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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS TO MECHANICS

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Lecture II : Kinematics of homogeneous turbulence.

Lecture III : Dynamical Invariants and Universal Equilibrium Theory.

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II Kinematics of homogeneous turbulence

2.1 Fourier transform of $\underset{\sim}{u}$ and relation to spectrum tensor

We now consider some simple aspects of the statistical description of a random solenoidal vector field $\underset{\sim}{u}(\underset{\sim}{x})$ whose statistical properties are invariant under ~~Galilean transformation~~ translation of the frame of reference. We may imagine an ensemble of realisations of such fields, $\underset{\sim}{u}(\underset{\sim}{x}, \omega)$ where the parameter ω $(\in \Omega)$ is a label for the members of the ensemble. If $P(\omega)$ is the probability density function for the ensemble, then the ensemble average of any functional of $\underset{\sim}{u}$, $\mathcal{F}\{\underset{\sim}{u}\}$, is defined as

$$\langle \mathcal{F}\{\underset{\sim}{u}\} \rangle = \int_\Omega \mathcal{F}\{\underset{\sim}{u}\}\, P(\omega)\, d\omega. \qquad (2.1)$$

We choose the frame of reference in which $\langle \underset{\sim}{u} \rangle = 0$. All boundary effects are ignored, it being supposed that the fluid fills all space. [e.g. $\Omega = \{1, 2, \cdots N\}$, $N \gg 1$; $P(\omega) = \frac{1}{N}$ would be a natural choice.] (and $\int_\Omega \to \Sigma$)

First let us define the Fourier transform of $\underset{\sim}{u}$:

$$\tilde{\underset{\sim}{u}}(\underset{\sim}{k}) = \frac{1}{(2\pi)^3} \int \underset{\sim}{u}(\underset{\sim}{x})\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, d^3\underset{\sim}{x} \qquad (2.2)$$

(the parameter ω being now implicit). This integral clearly does not converge in a conventional sense, and must be understood as a generalised function (or 'distribution'). We have the inverse relation

$$\underset{\sim}{u}(\underset{\sim}{x}) = \int \tilde{\underset{\sim}{u}}(\underset{\sim}{k})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, d^3\underset{\sim}{k} \qquad (2.3)$$

Since $\underset{\sim}{u}(\underset{\sim}{x})$ is real, we have for all $\underset{\sim}{k}$,

$$\hat{\underset{\sim}{u}}(-\underset{\sim}{k}) = \hat{\underset{\sim}{u}}^*(\underset{\sim}{k}) \qquad (2.4)$$

the star denoting a complex conjugate, and since $\nabla\cdot\underset{\sim}{u} = 0$, we have also for all $\underset{\sim}{k}$

$$\underset{\sim}{k}\cdot\hat{\underset{\sim}{u}}(\underset{\sim}{k}) = 0 \qquad (2.5)$$

Now consider the mean quantity

$$\langle \tilde{u}_i(\underset{\sim}{k})\, \tilde{u}_j^*(\underset{\sim}{k}') \rangle = \frac{1}{(2\pi)^6} \iint \langle u_i(\underset{\sim}{x})\, u_j(\underset{\sim}{x}') \rangle\, e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \underset{\sim}{k}'\cdot\underset{\sim}{x}')}\, d\underset{\sim}{x}\, d\underset{\sim}{x}'. \quad (2.$$

$$\langle u_i(\underset{\sim}{x})\, u_j(\underset{\sim}{x}')\rangle = R_{ij}(\underset{\sim}{r}), \tag{2.7}$$

the correlation tensor of the field $\underset{\sim}{u}$. Using the basic property of the δ-function

$$\int e^{-i(\underset{\sim}{k}-\underset{\sim}{k}')\cdot\underset{\sim}{x}}\, d\underset{\sim}{x} = (2\pi)^3\, \delta(\underset{\sim}{k}-\underset{\sim}{k}') \tag{2.8}$$

