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# Mathematical Results for Bingham Fluids

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Many problems arising in Mechanics and Physics lead to evolution problems which can be transform into variational inequalities. We shall consider here the flow of the visco-plastic Bingham fluid, which can be described by a variational inequality. We shall solve this inequality by using Galerkin's method and monotonicity and compactness arguments. The principal steps are the following:

1°. "Approximation" of the inequality by a family of non-linear partial differential equations ($\mathcal{E}_\varepsilon$)

2°. Resolution of $\mathcal{E}_\varepsilon$ by Galerkin's method i.e. by approximating $\mathcal{E}_\varepsilon$ by a system of ordinary differential equations whose solution is given by Cauchy-Lipschitz theorem. Then, we have to obtain a priori estimates to assure the existence of all solutions on all interval $(0, T)$. We can extract then a weakly convergent subsequence from the approximating solutions.

3°. Proof that the weak limit is indeed a solution of the initial problem i.e. the partial differential equation $\mathcal{E}_\varepsilon$.

4°. Taking $\varepsilon \to 0$ we show that a subsequence of the solutions of $\mathcal{E}_\varepsilon$ converge to a solution of the variational inequality (to doing that we use a weak-compactness argument similar to that employed in 2°)

Therefore the variational inequality of

Bingham fluid is a limit case of a family of partial differential equations. Moreover, these equations have a physical sense - actually they describe the flow of "pseudo-plastic" fluids.

## §1. Physical Problems.

We consider a fluid in $\mathbb{R}^3$ (or $\mathbb{R}^2$). Let denote by $\vec{u}(x,t)$ its velocity. ($\vec{u}=(u_1,u_2,u_3)$)

To obtain the equation which describe the motion of the fluid we have to add to the general conservation laws of a continuous media the constitutive (or specific) low of the fluid.

a). Conservation of mass (equation of continuity)

If $\rho(x,t)$ is the density of the fluid in the point $x\in\mathbb{R}^3$ at the time $t$ the equation given by this law is:

$$\frac{\partial\rho}{\partial t}+\sum_{i=1}^{3}\frac{\partial}{\partial x_i}(\rho u_i)=0$$

b). Conservation of momentum.

Let be $\vec{f}(x,t)$ the vector of exterior forces ($\vec{f}=(f_1,f_2,f_3)$). Let denote by $\sigma=(\sigma_{ij})_{i,j=1,\dots,3}$ the stress tensor. Then we have

$$\rho\left[\frac{\partial u_i}{\partial t}+\frac{\partial u_i}{\partial x_j}u_j\right]=f_i+\sum_j\frac{\partial\sigma_{ij}}{\partial x_j}$$

$\frac{\partial u_i}{\partial t}+\frac{\partial u_i}{\partial x_j}u_j=\frac{du_i}{dt}$ is the material derivative of $u$. We shall denote by $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ the vector given by $\gamma_i = \frac{du_i}{dt}$.

c). Conservation of angular momentum which gives:
$\sigma_{ij} = \sigma_{ji}$ (the symmetry of stress tensor)

d). Conservation of energy.

e). Constitutive equations.
Let denote by
$$D = (D_{ij})_{i,j=1,..,3}$$
the tensor of rate of deformation.
$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

For a viscous newtonian fluid the constitutive equation is given by:

(1.1) $$\sigma = 2\mu D - p\delta$$

where $\mu$ is the coefficient of viscosity, $p = p(x)$ is the pressure and $\delta = (\delta_{ij})_{i,j=1,...,3}$ $\delta_{ij}$ being Kronecker symbols.

We shall consider here incompressible fluids i.e $\rho$ = constant. By the continuity equation
$$\sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0 \quad (\text{or } \operatorname{div} u = 0)$$
For simplicity let be $\rho = 1$.

Let $\sigma^D$ denote the deviator of $\sigma$.
Generally speaking we call a fluid newtonian if the relation between $\sigma^D$ and $D$ is a linear one and non-newtonian if this relation

By (1)

(1.2) $\sigma^D = 2\mu D$ for a newtonian fluid.

(1.3) $\sigma^D = f(D)$ with $f$ some non linear function for non-newtonian fluids.

