:

$u(0) = 0 = u_0$. and that implies $\alpha(0) = 0$
$\dot{u}(0) = 0 = u_1$ " " " $\dot{\alpha}(0) = 0$.
$\sigma(0) = 0$.

- σ^α is then obtained as the solution of a variational inequality without associated functional - σ^α and $\dot{\sigma}^\alpha$ are together in this variational inequality so, to obtain the result equivalent to the proposition 1, we use the same method, but in place to show that
$$J(\bar{\sigma}) \leq J(\sigma^\alpha),$$
we have to prove $a(\bar{\sigma} - \sigma^\alpha, \bar{\sigma} - \sigma^\alpha) \leq 0$ to conclude $\bar{\sigma} = \sigma^\alpha$
- The problem corresponding to B_1 is then: Find.

(93) $\theta_\alpha(t) \in K$; $\langle \dot{\theta}_\alpha(t) - f_\alpha(t), \theta_\alpha(t) - \psi \rangle$ $\forall\, \psi \in K$.

It is resolved by the theory of maximal monotone operators and non linear semi groups of contraction. (cf H. Brezis in the references)

- If $\dot{\alpha}(t) \geq 0$, the solutions of (48) and (93) are just the same; that is to say: in this particular case give the same result. That is obtained again in a theorem of comparison given by H. Brezis in his thesis

IV. 3. Case of a fibers reenforced cylindrical bar with Hencky law.

If we adopt the same notations that in the multiphasy considered bar; the media occupies the domaine $\Omega \subset R^3$ and is composed occupying the multiply connected domain Ω^* and characterized by μ_m and g_m and the fibers, occupying the q domains Ω_i and characterized by μ_f and g_f. Then μ and g can be considered as bounded but discontinued functions defined on Ω.

The stress function solution, if it exists is the unique solution of

$$(74)\quad \begin{cases} \theta_\alpha \in K \;; \quad J(\theta_\alpha) \leq J(\varphi) \quad \forall\, \varphi \in K \\ \text{with} \\ \quad J_\alpha(\varphi) = \int_\Sigma \frac{1}{\mu} |\operatorname{grad} \varphi|^2 \, dx - 4\alpha \int_\Sigma \varphi \, dx \end{cases}$$

but here J_α and K are defined with μ and g non constant coefficients.

We introduce then over $H_0^1(\Sigma)$ a norme.

$$(75)\qquad \|\varphi\|^2 = \int_\Sigma \frac{1}{\mu} |\operatorname{grad} \varphi|^2 \, dx$$

equivalent to the classic one; and associated to the scalar product

$$(76)\qquad \langle \varphi, \psi \rangle = \int_\Sigma \frac{1}{\mu} \operatorname{grad} \varphi \operatorname{grad} \psi \, dx$$

and again θ_α is the unique solution of

$$(77)\quad \theta_\alpha \in K \;; \; \langle \theta_\alpha - \varphi, \theta_\alpha - f_\alpha \rangle \leq 0 \quad \forall\, \varphi \in K.$$

We obtain then easily, the elastic solution f_α defined as above by

$$(78)\qquad \langle \varphi, f_\alpha \rangle = 2\alpha \int_\Sigma \varphi \, dx \qquad \forall\, \varphi \in H_0^1(\Sigma)$$

the scalar product being the new one.

This elastic solution satisfied

$$(79)\qquad \begin{cases} f_\alpha \in H_0^1(\Sigma) \\ -\frac{\partial}{\partial x_i}\left(\frac{1}{\mu} \frac{\partial f_\alpha}{\partial x_i}\right) = 2\alpha \quad \text{on } \Sigma \end{cases}$$

We obtain also a new kind also of generalized distance (cf. some actual research of D. Cioranescu, H. Landon, J. Saint Jean Paulin)

IV 4. References

again: R. Glowinski - H. Lanchon and H. Lanchon. cf p7.

Gerhardt, Cl. Regularity of the solutions of non linear variational inequalities with a gradient bounded as constraint A.R.M.A 1974 or 1975

G. Duvaut - J. L. Lions. Inequalities in Mechanics and Physics Springer Verlag 1976

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