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GENERALIZED INTERPOLATION AND APPLICATIONS

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I.– GENERALIZED INTERPOLATION AND APPLICATIONS

1.1.– Interpolation in $\mathbb{R}_1$

If $p_n(z) \in \mathcal{P}_n$:

$$p_n(z) = a_0 + a_1 z + \cdots + a_n z^n$$

is such that for the $n+1$ distinct points $\bar{z} = \{z_i\}$

$$p_n(z_i) = \varphi(z_i) = w_i \quad , \text{with } \bar{w} = \{w_i\} \neq 0 \in \mathbb{R}_{n+1},$$

we say that p_n interpolates φ (or $\bar{w}$) at the discrete set of points $\bar{z}$.

Th 1.1: $\exists \, \mathcal{V} \; p_n \in \mathcal{P}_n \; : \; p_n(\bar{z}) = \bar{w}$

Proof: consider the homogeneous problem $p_n(z_i) = 0$ for $\bar{w} = 0 \in \mathbb{R}_n$ and show that it only has the trivial solution. Use the alternative theorem.

Remark 1: $w_i = p(z_i)$ can be regarded as the definition of a very particular functional f_i applied to p_n : the point functional which assigns to p_n its value at $z = z_i$:

$$(f_i, p_n) = f(p_n) = p_n(z_i)$$

1.2.– Generalized interpolation

Let E be a linear space of dimension $n+1 < \infty$ and E^* its algebraic dual. We say that

$$(f_i, p) = w_i \qquad i = 0(1)n$$

is a generalized interpolation for $p \in E$ and $\{f_i\} \in E^*$.

<u>Th 2.1</u> $\exists U$ $p \in E$: p interpolates $\bar{w}$ in a sense defined by $\{f_i\} \in E^*$ iff the f_i are linearly independent in E^*.

<u>Proof</u>: i) If $q_0 \cdots q_n \in E$ and $f_0 \cdots f_n \in E^*$ are, respectively, linearly independent $\Rightarrow$ $G = \det |f_i(q_j)| \neq 0$ for $i,j = 0(1)n$.

Take as p
$$p = \sum_0^n \alpha_j q_j$$
where the $\{\alpha_i\}$ are the solution of the linear system
$$(f_i, p) = \sum_0^n \alpha_j (f_i, q_j) = w_i \qquad i = 0(1)n$$
$\Rightarrow$ the $\{\alpha_i\}$ are well defined for all $\bar{w} \neq 0 \in \mathbb{R}_{n+1}$ and they are unique.

ii) If the problem ~~has~~ a non trivial solution $\bar{\alpha} = \{\alpha_i\}$ for all $\bar{w} \neq 0 \Rightarrow G \neq 0$ and $\{f_i\}$ linearly independent in E^*.

1.3.- <u>Concrete realizations of E and E^*</u> : <u>Examples</u>

1) The motivation of 1.2 was the case of discrete polynomial interpolation where $E = \mathcal{P}_n$ and f_i: $f_i(p) = p(z_i)$ $i = 0(1)n$.

2) $E = \mathcal{P}_n$ again but $f_i : (f_i, p) = p^{(i)}(0)$ $i = 0(1)n$. This is <u>Taylor</u> <u>interpolation</u> ; $n+1$ differential conditions (with $p^{(0)}(z) = p(z)$) are given at a fixed point, say $z = 0$.

Since $q_0 = 1, q_1 = z, \ldots, q_n = z^n$,
$$G = \det |f_i(q_j)| = \begin{vmatrix} 1 & & & \\ 0 & 1 & & 0 \\ 0 & 0 & 2! & \\ & & & \ddots \\ 0 & 0 & 0 & & n! \end{vmatrix} = 1 \cdot 2! \ldots n! \neq 0.$$

We have then verified that the proposed <u>interpolation</u> <u>method</u> is well defined : by Th 2 the f_i are linearly independent.

3) We now take E to be the set $\mathcal{H}_n$ of <u>trigonometric</u> polynomials of order n. Then $\dim(\mathcal{H}_n) = 2n+1$, $q_0 = 1$, $q_1 = \sin t$, $q_2 = \cos t, \ldots$

$q_{2n+1} = \cos nt$. We consider the functionals f_i:

$$(f_0, \psi) = \int_0^{2\pi} \psi(t)\,dt$$
$$(f_s, \psi) = \int_0^{2\pi} \psi(t) \sin st\,dt$$
$$(f_{s+n}, \psi) = \int_0^{2\pi} \psi(t) \cos st\,dt,$$

where $\psi \in \mathcal{H}_n$. The solution of the trigonometric interpolation problem

$$(f_i, \psi) = w_i \qquad i = 0(1)2n$$

is also possible since $G \neq 0$, as it follows from simple (but slightly lengthly) trigonometric manipulations. Notice now the analogy with a truncated Fourier expansion.