(We treat here the fluids verifying the Stokes's hypothesis: i.e. $\sigma^D$ is function of $D$)
In (2) $\mu$ may depend on temperature but for the linearity of the relation it is essential that $\mu$ is independent of the components of $D$.
A first step to consider nonnewtonian behaviour is to lett $\mu$ in (2) depend on the components of the rate of deformation. In this aim, let denote by $D_I$ and $D_{II}$ the first and the second invariants of the tensor $D$:

$$D_I = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0 \quad \text{for incompressible fluids}$$

$$D_{II} = \frac{1}{2} D_{ij} D_{ij}$$

Let denote by $\sigma_{II} = \frac{1}{2}\sigma_{ij}^D \sigma_{ij}^D$ (second invariant of $\sigma^D$)

We shall consider here a class on nonnewtonian fluids - by letting $\mu$ dependent of $D_{II}$. Examples

- Bingham fluid:

(1.4)
$$\begin{cases} \sigma_{II}^{1/2} < g \Rightarrow D_{ij} = 0 \\ \sigma_{II}^{1/2} \geqslant g \Rightarrow \sigma_{ij}^D = 2\mu D_{ij} + g \dfrac{D_{ij}}{D_{II}^{1/2}} \end{cases}$$

$g$ is called the plasticity coefficient.

- Dilatant fluids:

- Pseudo-plastic fluids:

$$\sigma_{ij}^{D} = 2\mu D_{ij} + 2\mu_1 D_{II}^{n-\frac{1}{2}} D_{ij} \qquad n < \tfrac{1}{2} \tag{1.6}$$

($\mu_1$ is a constant of the fluid)

- Yield dilatant fluids

$$\begin{cases} \sigma_{II}^{1/2} \le g \Rightarrow D_{ij} = 0 \\ \sigma_{II}^{1/2} \ge g \Rightarrow \sigma_{ij}^{D} = 2\mu D_{II}^{n-\frac{1}{2}} D_{ij} + g \dfrac{D_{ij}}{D_{II}^{1/2}} ; n > \tfrac{1}{2} \end{cases} \tag{1.7}$$

- Yield pseudoplastic fluids.

$$\begin{cases} \sigma_{II}^{1/2} < g \Rightarrow D_{ij} = 0 \\ \sigma_{II}^{1/2} \ge g \Rightarrow \sigma_{ij}^{D} = 2\mu_1 D_{II}^{n-\frac{1}{2}} D_{ij} + 2\mu D_{ij} + \\ \qquad + g \dfrac{D_{ij}}{D_{II}^{1/2}} \qquad n < \tfrac{1}{2} \end{cases} \tag{1.8}$$

Considering flows in pipes the equations (1.4) - (1.8) become (as $\sigma^D$ and $D$ have one component)

(1.4)' $\sigma_1^D - g = 2\mu D_1$ for Bingham fluids

(1.5)' $\sigma_1^D = \mu D_1^{2n}$ $(n > \frac{1}{2})$ for dilatant fluids

(1.6)' $\sigma_1^D = 2\mu D_1 + \mu_1 D_1^{2n}$ $n < \frac{1}{2}$

(1.7)' $\sigma_1^D - g = \mu D_1^{2n}$ $(n > \frac{1}{2})$ for yield dilatant fluids

(1.8)' $\sigma_1^D - g = 2\mu D_1 + \mu_1 D_1^{2n}$ $(n < \frac{1}{2})$ for yield pseudoplastic fluids

These equations have been verified in practice principally in the study of a lot of mixtures hydrocarbon greases. (1.4)' – (1.8)' were proposed by physicists after experimental observations. The equations (1.4) – (1.8) constitue in fact generalisations of (1.4)' – (1.8)' to higher dimensions.

Examples:

- Bingham fluids: some thickened hydrocarbon greases, certain asphalts and bitumens, some emulsions or water suspensions of quartz or mettallic oxydes, the tooth paste.
- Dilatant fluids: dilatant behaviour is observed in certain ranges of concentration in suspensions of irregularly shaped solids in liquids. The limited concentration range in which dilatancy is observed depends on the size distribution and on the shape of particles.
- Pseudo-plastic fluids (more common than dilatant fluids): some hydrocarbon greases certain aqueous dispersions of polyvinyl acetate or certain nonaqueous polymer solutions (and another fluids with strange names which can be found in the bibliography indicated at the end of this chapter)
- Yield pseudoplastic: many clay-water (see bibliography)

Bibliography:

G.W. Govier – K. Aziz: Flow of complex mixtures in Pipes (Van Nostrand

## §2. Variational formulation of problems of §1.

We consider henceforth the flow of some fluids in the interior of an open bound domain of $\mathbb{R}^3$ (or $\mathbb{R}^2$) supposing that at the solid boundary we have the adherence condition i.e.

$$u/_{\partial\Omega} = 0$$

Collecting all the informations given by the conservation laws and by the specific laws given in §1, we have to solve a set of problems. We shall formulate here, for example, the problem for newtonian fluids and the problem for Bingham fluids.

