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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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THE TAU METHOD

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These are preliminary lecture notes intended for participants only.
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room 112.

III. – THE TAU METHOD

3.1.– Canonical basis for an operator D

Let $\mathcal{D}$ be the class of linear differential operators with rational coefficients and J a compact set. We first assume that $J \subset \mathbb{R}_1$, $\varphi = \{\varphi_i\}$, $\varphi_i \in P_i$, $i \in \mathbb{N}$, is a basis for the space of polynomials $\mathbb{P}$

Let

$$D y = f \qquad (1)$$

be a given problem with supplementary conditions

$$(f_i, y) = w_i \qquad i = 1(1)\nu.$$

$Q = \{Q_i(x)\}$ is a basis such that

$$D Q_i = \varphi_i \qquad i \in \mathbb{N},$$

which we will call the canonical basis associated with D. Let assume now that problem (1) is replaced by a perturbed problem

$$D y_n^* = f + H_n \qquad (2)$$

where $H_n \in P_n$ and assume for simplicity that also $f \in P_n$. Then, if $f = \sum_0^n \alpha_i \varphi_i$,

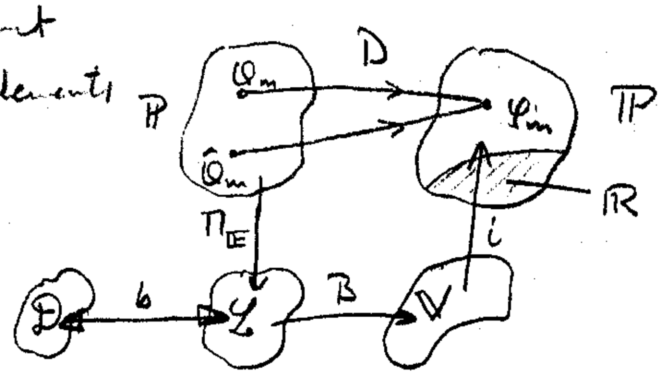
$$y_n^* = \sum_0^n \alpha_i Q_i + \sum_0^n \beta_i Q_i,$$

where $H_n = \sum_0^n \beta_i \varphi_i$, would be the exact solution of problem (2).

If we consider the mapping $D : \varphi \circlearrowleft$, it is clear that the elements Q_i of the basis Q could be evalu...

ted out of that transformation.

However, such inversion is not immediate, due to the fact that there are multiple elements in Q and, secondly, that a subset $\mathbb{R}$ of $\mathbb{P}$



$$\mathbb{R} = \underset{s\in S}{sp}\{\varphi_s\}$$

where
$$S = \{s : \nexists\ \varphi_n, n\in \mathbb{N} : D\varphi_n \in \mathcal{P}_n\}$$
may not be in the image of D.

We will see now how to deal with this situation

<u>Def 1.3</u>. $\mathbb{E}$ is an equivalence relation defined in Q:
$$Q_{m_i} \sim Q_{m_j} \quad \text{if} \quad Q_{m_i}-Q_{m_j} \in V_D$$

where V_D is the algebraic kernel of D.

<u>Def 2.3</u>: $\Pi_{\mathbb{E}}$ is a projector $\Pi_{\mathbb{E}}: Q \mapsto Q/\mathbb{E} = \mathcal{L}$ and $\mathcal{L}$ is the sequence of canonical classes. The elements $\mathcal{L}_n$ of $\mathcal{L}$ are classes of equivalence of polynomials. $\mathcal{L}$ is itself a linear space.

<u>Def 3.3</u>: i is an injection $i: \mathbb{V} \mapsto \mathbb{P}$

<u>Theorem 1.3</u>: $b: \mathcal{L} \mapsto \mathcal{D}$ is a bijection

$\rightarrow$) Let $(D_1 \neq D_2) \in \mathcal{D}$, $\Rightarrow$ $D_1 - D_2 = D^* \subset \mathcal{D}$:

for $Q_{m_i} \in \mathcal{L}_m$, $m \in \mathbb{N}-S$, $D^* Q_{m_i} = R_m \subset \mathbb{R}$

As card$(S) < \infty$, $\Rightarrow$ $D_1 \equiv D_2$

$\Leftarrow$) Let $\mathcal{L}^*_m, \mathcal{L}^+_m$ be distinct classes in $\mathcal{L}$, $\Rightarrow \exists$

m_i, m_j : $Q^*_{m_i} \in \mathcal{L}^*_m$, $Q^*_{m_i} \notin \mathcal{L}^+_m$

$Q_{m_j} \in \mathcal{L}^+_m$, $Q_{m_i} \notin \mathcal{L}^*_m$

$\Rightarrow Q_{m_i} - Q_{m_j} \notin V_D$ which contradicts the definition of $\mathcal{L}$.