4) $E \equiv \mathcal{P}_{2n+2}$ and $(f_i, p) = p(z_i) \quad i = 0(1)n$

$$(f_{i+n}, p) = p'(z_i) \quad i = 1(1)n+2$$

This generalized interpolation problem is Hermite's interpolation at $n+1$ points z_i. Show that the f' are well defined.

5) Other known (or unknown) procedures follow easily if E is taken as a set of exponentials special polynomials, etc and/or f define combinations of function and derivative values at different points z_i.

1.4.- Preferential bases $\mathcal{B} = \{\hat{q}_j\} \in E$ is preferential with respect of a set $\{f_i\} \in E^*$ if

$$(f_i, \hat{q}_j) = \delta_{ij} \qquad i,j = 0(1)n.$$

Th 3.1: For any linearly independent $\{f_i\} \in E^*$, $i = 0(1)n$, $\exists$ a $\cup$ preferential basis $\{\hat{q}_j\}$, $j = 0(1)n$.

Proof: Let $\{q_s\} \in E$, $s = 0(1)n$ be a basis for E,

$\Rightarrow \forall \hat{q}_j \in E$: $\quad \hat{q}_j = \sum_{s=0}^{n} \xi_{sj} q_s$,

with unknown coefficients ξ_{sj}. The condition $(f_i, q_j) = \delta_{ij}$

$\Rightarrow$
$$j = 0(1)n: \quad \sum_{s=0}^{n} \xi_{sj} (q_s, f_i) = \delta_{ij}$$

a series of $n+1$ linear systems of algebraic equations which defines uniquely the ξ_{sj}, for $i = 0(1)n$.

Def: If $\{q_j\} \in E$ and $\{f_i\} \in E^*$ are, respectively, linearly independent and $(f_i, q_j) = \delta_{ij}$ we say that the 2 sequences are biorthogonal

Th 4.1: (Lagrange's representation theorem): If $\{f_i\}$ and $\{q_j\}$ are biorthogonal $\Rightarrow \forall\, p \in E$ admits the following Lagrange's representation

$$p = \sum_{j=0}^{n} (f_j, p)\, q_j$$

Proof: apply f_i to the rhs:

$$\left(\sum_{j=0}^{n} (p, f_j)\, q_j, f_i \right) = (p, f_i) \qquad \forall\, f_i \in \{f_i\}$$

$\Rightarrow$ rhs defines p

Cor 4.a (Lagrange's interpolation formula) if

$$(f_i, p) = w_i \qquad i = 0(1)n,$$

$\Rightarrow$ the solution of the generalized interpolation problem is

$$p = \sum_{j=0}^{n} w_j q_j$$

where $\{q_j\}$ is biorthogonal to $\{f_i\}$ (i.e. the functionals which specify the type of interpolation method in question).

Remark 2 The prefixed basis q_j is now seen as a generalization of the well known Lagrange's fundamental polynomials $l_i(x) \in \mathcal{P}_n$, $i = 0(1)n$:

$$l_i(x) = \frac{\prod_{r=0}^{n} (x-x_r)}{(x-x_i)\prod_{r=0}^{n}{}'(x_i-x_r)} \quad ; \quad \prod_{r=0}^{n}(x-x_r) = (x-x_0)\cdots(x-x_n)$$

such that $l_i(x_j) = \delta_{ij}$ and $p(x) = \sum_{i=0}^{n} y_i\, l_i(x)$.

The $l_i(x)$ are the preferential basis in $E = \mathcal{P}_n$ for the point interpolation problem where f_i:

$$(f_i, p) = p(z_i)$$

Since the preferential basis has to be evaluated for each (interpolation procedure) f_i, it is convenient to express the solution in a compact form in terms of an arbitrary (not necessarily preferential) basis q_j. After some determinantal manipulations we find that

Cor 4-b

$$p = \frac{-1}{G}\begin{vmatrix} 0 & q_0 & q_1 & \cdots & q_n \\ \omega_1 & (q_0,f_0) & (q_1,f_0) & \cdots & (q_n,f_0) \\ \omega_2 & (q_0,f_1) & (q_1,f_1) & \cdots & (q_n,f_1) \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ \omega_n & (q_0,f_n) & (q_1,f_n) & \cdots & (q_n,f_n) \end{vmatrix}$$

where $G = \det |(f_i, q_j)|$, $i,j = 0(1)n$.