(2.6) then takes the form

$$\langle u_i(\underset{\sim}{k})\, \tilde{u}_j^*(\underset{\sim}{k}')\rangle = \Phi_{ij}(\underset{\sim}{k})\, \delta(\underset{\sim}{k}-\underset{\sim}{k}') \tag{2.9}$$

where
$$\Phi_{ij}(\underset{\sim}{k}) = \frac{1}{(2\pi)^3}\int R_{ij}(\underset{\sim}{r})\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}}\, d\underset{\sim}{r}\,. \tag{2.10}$$

The tensor $\Phi_{ij}(\underset{\sim}{k})$ is the <u>spectrum tensor</u> of the field $\underset{\sim}{u}(\underset{\sim}{x})$ and it plays a fundamental rôle in theories of turbulence. From (2.4), it satisfies the condition of Hermitian symmetry

$$\Phi_{ij}(\underset{\sim}{k}) = \Phi_{ji}(-\underset{\sim}{k}) = \Phi_{ji}^*(\underset{\sim}{k}) \tag{2.11}$$

while from (2.5), we have for all $\underset{\sim}{k}$,

$$k_j\, \Phi_{ij}(\underset{\sim}{k}) = k_i\, \Phi_{ij}(\underset{\sim}{k}) = 0\,. \tag{2.12}$$

Also the following theorem (Cramer's theorem) is of basic importance: the necessary and sufficient condition that $\Phi_{ij}(\underset{\sim}{k})$ be the spectrum tensor of a <u>real</u> random field $\underset{\sim}{u}(\underset{\sim}{x})$ of the type considered here is that

$$X_i X_j^*\, \Phi_{ij}(\underset{\sim}{k}) \geqslant 0, \tag{2.13}$$

for <u>arbitrary</u> complex vectors X_i.

The vorticity field $\underset{\sim}{\omega} = \nabla\wedge\underset{\sim}{u}$ evidently has Fourier transform $\tilde{\underset{\sim}{\omega}} = i\underset{\sim}{k}\wedge\tilde{\underset{\sim}{u}}$, and spectrum tensor

$$\Omega_{ij}(\underset{\sim}{k}) = \epsilon_{imn}\,\epsilon_{jpq}\, k_m k_p\, \Phi_{nq}(\underset{\sim}{k}) \tag{2.14}$$

In particular, using (2.12),

we have
$$\Omega_{ii}(\underset{\sim}{k}) = k^2 \Phi_{ii}(\underset{\sim}{k}). \tag{2.16}$$

2.2 Energy and helicity spectrum functions

The energy spectrum function $E(k)$ is defined by
$$E(k) = \frac{1}{2}\int_{S_k} \Phi_{ii}(\underset{\sim}{k})\, dS_k \tag{2.17}$$
the integration being over the sphere of radius k in $\underset{\sim}{k}$-space.
Note that
$$\tfrac{1}{2}\langle \underset{\sim}{u}^2\rangle = \tfrac{1}{2}R_{ii}(0) = \tfrac{1}{2}\int \Phi_{ii}(\underset{\sim}{k})\, d^3\underset{\sim}{k} = \int_0^\infty E(k)\, dk \,, \tag{2.18}$$
so that $\rho E(k)\,dk$ represents the contribution to kinetic energy density from the wave-number range $(k, k+dk)$ in $\underset{\sim}{k}$-space.
Choosing $\underset{\sim}{X}$ real and in the direction of the 1-axis in (2.13) gives $\Phi_{11}(\underset{\sim}{k}) \geq 0$ and similarly $\Phi_{22}(\underset{\sim}{k}) \geq 0$, $\Phi_{33}(\underset{\sim}{k}) \geq 0$.
Hence in particular
$$E(k) \geq 0 \quad \text{for all } k. \tag{2.19}$$
Note from (2.16), by analogy with (2.18), the mean square vorticity is given by
$$\tfrac{1}{2}\langle \underset{\sim}{\omega}^2\rangle = \int_0^\infty k^2 E(k)\, dk \tag{2.20}$$