Theorem 2.3 : $B : \mathcal{L} \leftrightarrow V$ is a bijection

It follows from the universal decomposition of a mapping as a product $i \cdot B \cdot \Pi_E$ where i is an injection, B is a bijection and Π_E a projection.

Remark 1.3 : Theorems 1 and 2 suggest a procedure for the computation of $\mathcal{L}$:

Theorem 3.3 The canonical sequence $\mathcal{L}$ can be computed by recursion.

Proof : consider the bijection $B : \mathcal{L} \to V$ and notice that

$$D\varphi_m = \sum_{r=0}^{n} a^{(m)}_r \varphi_r \quad , n = n(m)$$

$\Rightarrow$ inverting B over V, B^{-1}:

$$\mathcal{L}_m = \frac{1}{a^{(m)}_n}\left[\varphi_m - \sum_{A_m} a_r^{(m)} \mathcal{L}_r\right]$$

$A_m = \{r : r \in \mathbb{N} - S, r < n\}$.

3.2.- Application to the numerical solution of differential equations.

1) Let us consider the case of $Dy = (x^2+1)y'' - 6y = 0$, $(f_i, y) = w_i$ $i = 9,1$. Steps in the procedure : i) $Dx^n = (n+2)(n-3)x^n + n(n-1)x^{n-2}$

for $n = 0,1,2$ $\quad Q_0 = -1/6,\ Q_1 = -x/6,\ Q_2 = -(3x^2+1)/16$

$n = 3 \qquad Dx^3 = 6x \implies \hat{Q}_1 = x^3/6$

From $Q_1, \hat{Q}_1 \implies k(x+x^3) = U_D$ (i.e. $(x+x^3)$ exact polynomial solution of $Dy=0$, if it satisfies the supplementary conditions, the problem is solved, if not, compute $Q_4, \ldots, Q_n$ (Q_m particular realization of $\mathcal{Y}_m$) and replace by the perturbed problem:

ii) $\qquad D\, y_n^* = H_n = \tau_0 A(x) + \tau_1 B(x) \qquad$ (A, B lin. indep.)

3 parameters are required to fix the solution according to the initial or boundary conditions and to make the coefficient of the undefined canonical polynomial Q_3 equal to zero: 2 of them are provided by the perturbation term H_n, the third by the $k(x+x^3) \equiv U_D$ which must be added to y_n^*.

2) Eigenvalue problem: $D_\lambda y = y'' + \lambda^2 y = 0$, $y(0) = y(1) = 0$, $x \in [0,1]$, $\varphi_i = x^i$, $i \in \mathbb{N}$.

Steps in the procedure: i) $D x^n = \lambda^2 x^n + n(n-1)x^{n-2}$, $U_D \equiv \emptyset$.

$$\mathcal{Y}_n = Q_n = 1/\lambda^2 \{ x^n - n(n-1) Q_{n-2} \}.$$

$Q_0 = 1/\lambda^2,\ Q_1 = x/\lambda^2,\ Q_2 = (x^2 - 2/\lambda^2)/\lambda^2, \ldots$

ii) Perturbed problem:

$$D_\lambda y_n^* = H_n = \tau_0 A(x) + \tau_1 B(x)$$

Say $A(x) = T_2^*(x) = 8x^2 - 8x + 1$, $B(x) = T_1^*(x) = 2x - 1$, two linearly independent polynomials are required in order to satisfy the 2 boundary conditions

$$y(0) = y(1) = 0.$$

iii) Construction of the solution:

The solution of $D_\lambda y^* = H_n$ is given by a polynomial $y_n^*(x)$:

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(1) See: E.L. Ortiz, SIAM Numerical Analysis $\underline{6}(3)$ p 480.
Studies in Numerical Analysis (Editor R. Scaife)

$$y_n^* = \sum_0^4 c_i Q_i \xmapsto{D} H_n$$

In our case

$$y_n^*(x,\lambda) = \tau_0 \sum_{i=0}^{2} c_i^{(2)} Q_i + \tau_1 \sum_{i=0}^{1} c_i^{(1)} Q_i$$

$$= \tau_0 \left[8Q_2 - 8Q_1 + Q_0 \right] + \tau_1 \left[2Q_2 - Q_0 \right]$$

$$= \frac{1}{\lambda^4} \left\{ \tau_0 \left[(8x^2 - 8x + 1)\lambda^2 - 16 \right] + \tau_1 \left[(2x-1)\lambda^2 \right] \right\} \quad (3)$$