Problem A.

Find $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ such that

$$\begin{cases} \operatorname{div} u = 0 \\ u/_{\partial\Omega} = 0 \\ (f-\gamma)_i = \sigma_{ij,j} \\ \sigma_{ij} = 2\mu D_{ij} - p\delta_{ij} \end{cases} \qquad \sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j}$$

Problem B.

Find $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ such that

$$\begin{cases} \operatorname{div} u = 0 \\ u/_{\partial\Omega} = 0 \\ (f-\gamma)_i = \sigma_{ij,j} \\ \sigma_{ij} = 2\mu D_{ij} + g\dfrac{D_{ij}}{D_{II}^{1/2}} - p\delta_{ij} \end{cases}$$

Some important quantities.

$\int_\Omega \sum_i |u_i|^2 dx = \|u\|^2_{L^2(\Omega)}$ — the kinetic energy

$\int_0^T \int_\Omega \sum_{i,j} \left(\frac{\partial u_i}{\partial x_j}\right)^2 dx\,dt = \int_0^T \|\frac{\partial u}{\partial x}\|^2 dt$ — the energy lost by viscosity during the period ~~of~~ of time T.

It is natural to look for solutions with kinetic energy bounded in time and such that the energy lost by viscosity is bounded. We shall ask hence:

$$u \in L^\infty(0,T;\ L^2(\Omega)) \cap L^2(0,T;X)$$

$X = \{u \mid \frac{\partial u_i}{\partial x_j} \in L^2(\Omega)\}$

(Functional spaces will be precised later)

The Problem A lead to the Navier - Stokes equations. In fact, replacing $\sigma_{ij}$ by its value given by the specific law, we obtain easily that $u$ satisfies to:

$$(2.1)\quad \begin{cases} \dfrac{\partial u_i}{\partial t} + \sum \dfrac{\partial u_i}{\partial x_j} u_j - \mu \Delta u_i = f_i - \dfrac{\partial p}{\partial x_i} & i = 1,2,3 \\ u|_{\partial\Omega} = 0 \\ \operatorname{div} u = 0 \end{cases}$$

Let be now $v$ a test function (i.e $v|_{\partial\Omega} = 0$, $\operatorname{div} v = 0$)

By multiplication by $v$ (2.1) becomes:

$$\left(\frac{du}{dt}, v\right) + \mu\, a(u,v) + \mathscr{b}(u,u,v) = (f,v)$$

with the classical notations:

$$(u,v) = \sum_{i=1}^{3} \int_\Omega u_i v_i\, dx$$

$$a(u,v) = 2 \sum_{ij=1}^{3} \int D_{ij}(u)\, D_{ij}(v)\, dx$$

$$b(u,v,w) = \sum_{ij=1}^{3} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx$$

It is the weak formulation (or variational formulation) of Navier - Stokes equations. Therefore if $u$ is solution of Problem A it is solution of the following problem too:

Problem A': Find $u(x,t) = (u_i(x,t))_{i=1,..,3}$ such that

$$\begin{cases} (u'(t), v) + \mu a(u,v) + b(u,u,v) = (f,v) \\ \text{for } \forall v \ni \cdot \quad \operatorname{div} v = 0, \quad v/_{\partial\Omega} = 0 \\ u/_{\partial\Omega} = 0, \quad \operatorname{div} u = 0 \end{cases}$$

On the other hand if $(F,v) = 0$ for all $v$ such that $\operatorname{div} v = 0$ then $F = \operatorname{grad} p$ ( $p$ some function defined moduli one constant). Using this result reintegrating by parts in the equation of Problem A' it is easy to see that if $u$ is solution of Problem A' it is solution of Problem A. Therefore the two problems are equivalent.

We shall try now to find what kind of variational problem can be related to the Problem B.

Theorem: The problem B is equivalent to the Problem B'.

