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# MATHEMATICAL RESULTS FOR BINGHAM FLUIDS
(Part II)

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## §3. Some complements on convex differentiable and nondifferentiable functionals.

Let be $B$ a Banach space and $B'$ its dual.
Let be $f$ a function $v \to \Phi(v)$ from $B \to \mathbb{R}$.

<u>Definition 1</u> The function $v \to \Phi(v)$ is <u>convex</u> if $\forall u, v \in B$ and $\forall \alpha \in ]0,1[$ :

$$\Phi(\alpha u + (1-\alpha)v) \leq \alpha \Phi(u) + (1-\alpha)\Phi(v)$$

It is called <u>strictly convex</u> if $\forall u, v \in B$ and $\forall \alpha \in ]0,1[$ :

$$\Phi(\alpha u + (1-\alpha)v) < \alpha \Phi(u) + (1-\alpha)\Phi(v)$$

<u>Definition 2</u>. The function $\Phi$ is Gateaux-differentiable (or G-differentiable) at $u \in B$ if there exists one element of $B'$, denoted by $\Phi'(u)$ such that

$$\lim_{\lambda \to 0} \frac{\Phi(u+\lambda v) - \Phi(u)}{\lambda} = \langle \Phi'(u), v \rangle \quad \forall v \in B.$$

($\Phi'(u)$ is called also the gradient of $\Phi$ at $u$)

<u>Lemma 1</u>. If $\Phi$ is convex and G-differentiable

$$\Rightarrow \quad \Phi(v) - \Phi(u) \geq \langle \Phi'(u), v-u \rangle \quad \forall u, v \in B$$

<u>Proof</u>. By the convexity of $\Phi$ we have

$$\Phi(u + \theta(v-u)) \leq \Phi(u) + \theta(\Phi(v) - \Phi(u))$$

$$\Rightarrow \quad \Phi(v) - \Phi(u) \geq \frac{\Phi(u+\theta(v-u)) - \Phi(u)}{\theta} \xrightarrow[\text{as } \theta \to 0]{}$$

Lemma 2 (converse of lemma 1)

If $\Phi$ is $G$- differentiable and if

$$\Phi(v) - \Phi(u) \geq \langle \Phi'(u), v-u \rangle \quad \forall v, u \in B$$

$\Rightarrow$ $\Phi$ is convex.

Proof

We have to prouve that if $\theta \in (0,1)$

$$\Phi(\theta u + (1-\theta)v) \leq \theta \Phi(u) + (1-\theta)\Phi(v)$$

Let be $w = \theta u + (1-\theta) v = \theta(u-v) + v$.

By hypothesis

$$\Phi(v) - \Phi(w) \geq \langle \Phi'(w), v-w \rangle = \tag{3.1}$$

$$= -\theta \langle \Phi'(w), u-v \rangle$$

$$\Phi(u) - \Phi(w) \geq \langle \Phi'(w), u-w \rangle = \tag{3.2}$$

$$= (1-\theta) \langle \Phi'(w), u-v \rangle$$

Multiply (3.1) by $(1-\theta)$ and (3.2) by $\theta$ and add:

$$(1-\theta)\Phi(v) - (1-\theta)\Phi(w) + \theta\Phi(u) - \theta\Phi(w) \geq 0$$

and thus the lemma is proved. ∎

Lemma 3

If $\Phi$ is convex and $G$-diff $\Rightarrow$

$u \longrightarrow \Phi'(u)$ is monotone.

Proof.

We have to prouve that

$$\langle \Phi'(u) - \Phi'(v), u-v \rangle \geq 0 \quad \forall u, v \in B.$$

By lemma 1 we get

$$\langle \Phi'(u), u-v \rangle \geq \Phi(u) - \Phi(v)$$

## Lemma 4

If $\Phi$ is $G$-differentiàl on $B$ then there $\exists$ $\theta \in (0,1) \ni$:

$$\Phi(u+v) = \Phi(u) + \langle \Phi'(u+\theta v), v \rangle \quad \forall u,v \in B.$$

## Proof.

Consider the real function $\lambda \longrightarrow \varphi(\lambda) = \Phi(u+\lambda v)$
Then $\varphi'(\lambda) = \langle \Phi'(u+\lambda v), v \rangle$. The result follows now by the finite increment theorem for real functions:

$$\varphi(1) - \varphi(0) = 1 \cdot \varphi'(\theta) \text{ with } \theta \in (0,1)$$

so that we have proved the lemma observing that $\varphi(1) = \Phi(u+v)$ and $\varphi(0) = \Phi(u)$ ∎

## Lemma 5 (converse of lemma 3)

If $\Phi$ is $G$-differentiable on $B$ and if $u \longrightarrow \Phi'(u)$ is a monotone operator $\Longrightarrow$

$\Phi$ is convex.