(v) Adjustment of the solution to the supplementary conditions of Problem I: $y(0) = y(1) = 0$: since the system is homogeneous,

$$\begin{vmatrix} \lambda^2 - 16 & -\lambda^2 \\ \lambda^2 - 16 & \lambda^2 \end{vmatrix} = 2\lambda^2(\lambda^2 - 16) = 0 \Rightarrow \begin{cases} \lambda = 0 & \text{trivial sol.} \\ \underline{\lambda = 4} \end{cases}$$

A second degree approximation gave an error $e = |\pi - 4| = 0.86$

Higher order approximations give the following approximate spectrum of D:

n	$\lvert\lambda_1 - \bar\lambda_1\rvert$	$\lvert\lambda_2 - \bar\lambda_2\rvert$	$\lvert\lambda - \lambda_3\rvert$	$\lvert\lambda - \lambda_4\rvert$
4	0.05	0.27		
8	0.000'0000'04	0.000'006	0.01	0.13 . . . .

($\bar\lambda$ stand for the exact value of λ).[2]

With the value of λ we go back to (3) and obtain the corresponding eigenfunction (τ_1 is put in terms of τ_0 and we get a solution up to an amplitude factor as usual)

In the case of an initial or boundary value problem for $y'' + y = 0$ the approximate solution is constructed

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(2) See Chaves, T - Ortiz, E.L. Zeitschrift für Angewandte Mathematik (ZAMM) 48(6) p. 415.

in the same form : take $\lambda = 1$ in the expression of the Q's and get

$$y_n^*(x) = z_0(8x^2 - 8x - 15) + z_1(2x-1)$$

i) IVP

where $y(0) = 1$, $y'(0) = 0$ we get an inhomogeneous system

$$\begin{cases} -15z_0 - z_1 = 1 \\ -8z_0 + 2z_1 = 0 \end{cases} \Rightarrow z_0 = -\frac{1}{19} \quad z_1 = -\frac{4}{19}$$

from which $y_n^*(x) = \frac{-1}{19}(8x^2 - 19) = 1 - \frac{8}{19}x^2$

ii) BVP

where $y(0) = 2$, $y(1) = 1$ $\Rightarrow$

$$\begin{aligned} -15z_0 - z_1 &= 2 \\ -15z_0 + z_1 &= 1 \end{aligned} \qquad z_0 = -1/10, \; z_1 = -1/2$$

from which $y_n^*(x) = 2 - \frac{1}{5}x - \frac{4}{5}x^2$

3.3.- The expansion of a given function in terms of a given polynomial basis

Let us assume now that $y(x)$ is known and we want an expansion of it in terms of a given polynomial basis $\varphi = \{\varphi_i(x)\}$. We shall use the Tau method to give a solution to this problem.

To be concrete, we shall assume that $f(x) = e^{-x}$, $x \in [-1,1]$ and $\varphi = \{P_n(x)\}$, the Legendre polynomials defined in $[-1,1]$.

In this case we have to choose an operator D such that f, i.e. e^{-x}, is the solution of $Dy=0$ (or $Dy=g$). We clearly take advantage of the fact that we can choose, to choose the simplest possible operator: $Dy=y'+y$, $y(0)=1$, in our case, which is such that $U_D=\phi$.

We start constructing the sequence $\mathcal{L}\equiv Q$ following theorem 3.3[3]:

$$D\varphi_n = DP_n = P_n + \sum_{k=1}^{M}[2(n-2k)+3]\,P_{n+1-2k}(x)$$

$$\Rightarrow \quad \mathcal{L}_n(x) \equiv Q_n(x) = P_n(x) - \sum_{1}^{M}(2(n-k)+3]\,Q_{n+1-2k}(x)$$

As a perturbation we take $T_n(x)$ expressed in Legendre polynomials and solve the perturbed problem:

$$Dy_n^* = \tau T_n(x) = \tau\sum_{0}^{n} c_i P_i(x)$$

as before. The desired expansion is:

$$y_n^* = \tau\sum_{0}^{n} c_i Q_i(x) = \frac{\sum_{i=0}^{n} c_i Q_i(x)}{\sum_{i=0}^{n} c_i Q_i(0)}$$