Proof: notice that, for example, $(f_1, p) = \omega_1 \frac{-G}{-G} = \omega_1$ and generalize for all ω_i.

5.- Example Take $q_0 = 1$, $q_1 = x$, $Z = \{x_0, x_1\}$, $p \in \mathcal{P}_1$ and

$$(f_i, p) = y_i \qquad i = 0,1$$

Then

$$p = \frac{1}{x_0 - x_1}\begin{vmatrix} 0 & 1 & x \\ y_0 & 1 & x_0 \\ y_1 & 1 & x_1 \end{vmatrix} = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x_0 - x}{x_0 - x_1} y_1$$

Give similar concrete examples for the other realizations of

Remark 4.1 Notice that all the points $\{z_i\}$ are involved in the definition of each of the Lagrange's fundamental polynomial $l_i(x)$. This is a serious inconvenience in practical computations, since the addition of one more element to the set $Z = \{z_i\}$ forces us to re-compute all the $l_i(x)$'s. The same applies to the preferred basis $\{\hat{q}_j\}$.

Newton polynomials for Z: $1, z - z_0, (z-z_0)(z-z_1), \dots, (z-z_0)\cdots(z-z_{n-1})$, however, constitute a basis which is permanent, i.e. if more points are added to the set Z no change is required in the elements already compiled. The functionals f_i which effect Newton's interpolation method are, however, not as simple as the ones associated with Legendre's method of interpolation.

We shall now try to extend such type of permanent basis to our generalized interpolation situation. This will also allow us to treat the infinite dimensional case.

Th 5.1 (Newton's representation theorem). E is a linear space of infinite dimension, $E_{n+1} = sp\{q_0 \cdots q_n\} \subset E$ and $E^*_{n+1} = sp\{f_0 \cdots f_n\} \subset E^*$
$\Rightarrow$ for all n, $\exists\, \mathcal{V}$

$$\left.\begin{array}{l} \hat{q}_j \in E_{n+1} \\ \hat{f}_i \in E^*_{n+1} \end{array}\right\} : \ (\hat{f}_i, \hat{q}_j) = \delta_{ij}$$

and

$$sp\{\hat{q}_0 \cdots \hat{q}_n\} = E_{n+1}, \quad sp\{\hat{f}_0 \cdots \hat{f}_n\} = E^*_{n+1}$$

$$p = \sum_{i=0}^{n} (\hat{f}_i, p)\, \hat{q}_i$$

<u>Remark 5.1</u> The first implication of the last Theorem tells us of the existence of linear transformations

$$A : q_i \mapsto \hat{q}_i$$
$$B : f_j \mapsto \hat{f}_j$$

defined <u>lower</u> <u>triangular</u> matrices, that is why they are <u>permanent</u>.

<u>Cor 5-1</u> (Newton's permanent interpolation formula.)

If

$$(\hat{f}_i, p) = w_i \qquad i = 0(1)n$$

$\Rightarrow$

$$p = \sum_{0}^{n} w_i \hat{q}_i$$

solves the interpolation problem and is permanent

<u>Cr 5-2</u> $\hat{f}_i$ and $\hat{q}_j$ are given by the following expressions

$$f_i = \frac{1}{G_j} \begin{vmatrix} (f_0, q_0) & (f_1, q_0) & \cdots & (f_i, q_0) \\ \text{---} & \text{---} & & \text{---} \\ (f_0, q_{i-1}) & (f_1, q_{j-1}) & \cdots & (f_i, q_{j-1}) \\ f_0 & f_1 & \cdots & f_i \end{vmatrix}$$

and

$$\hat{q}_j = \frac{1}{G_{j-1}} \begin{vmatrix} (f_0, q_0) & (f_0, q_1) & \cdots & (f_0, q_j) \\ \text{---} & & & \text{---} \\ (f_{j-1}, q_0) & (f_{j-1}, q_1) & \cdots & (f_{j-1}, q_j) \\ q_0 & q_1 & \cdots & q_j \end{vmatrix}$$

where $\{f_i\}, \{q_j\}$ are linearly independent sets