## Proof.

By lemma 4

$$\Phi(v) = \Phi(u) + \langle \Phi'(u+\theta(v-u)), v-u \rangle \quad \forall u,v \in B$$

$$= \Phi(u) + \langle \Phi'(u), v-u \rangle +$$

$$+ \langle \Phi'(u+\theta(v-u)) - \Phi'(u), v-u \rangle =$$

$$= \Phi(u) + \langle \Phi'(u), v-u \rangle +$$

$$+ \frac{1}{\theta} \underbrace{\langle \Phi'(u+\theta(v-u)) - \Phi'(u), (u+\theta(v-u)) - u \rangle}_{\geq 0 \text{ by hypothesis}}$$

Thus $\Phi(v) - \Phi(u) \geq \langle \Phi'(u), v-u \rangle$
and the statement of the lemma follows from lemma 2 ∎.

Definition

$\theta \in B'$ is an element of the subgradient of $\Phi$ att the point $u \in B \iff \Phi(v) - \Phi(u) \geq \langle \theta, v-u \rangle \ \forall v \in B$

Theorem (see [1] - Bibliography at the end of this chapter)

If $\Phi$ is a convex function such that the subgradient of $\Phi$ at the point $u$ is reduced at a point $\xi \in V' \Rightarrow$ $\Phi$ is G-differentiable at $u$ and

$$\xi = \Phi'(u)$$

∎

Usually the subgradient of $\Phi$ at the point $u$ is denoted by $\partial\Phi(u)$ (by the definition of the G-differentiability, if $\partial\Phi(u)$ is a set of points in $V'$, $\Phi$ is not G-differentiable in $u$!)

Definition $\Phi$ is weak lower semicontinuous (l.s.c) if $\underline{\lim}\ \Phi(u_n) \geq \Phi(u)$ if $u_n \xrightarrow{weakly} u$ in the topology of $B$.

Examples: 1). $u \longrightarrow \|u\|$ is weak l.s.c in a Banach space

2). Let be $B = H^2(\Omega)$. We define for $u \in B$

$$\varphi(u) = \begin{cases} \frac{1}{2}\int_\Omega |\text{grad}\, u|^2 dx + \int_\Omega |\text{grad}\, u|\, dx; & \text{if } u \in H_0^1(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

Then: $\varphi(u)$ is convex and l.s.c on $B$.

3). $j(v) = \int_\Omega D_{II}^{1/2}(v)\, dx$ if $v \in H_0^1(\Omega)^2 \Rightarrow j$ is l.s.c. ($j$ is continuous convexe on $H_0^1(\Omega)^2$ so that is weak l.s.c for the weak topology of $H_0^1(\Omega)^2$) ∎

Suppose we have to solve the following problem

$$(3.3) \begin{cases} \mu a(u, v-u) + \Phi(v) - \Phi(u) \geq (f, v-u) \\ \forall v \in V(\Omega) \text{ - a Banach space }, \ \Omega \subset \mathbb{R}^3 \end{cases}$$

$a(u,v) = \int \sum^3 \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}$ ...

<u>Lemma 6</u>

If $\Phi$ is a G-diff. convex functional for $\forall v \in V$ (3.3) is equivalent to the equation:

$$(3.4) \qquad \mu a(u, v-u) + \langle \Phi'(u), v \rangle = (f, v) \qquad \forall v \in V$$

<u>Proof</u>. Let $v$ be $v = u + \lambda w$ in (3.3) with $w$ arbitrary in $V$ and $\lambda > 0$. We get, after division by $\frac{1}{\lambda}$:

$$\mu a(u, w) + \lambda^{-1} [\Phi(u+\lambda w) - \Phi(u)] \geq (f, w)$$

Now letting $\lambda \to 0$, we have

$$\mu a(u, w) + \langle \Phi'(u, w\rangle \geq (f, w)$$

By changing $w$ in $-w$ we get the first implication, Let suppose now that $u$ is solution of (3.4.) We have

$$\mu a(u, v-u) + \langle \Phi'(u), v-u \rangle = \langle f, v-u)$$

and
$$\mu a(u, v-u) + \Phi(u) - \Phi(v) = (f, v-u) + \Phi(u) - \Phi(v) - (\Phi'(u), v-u\rangle$$

By lemma 1

$$\Phi(u) - \Phi(v) - \langle \Phi'(u), v-u) \geq 0$$

so that

$$\mu a(u, v-u) + \Phi(u) - \Phi(v) \geq (f, v-u)$$

and $u$ is solution of (3.3).