As a second example we expand the same function, e^{-x}, in $x\in[0,1]$, now in terms of Hermite polynomials $H_n(x)$. Since

$$\int_0^x H_n(t)\,dt = \frac{1}{2(n+1)}[H_{n+1}(x)-H_{n+1}(0)]$$

it seems more appropriate use an <u>integral</u> operator for D: we take then

$$Dy = y(x)+\int_0^x y(t)\,dt = C$$

where C is, in this case, equal to 1. The Q_n are computed as before: $DH_n = H_n(x)+[H_{n+1}(x)-H_{n+1}(0)]/2(n+1) \Rightarrow$
$Q_n = 2n\{H_{n-1}(x)-Q_{n-1}(x)+\alpha\}$, where $\alpha = H_n(0)Q_0(x)$, is evaluated by recursion and the solution of the

(3) [illegible]

perturbed problem

$$y(x)+\int_0^x y(t)\,dt = 1 + h_n(x), \quad \text{(perturbation term)}$$

with $h_n = \tau T_n(x)$ expressed in terms of the Hermite polynomials, is calculated as before.

For $n=4$ the errors in the coefficients obtained with the Tau method and those given by the $L^2[0,1]$ expansion of e^{-x} in terms of Hermite polynomials are:

$k=0$	1	2	3	4
0.01	0.03	0.01	0.001	0.00004

the maximum error (f- approximate Hermite expansion) in $[0,1]$ is

$$0.3\times 10^{-7}.$$

A particularly interesting case is that when $\varphi=\{T_n(x)\}$, the Chebyshev basis. Since the perturbation term is itself a Chebyshev (or a linear combination) polynomial, the solution is now simply

$$y_n^* = \tau\, Q_n(x)$$

The recursive procedure we have described above allows us then to construct recursively approximate expansions of functions in terms of Chebyshev polynomials. These expansions are of considerable interest in numerical mathematics and tables have been computed for the most used mathematical functions; these tables can be constructed recursively using the present

3.4.- Step by Step formulation: piecewise approximation

$Dy = 0$ is, as before, our problem. We put $f \equiv 0$ for simplicity. The problem has given supplementary conditions and the solution is required in the interval $J \subset \mathbb{R}_1$.

We shall construct the solution by dividing J in subintervals d_i : $\bigcup_{i=1}^{n} d_i = J$ and construct Tau solutions in the subintervals d_i:

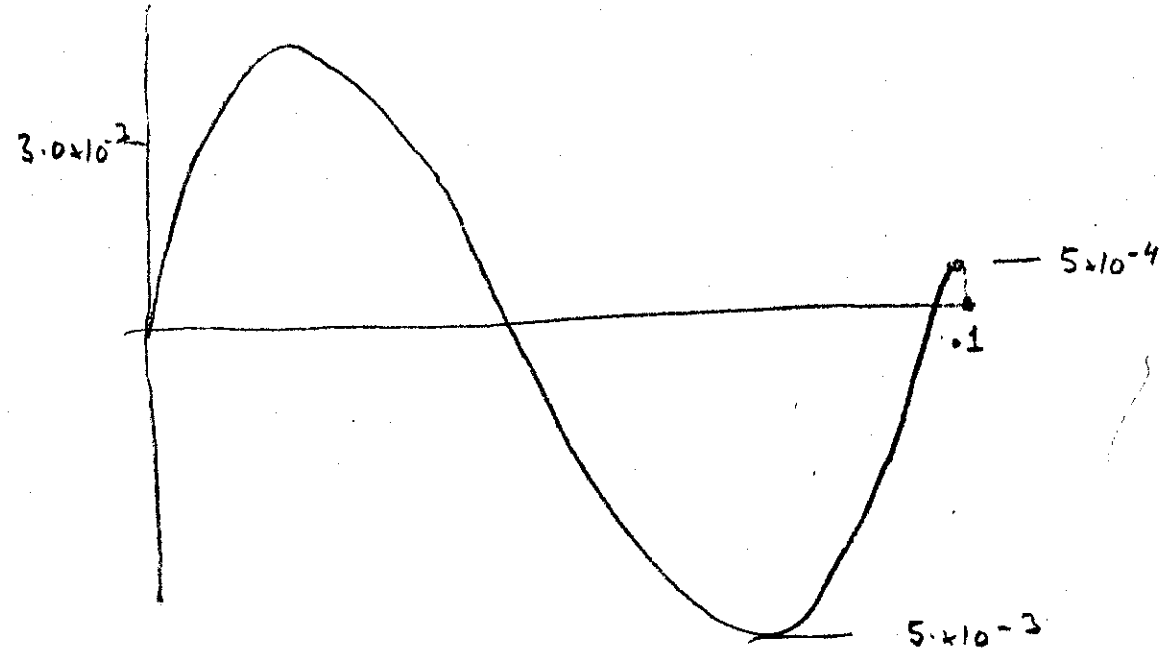
The first of them is constructed with the given conditions and for the construction of the solution in d_1 we first choose the perturbation $H_{n_1}(x)$, $x \in d_1$, in such a way that the error in the solution is minimized at the end point of d_1. The solution of this problem is used as the starting point of the approximate solution in d_2. We repeat the process until $i = n$.