Find $u(x,t) = (u_i(x,t))_{i=1,..,3}$ such that

$$\begin{cases} (u'(t), v - u(t)) + \mu a(u(t), v - u(t)) + b(u(t), u(t), v - u(t)) + \\ \qquad + g j(v) - g j(u(t)) \geq (f, v - u(t)) \\ \forall v \ni: \operatorname{div} v = 0, \quad v/_{\partial\Omega} = 0 \\ \text{and } \operatorname{div} u = 0, \quad u/_{\partial\Omega} = 0 \end{cases}$$

where:

$$j(v) = 2 \int_{\Omega} D_{II}^{1/2}(u) \, dx$$

Proof.

Let be $v$ a test function, by multiplication in the equation of motion

$$f_i - \gamma_i = - \sigma_{ij,j}$$

with $v-u$ we get:

$$(f-\gamma, v-u) = -\sum_{i,j=1}^{3} \int_\Omega \sigma_{ij,j} \cdot (v-u)_i \, dx = \text{by integration by parts}$$

$$= \int_\Omega \sigma_{ij}(u) \frac{\partial (v-u)_i}{\partial x_j} = \text{by the symmetry of } \sigma$$

$$= \int_\Omega D_{ij}(v-u) \, \sigma_{ij}(u) \, dx$$

In the last term we must replace $\sigma_{ij}$ by its value given by the constitutive law:

$$\int_\Omega D_{ij}(v-u) \sigma_{ij}(u) \, dx = 2\mu \int_\Omega D_{ij}(u) \, D_{ij}(v-u) \, dx +$$

$$+ g \int_\Omega D_{II}^{-1/2}(u) \, D_{ij}(u) \, D_{ij}(v-u) \, dx =$$

$$= \mu \, a(u, v-u) + g \int_\Omega D_{II}^{-1/2}(u) \, D_{ij}(u) \, D_{ij}(v) \, dx -$$

$$-2g \int_\Omega D_{II}^{-1/2}(u) \, D_{II}(u) \, dx \leq$$

$$\leq \mu \, a(u, v-u) + 2g \int_\Omega D_{II}^{-1/2}(u) \, D_{II}^{1/2}(u) \, D_{II}^{1/2}(v) \, dx -$$

$$- 2g \int_\Omega D_{II}^{1/2}(u) \, dx = \mu \, a(u, v-u) + g j(v) - g j(u)$$

We have used the inequality (which is very easy to prouve):

$$D_{ij}(u) \, D_{ij}(v) \leq 2 D_{II}^{1/2}(u) \, D_{II}^{1/2}(v)$$

We have therefore prouved that
Problem B $\Longrightarrow$ Problem B'.

The converse implication is a little more delicate, due essentially to the fact that in the variational inequality in the Problem B' we have a functional $v \longrightarrow j(v)$ which is not differentiable in $u$ if $D_{II}(u)$ ~~is~~ not $\neq 0$.

Suppose now that $u$ solution of Problem B' is such that $D_{II}(u) \neq 0$. Therefore $j'(u)$ is well defined by

$$(j'(u), v) = \int_{\Omega} D_{II}^{-1/2}(u)\, D_{ij}(u)\, D_{ij}(v)\, dx$$

and the variational inequality satisfied by $u$ is equivalent to the equation

$$(u', v) + \mu a(u, v) + b(u, u, v) + g(j'(u), v) = (f, v) \tag{2.2}$$

We can now integrate by parts in (2.2):

$$\sum_{i=1}^{3}\int_{\Omega}(f_i - \gamma_i)\, v_i\, dx = -\sum_{j=1}^{3}\int_{\Omega} 2\mu \frac{\partial}{\partial x_j}(D_{ij}(u))\cdot v_i\, dx \; +$$

$$+ \sum_{j=1}^{3}\int_{\Omega} g\left(\frac{\partial}{\partial x_j}(D_{II}^{-1/2}(u)\, D_{ij}(u))\right) v_i\, dx \qquad \forall v = \{v_i\}_{i=1,3}$$

with $\operatorname{div} v = 0$

and by the same argument as for the proof of the equivalence of the problem A and A' we get that $u$ satisfies to

$$\sigma_{ij,j} = f_i - \gamma_i$$

with $\sigma_{ij}$ given by the Bingham constitutive law, hence Problem B' $\Longrightarrow$ Problem B.

<u>Remark</u>. All the proofs done here are completely formal. We shall precise later the formulation, in appropriate