We may further define the helicity spectrum function $F(k)$ by
$$F(k) = i\int_{S_k} \epsilon_{ikl}\, k_k\, \Phi_{il}(\underset{\sim}{k})\, dS_k \tag{2.21}$$
so that, with $\underset{\sim}{\omega} = \nabla \wedge \underset{\sim}{u}$,
$$\langle \underset{\sim}{u}\cdot\underset{\sim}{\omega}\rangle = i\epsilon_{ikl}\int k_k\, \Phi_{il}(\underset{\sim}{k})\, d\underset{\sim}{k} = \int_0^\infty F(k)\, dk \tag{2.22}$$
Unlike $E(k)$, $F(k)$ can be positive or negative; in fact since it is a pseudo-scalar rather than a pure scalar, it changes sign under

In turbulence that is reflexionally symmetric, pseudo-scalar quantities such as $F(k)$ must be identically zero, and this is the situation that has traditionally been most intensively studied both theoretically and in the laboratory. In these lectures however I wish to retain, and to consider explicitly, the effects of departures from reflexional symmetry, firstly because such effects turn out to be of crucial importance in the challenging problem of the generation of magnetic fields by turbulent motion in conducting fluids, and secondly because consideration of relatively 'new' effects, such as non-zero mean helicity in turbulence, forces us to look at old problems from a fresh point of view.

2.3 Isotropic turbulence

The field $\underset{\sim}{u}(\underset{\sim}{x})$ is statistically isotropic (as well as homogeneous) if its statistical properties are invariant under rotations (as well as translations) of the frame of reference. Note here that we employ 'isotropy' in the weak sense, so that a field can be isotropic and yet lack reflexional symmetry. In particular $\Phi_{ij}(\underset{\sim}{k})$ must be an isotropic function of $\underset{\sim}{k}$, and the most general possibility satisfying (2.12) is then

$$\Phi_{ij}(\underset{\sim}{k}) = a(k)\,(k^2\delta_{ij} - k_i k_j) + i\,b(k)\,\epsilon_{ijk} k_k \,, \qquad (2.23)$$

where $a(k)$ and $b(k)$ are real (by virtue of 2.11). Equations (2.17) and (2.21) now show that

$$a(k) = \frac{E(k)}{4\pi k^4}\,, \qquad b(k) = \frac{F(k)}{8\pi k^4}\,. \qquad (2.24)$$

How can we conceive of turbulence that is statistically isotropic, and yet has non-zero helicity? Suppose that fluid is contained within a sphere whose surface is perforated with a large number of small holes distributed randomly. Suppose that a little right-handed screw propellor is mounted in each hole and that fluid is injected at high speed through a subset of the holes, emerging through the complement of the subset. The flow in the neighbourhood of the centre of the sphere is then certainly turbulent and isotropic (statistically there is no preferred direction) but nevertheless there should clearly be a detectable ~~difference~~ ~~between~~ preference for positive values of $\underset{\sim}{u}\cdot\underset{\sim}{\omega}$ since each fluid particle entering the sphere is forced to follow a right-handed helical path. I give this idealised example merely to indicate that the situation considered is at least conceivable; I shall indicate later how such non-reflexionally symmetric situations do indeed arise in natural flow systems in astrophysical and geophysical contexts.

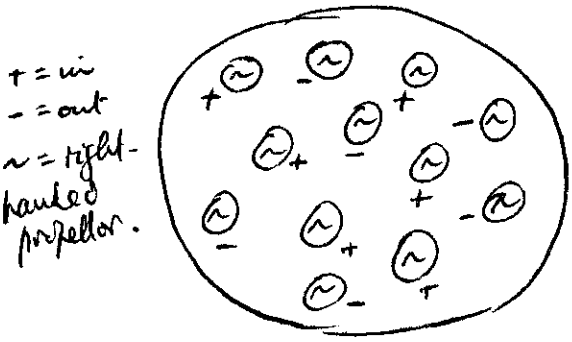


Figure 4.