<u>Remarks</u>. 1). It is the same argument which was employed in the proof of the theorem § 2 (on the equivalence between the problems B e B')

2). (3.3) may be written after integration by parts (if allowed!) as follows:

$$-\mu \Delta u - f \in \partial \Phi(u)$$

<u>Bibliography</u>

[1]. R.T. Rockafellar – Convex analysis, Princeton Acad. Press 1970

[2]. H. Brézis – Monotonicity methods in Hilbert spaces e some applications to Nonlinear Partial

## § 4. Variational problem. Functional spaces.

Back to the variational inequality of the flow of a Bingham fluid. We have to find solutions for the Problem $B'$ (§2) such that $u \in L^\infty(0,T, L^2(\Omega))$ and $u \in L^2(0,T; X)$. For giving a usual (and justified) variational formulation we shall introduce some functional spaces.

Let $\mathcal{D}(\Omega)$ be the space of indefinitely differentiable functions on $\Omega$ with compact support. $\Omega$ is a smooth bounded set in $\mathbb{R}^3$ (or $\mathbb{R}^2$).

Definition

$$\mathcal{V} = \{ \varphi \in \mathcal{D}(\Omega)^2 \ni : \operatorname{div} \varphi = 0 \}$$

$$V = \text{closure of } \mathcal{V} \text{ in } H^1(\Omega)^2$$

$$H = \text{closure of } \mathcal{V} \text{ in } L^2(\Omega)^2$$

The norm in $H$ is given by:

$$|u| = \left( \sum_{i=1}^{3} |u_i|^2_{L^2(\Omega)} \right)^{1/2}$$

and the norm in $V$ is given:

$$\|u\| = \left( \sum_{i=1}^{3} \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right)^{1/2}$$ (which is equivalent to the norm induced by $H^1(\Omega)^2$ in $V$ as $u/_{\partial\Omega} = 0$; the norm $H^1(\Omega)^2$ is given by: $|u|_{H^1(\Omega)} = \left( |u|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \right)^{1/2}$)

We shall admit the following lemmas:

Lemma 1:

$$V = \{ v \in H_0^1(\Omega)^2 \mid \operatorname{div} v = 0 \}$$

$$H = \{ v \in L^2(\Omega)^2 \mid \operatorname{div} v = 0 \ \& \ v.n = 0 \text{ on } \partial\Omega \}$$

($v.n = \sum v_i \cos(\vec{n}.x_i)$, $\vec{n}$ is the exterior normal on $\partial\Omega$) ∎

Lemma 2.

If $f \in (H_0^1(\Omega)^2)' = H^{-1}(\Omega)^2$ is such that $(f, v) = 0 \quad \forall v \in V \implies$ there exists $p \in L^2(\Omega)$ such that $f_i = \frac{\partial p}{\partial x_i}$ $(f = \text{grad}\, p)$

We will now identify $H$ with its dual. We have thus:

$$V \underset{\text{dense}}{\subset} H \underset{\text{dense}}{\subset} V'$$

$V'$ is the dual of $V$. We shall denote by $\|\cdot\|_*$ the norm on $V'$.

If $f \in V'$ :

$$\|f\|_* = \sup_{v \in V} \frac{|(f, v)|}{\|v\|}$$

With these notations we can get now the definitive variational formulation of the problem B.

Problem 1

Find $u$ such that

$$(4.1)\quad \begin{cases} (u', v-u) + \mu a(u, v-u) + b(u, u, v-u) + \\ \qquad\qquad + g j(v) - j(u) \geq (f, v-u) \\ \forall v \in V \end{cases}$$

and

$$(4.2)\quad \begin{cases} u \in L^2(0,T;\, V) \cap L^\infty(0,T;\, H) \\ \dfrac{du}{dt} \in L^2(0,T;\, V') \qquad u(0,x) = u_0(x) \in H \end{cases}$$

$L^2(0,T;\, V')$

We shall prouve in the following chapter that if $\Omega \subset \mathbb{R}^2$ this problem has a solution which is unique. For $\Omega \subset \mathbb{R}^3$ we must introduct a class of solution which are "weaker" than those for $\Omega \subset \mathbb{R}^2$ and we will prouve an existence theorem for this case. The problem of the uniqueness for $\Omega \subset \mathbb{R}^3$ is open.