Example: we take $Dy = y' + y = 0$, $x \in [0, ?]$, $|d_i| = 1/10$.

and solve $Dy^*_{n_1} = y^{*\prime}_{n_1} + y^*_{n_1} = \tau T_n$

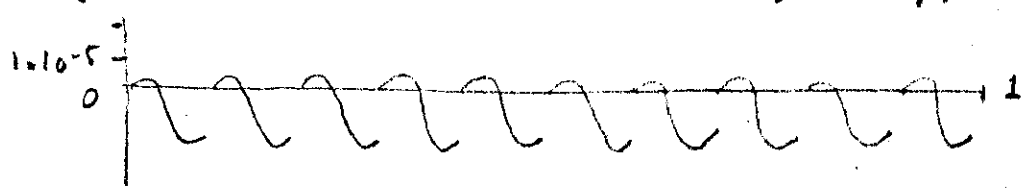
where T_n is the Chebyshev polynomial of order n defined in $[0, 0.1]$. We then solve the same problem (notice that the Q_n's don't change!) with $H_n = \tau P_n$, where P_n is the Legendre polynomial defined for $x \in [0, 0.1]$.

The maximum error of the last solution is



considerably reduced. Figure 1 shows the error curve for $Dy = \tau P_n$ with $P_n \in [0,1]$: it clearly shows the marked reduction of the error at the end point of the interval.[4]

Figure 2 shows a series of step by step Tau solutions. The error over $[0,1]$ has been reduced from 5×10^{-3} to 10^{-5}. However, the approximate solu-



tion is not continuous at the nodes $0.1, 0.2, \ldots, 0.9$. Finally Figure 3 shows the error curve for the case when only the Legendre solution is used: with a slight loss in accuracy we get a continuous solution:

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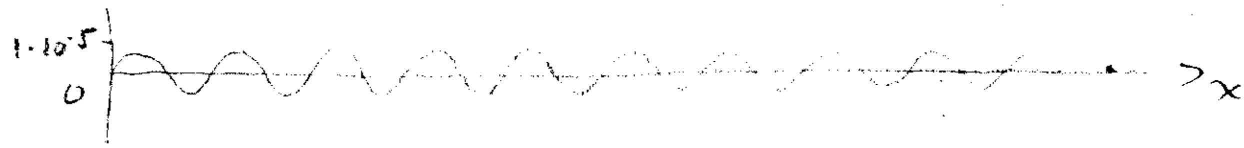
(4) See E. L. Ortiz, Seminaire Lions: Analyse Numérique et Contrôle [illegible]



Notice the balancing of the [illegible] about the $\overrightarrow{OX}$. (The irregularities are due to our tracing of the curve by hand !).

Details of a method based on the [illegible] of the segmented in a <u>single</u> <u>finite</u> <u>element</u> that is then mapped succesively over the d_i's is discussed elsewhere (5)

<u>Remark 2·3</u> i) We notice finally that the basis Q (or [illegible] if $U_D \neq \emptyset$) can be interpreted as a <u>permanent</u> basis constructed by means of Th. 5.1

ii) An extension to PDE is found in the Thesis of Mr. C. Wright (Imperial College, London 1976). Systems of DE and simultaneous approximation of a function and its derivative with the Tau method are discussed in the Thesis of Mr. J. Freilich (Imperial College, to be submitted in 1977) and in a forthcoming paper of Freilich - Ortiz.

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(5) E.L. Ortiz, J. of Maths. and Comp. with Appl. Pergamon Press, 1975