III Dynamical Invariants and Universal Equilibrium Theory

3.1 Conservation of mean energy and helicity when $\nu = 0$.

Let us return now to eqn. (2.) in the conceptual limit $\nu = 0$, ~~and~~ and with $\underset{\sim}{f} = 0$, viz.

$$\frac{D\underset{\sim}{u}}{Dt} \equiv \frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u} = -\nabla P \qquad (2.25)$$

As long as $\underset{\sim}{u}$ remains analytic at all points, it is clear that this equation guarantees conservation of mean kinetic energy density; this follows from the result

$$\frac{d}{dt}\,\frac{1}{2}\langle \underset{\sim}{u}^2\rangle = \langle \underset{\sim}{u}\cdot D\underset{\sim}{u}/Dt\rangle = -\nabla\cdot\langle \underset{\sim}{u}P\rangle \qquad (2.26)$$

and the fact that an ensemble average quantity such as $\langle \underset{\sim}{u}P\rangle$ is independent of $\underset{\sim}{x}$ in the (supposed) homogeneous situation. It is less well-known that the mean helicity $\langle \underset{\sim}{u}\cdot\underset{\sim}{\omega}\rangle$ is also invariant. To prove this, note that the curl of (2.25) is the well-known vorticity equation of 'ideal' fluid mechanics

$$\frac{D\underset{\sim}{\omega}}{Dt} = \underset{\sim}{\omega}\cdot\nabla\underset{\sim}{u}. \qquad (2.27)$$

From (2.25) and (2.27),

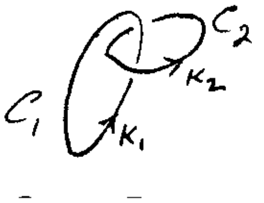
$$\begin{aligned}\frac{D}{Dt}(\underset{\sim}{u}\cdot\underset{\sim}{\omega}) &= -\underset{\sim}{\omega}\cdot\nabla P + \underset{\sim}{u}\cdot(\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{u}\\ &= (\underset{\sim}{\omega}\cdot\nabla)\left(-P+\tfrac{1}{2}\underset{\sim}{u}^2\right),\end{aligned}$$

whence

$$\frac{d}{dt}\langle \underset{\sim}{u}\cdot\underset{\sim}{\omega}\rangle = \nabla\cdot\langle \underset{\sim}{\omega}(-P+\tfrac{1}{2}\underset{\sim}{u}^2)\rangle = 0 \qquad (2.28)$$

for the same reason as above.

What is the physical interpretation of this invariance of mean helicity? To appreciate this, consider the very special vorticity distribution illustrated in figure 5



in which vorticity is confined to two linked vortex tubes of strengths κ_1 and κ_2, and of vanishingly small cross-sections. ... well-known that in the ideal fluid context, 'vortex lines move with the fluid', so that any topological linkage such as that illustrated in figure 5 is necessarily preserved. The invariance of helicity in fact reflects this basic property; for the configuration of figure 5, the <u>total</u> helicity, defined as

$$I = \int \underset{\sim}{u} \cdot \underset{\sim}{\omega} \, d^3\underset{\sim}{x} \qquad (2.29)$$

is given by

$$I = \kappa_1 \oint_{C_1} \underset{\sim}{u} \cdot d\underset{\sim}{x} + \kappa_2 \oint_{C_2} \underset{\sim}{u} \cdot d\underset{\sim}{x} \qquad (2.30)$$

and since

$$\oint_{C_1} \underset{\sim}{u} \cdot d\underset{\sim}{x} = \kappa_2 , \quad \oint_{C_2} \underset{\sim}{u} \cdot d\underset{\sim}{x} = \kappa_1 \qquad (2.31)$$

we have

$$I = 2 \kappa_1 \kappa_2 . \qquad (2.32)$$

If the vortex tubes were unlinked we would have $I = 0$; for more general linkages we have generally

$$I = \pm 2 n \kappa_1 \kappa_2 , \quad n = 0, 1, 2, \ldots \qquad (2.33)$$

depending on the 'degree of linkage'. The parameter n is in fact the fundamental topological invariant of the two curves C_1 and C_2, and the helicity of the velocity field is intimately related to this invariant.

3.2 <u>Universal equilibrium theory in reflexionally symmetric turbulence</u>

The traditional picture of turbulence (Kolmogorov 1941) may be briefly summarised as follows: suppose (to fix ideas) that a random body force $\underset{\sim}{f}(\underset{\sim}{x}, t)$ injects energy into a fluid on a length-scale l_0 and at a rate ϵ per unit mass. On dimensional grounds it is to be expected that a mean kinetic energy $\frac{1}{2}u_0^2 = \frac{1}{2}\langle \underset{\sim}{u}^2 \rangle$ will be generated where

where A is a dimensionless constant of order unity. We suppose that

$$R = u_0 l_0/\nu \gg 1 \qquad (3.2)$$

The eddies generated on the scale l_0 may then be expected to be unstable to disturbances on a smaller length-scale l_1, which will then in turn be unstable to smaller disturbances on a scale l_2, and so on, with always $l_{n+1} < l_n$. The energy generated may be conceived as 'cascading' through ever-decreasing length-scales until ultimately, on some 'inner' length-scale l_ν, viscous effects begin to take effect. In terms of the energy spectrum $E(k)$, the situation is as illustrated in figure 6:

Fig. 6

On this picture the wave-number k_ν is determined in order of magnitude solely by the parameters ϵ and ν; on dimensional grounds,

$$k_\nu \sim (\epsilon/\nu^3)^{1/4} \qquad (3.3)$$

or equivalently

$$k_\nu^{-1} \sim l_\nu \sim (\nu^3/\epsilon)^{1/4} \qquad (3.4)$$

Note now that, from (3.1) and (3.4),

$$l_\nu/l_0 \sim R^{-3/4} \ll 1 \qquad (3.5)$$

i.e. the picture is so far self-consistent!

In the 'inertial range of wave-numbers' $k_0 \ll k \ll k_\nu$, it is evident that (provided motions on widely different [illegible] $E(k)$ should be determined by ϵ and k and by no other dimensional parameters (viscosity being operative only for $k \gtrsim k_\nu$); on dimensional grounds it follows that

$$E(k) = C\,\epsilon^{2/3}\,k^{-5/3} \qquad (k_0 \ll k \ll k_\nu) \qquad (3.6)$$

where C is a universal constant (the same for all turbulent flows); most experimental investigations place this constant in the range $C \approx 0.5 \pm 0.2$ (See A.A. Townsend: Turbulent Shear Flow, 1976, p. 98). Equation (3.6) is the celebrated Kolmogorov spectrum, widely supported by experimental observation in atmospheric and oceanographic turbulence and in laboratory turbulence, but also known not to be quite asymptotically exact in the limit $R = (k_\nu/k_0)^{4/3} \to \infty$, for reasons discussed at length in Monin & Yaglom, [Statistical Fluid Mechanics II (1975), §25].

The picture then is that energy is 'injected' on a scale l_0, and dissipated at the same rate ϵ on scales $\lesssim l_\nu = R^{-3/4} l_0$. The energy balance equation is

$$\frac{d}{dt}\,\tfrac{1}{2}\langle u^2\rangle = -\nu\langle \underset{\sim}{u}\cdot\nabla\wedge(\nabla\wedge\underset{\sim}{u})\rangle + \langle \underset{\sim}{u}\cdot\underset{\sim}{f}\rangle$$

$$= -\nu\langle \underset{\sim}{\omega}^2\rangle + \langle \underset{\sim}{u}\cdot\underset{\sim}{f}\rangle$$

and if statistical equilibrium is established then

$$\epsilon = \langle \underset{\sim}{u}\cdot\underset{\sim}{f}\rangle = \nu\langle \underset{\sim}{\omega}^2\rangle \qquad (3.7)$$

Note that $\frac{1}{2}\langle\underset{\sim}{\omega}^2\rangle$, which satisfies (2.20), has spectrum function

$$k^2 E(k) = C\,\epsilon^{2/3}\,k^{1/3} \qquad (k_0 \ll k \ll k_\nu) \qquad (3.8)$$

in the inertial range. Unlike $\langle\underset{\sim}{u}^2\rangle$, $\langle\underset{\sim}{\omega}^2\rangle$ does depend on ν, and in fact from (3.7) $\langle\underset{\sim}{\omega}^2\rangle = \epsilon/\nu$ increases

We may define a further length-scale l_c by

$$l_c^2 = \langle u^2 \rangle / \langle \omega^2 \rangle = \frac{u_0^2 \nu}{\epsilon} \sim l_0^2 / R \qquad (3.9)$$

from (3.1). (Constants of order unity are unimportant in such estimates). Hence $l_c = l_0 R^{-1/2}$, and $k_c = l_c^{-1} = k_0 R^{1/2}$ is a wave-number in the inertial range (since $k_0 \ll k_c \ll k_v$). It may be shown from (3.9) that l_c provides a measure of the mean radius of curvature of an instantaneous streamline of $\underset{\sim}{u}(\underset{\sim}{x}, t)$. Such a streamline is depicted in figure 7:

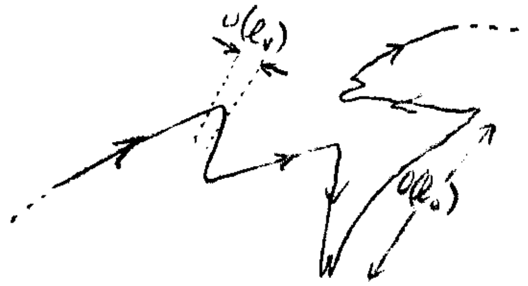


Fig. 7

The two-length scales l_0 and l_v are apparent here; l_c is concealed, but note simply that $l_c \sim (l_0 l_v^2)^{1/3}$.

For $k \gtrsim k_v$, $E(k)$ may be expected to decrease rapidly (e.g. like $e^{-(k/k_v)^n}$ for some $n > 0$), because of the increasing influence of viscosity. Even in this "far dissipation range" however, there is no completely satisfactory theory for the nature of this "viscous cut-off", an indication of the peculiar intractability of the subject of turbulence dynamics!

3.3 The spectrum for a passive scalar field.

Let $\theta(\underset{\sim}{x}, t)$ be a scalar field (e.g. temperature, dye concentration radioactive contaminant etc.) assumed dynamically passive (i.e. its presence does not affect the above arguments concerning $\underset{\sim}{u}$.

Suppose for the moment also that $\theta(\underset{\sim}{x}, t)$ is a homogeneous random function with $\langle\theta\rangle = 0$. [Later we shall consider the more interesting situation when θ is non-homogeneous.] The equation governing the evolution of θ is

$$\frac{\partial\theta}{\partial t} + \underset{\sim}{u}\cdot\nabla\theta = \kappa\nabla^2\theta, \qquad (3.10)$$

where κ is the appropriate molecular diffusivity. It follows from (3.10) that

$$\frac{d}{dt}\,\frac{1}{2}\langle\theta^2\rangle = -\kappa\langle(\nabla\theta)^2\rangle \qquad (3.11)$$

Let us suppose that $u_0 l_0 \gg \kappa \gg \nu$; then (i) diffusion is negligible for some range of wave-numbers $k_0 \ll k \ll k_d$, k_d representing the wave-number (in order of magnitude) where diffusion begins to be important, and (ii) since $\kappa \gg \nu$, we can be sure on physical grounds that $k_d \ll k_\nu$. It follows that k_d is determined solely by ϵ and by κ and is therefore given by (cf. 3.3)

$$k_c \sim (\epsilon/\kappa^3)^{1/4} \qquad (3.12)$$

Now suppose that the level of $\langle\theta^2\rangle$ (let me call this 'thetergy' by analogy with energy, and for want of a better term!) is maintained by injection at a rate η at length-scales of order l_0. Since (3.10) is linear in θ, ~~[illegible]~~ the spectrum function $\Gamma(k)$ of $\theta(\underset{\sim}{x}, t)$, defined in such a way that

$$\langle\theta^2\rangle = \int_0^\infty \Gamma(k)\,dk, \qquad (3.13)$$

is proportional to η and is otherwise determined solely by

$$\Gamma(k) = C' \eta \epsilon^{-1/3} k^{-5/3} \qquad (k_0 \ll k \ll k_d) . \qquad (3.14)$$

This is the Corrsin-Obukhov spectrum; C' is a further universal dimensionless constant, of order unity.

Consider now the form of the spectrum in the wave-number range $k_d \ll k \ll k_\nu$. Here ~~[illegible]~~ strong diffusion eliminates field fluctuations very rapidly, because of the small length-scale. The surfaces $\theta = $ cst. are only weakly perturbed by the turbulence. The fluctuations $\theta'(\underset{\sim}{x}, t)$ on such scales are determined by the equation

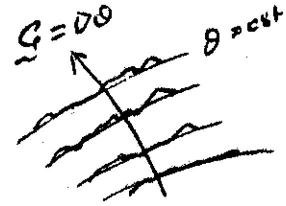


Fig. 8

$$\kappa \nabla^2 \theta' \approx - \underset{\sim}{u} \cdot \underset{\sim}{G} , \qquad (3.15)$$

where $\underset{\sim}{G}$ is the local mean value of $\nabla\theta$. The Fourier transform of (3.15), regarding $\underset{\sim}{G}$ as constant is

$$\cancel{-\kappa \nabla^2 \theta} \quad \kappa k^2 \tilde{\theta}' = - \underset{\sim}{\hat{u}} \cdot \underset{\sim}{G}$$

and so

$$4\pi k^2 \Gamma(k) = \frac{1}{\kappa^2 k^4} \Phi_{ij}(\underset{\sim}{k}) G_i G_j \qquad (3.16)$$

This purely local treatment is incorrect in so far as it fails to take account of the variation in direction of $\underset{\sim}{G}$ on scales $\gtrsim l_d$. To correct for this, we suppose now in (3.16) that $\underset{\sim}{G}$ is isotropically distributed in direction, and average over these directions to obtain

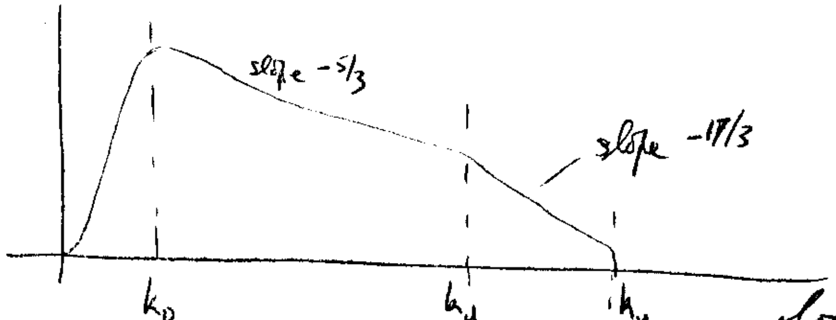
$$\Gamma(k) = \frac{1}{\kappa^2 k^4} \langle \underset{\sim}{G}^2 \rangle E(k) \qquad \text{for } k \gg k_d \qquad (3.17)$$

In particular, for $k_d \gg k \gg k_\nu$, we have

$$\Gamma(k) \propto k^{-17/3}$$

showing the fairly rapid decrease of Γ with k associated with [illegible] smaller length-scales.

The total picture as far as $T(k)$ is concerned is now as in figure 9: again there is some kind of exponential cut-off for $k \gtrsim k_v$.



3.4. <u>Effect of helicity on energy cascade; form of helicity spectrum</u>

Let us now return to the purely dynamical considerations concerning energy cascade, and let us suppose that helicity as well as energy is injected at the scale l_0 (as in the idealised experiment described on p. 14). Consider the dimensionless quantity

$$\mathcal{H} = \langle \underset{\sim}{u} \cdot \underset{\sim}{\omega} \rangle \Big/ \langle \underset{\sim}{u}^2 \rangle^{\frac{1}{2}} \langle \underset{\sim}{\omega}^2 \rangle^{\frac{1}{2}} \qquad (3.9)$$

which I shall describe as the "relative helicity".

By the Schwarz inequality,

$$|\mathcal{H}| \leq 1, \qquad (3.10)$$

with equality only if $\underset{\sim}{u}$ and $\underset{\sim}{\omega}$ are everywhere parallel. An example of a motion for which $\mathcal{H} = 1$ is the space-periodic velocity field (in Cartesian coordinates)

$$\underset{\sim}{u} = u_0(\sin kz,\ \cos kz,\ 0) \qquad (3.11)$$

for which

$$\underset{\sim}{\omega} = k\underset{\sim}{u}, \qquad (3.12)$$

a motion that may be described as a 'pure helicity mode'.

The situation $|\mathcal{H}| = O(1)$ is however totally incompatible with the existence of a Kolmogorov spectrum, for the following reason.