Let therefore $\Omega \subset \mathbb{R}^2$. We need the following results.

Lemma 3

If $u \in H_0^1(\Omega)$ then:

$$(4.3) \qquad \| u \|^2_{L^4(\Omega)} \leq C \| u \| \, | u |$$

Proof.

It is sufficient to prouve (4.3) for $u \in \mathcal{D}(\Omega)$ and by extending $u$ by 0 outside $\Omega$ it is sufficient to prouve (4.3) for $u \in \mathcal{D}(\mathbb{R}^2)$.

We have

$$u^2(x_1, x_2) = 2 \int_{-\infty}^{x_1} u(x_1, s) \frac{\partial u}{\partial x_2}(x_1, s)\, ds \text{ from here:}$$

$$|u(x_1, x_2)|^2 \leq 2 \int_{-\infty}^{+\infty} |u(x_1, s)| \left| \frac{\partial u}{\partial x_2}(x_1, s) \right| ds = \varphi(x_1)$$

By the same way:

$$|u(x_1, x_2)|^2 \leq 2 \int_{-\infty}^{+\infty} |u(s, x_2)| \left| \frac{\partial u}{\partial x_1}(s, x_2) \right| ds = \psi(x_2)$$

It follows

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |u(x_1, x_2)|^4 \, dx_1 \, dx_2 \leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi(x_1) \psi(x_2) \, dx_1 \, dx_2 =$$

$$= \int_{-\infty}^{+\infty} \varphi(x_1)\, dx_1 \int_{-\infty}^{+\infty} \varphi(x_2)\, dx_2 =$$

$$= 4 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |u| \left| \frac{\partial u}{\partial x_2} \right| dx_1 dx_2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |u| \left| \frac{\partial u}{\partial x_1} \right| dx_1 dx_2 \leq$$

$$\leq 4 |u|^2_{L^2} \left| \frac{\partial u}{\partial x_1} \right|_{L^2} \left| \frac{\partial u}{\partial x_2} \right|_{L^2} \leq 4 \|u\|^2_{L^2(R^2)} \|u\|^2_{H^1(R^2)} \quad \blacksquare$$

<u>Lemma 4</u>

$b(u,v,w)$ is a continuous trilinear form on $V \times V \times V$.
and $b(u,v,w) = -b(u,w,v)$ $\blacksquare$

<u>Proof.</u>

$$|b(u,v,w)| \leq \sum_{ij} |u_j|_{L^4(\Omega)} \left| \frac{\partial w_i}{\partial x_j} \right|_{L^2(\Omega)} |v_i|_{L^4(\Omega)}$$

$\leq$ by Sobolev imbedding theorem: $H_0^1(\Omega) \subset L^p(\Omega)$ $\forall p < +\infty$

$\leq$ Const. $\|u\| \, \|w\| \, \|v\|$

Then

$$b(u,v,w) + b(u,w,v) = \sum_{ij} \int_\Omega u_i \left( \frac{\partial v_j}{\partial x_i} w_j + \frac{\partial w_j}{\partial x_i} v_j \right) dx =$$

$$= \sum_{ij} \int_\Omega u_i \frac{\partial}{\partial x_i} (v_j w_j) = - \sum_{ij} \int_\Omega \frac{\partial u_i}{\partial x_i} (v_j w_j)\, dx =$$

$$= \int_\Omega \left( \sum_i \frac{\partial u_i}{\partial x_i} \right) \sum_j (v_j \cdot w_j) \xrightarrow{\text{as } \operatorname{div} u = 0} 0 \quad \blacksquare$$

Let be $B(u,v)$ defined by

$$b(u,v,w) = (B(u,v),w) \quad \forall u,v,w \in V.$$

We have the following result.

<u>Lemma 5</u>

$B$ is a continuous bilinear form from $V \times V \longrightarrow V'$ and if $u,v \in L^2(0,T;V) \cap L^\infty(0,T;H) \Rightarrow$

$$B(u,v) \in L^2(0,T;V')$$

<u>Proof</u>.

$$|(B(u,v),w)| = |b(u,v,w)| \underset{=}{\text{by lemma 4}} |b(u,w,v)| \leq$$

$$\leq \sum_{i,j} \int_\Omega |u_i| \left|\frac{\partial w_j}{\partial x_i}\right| |v_j| \leq \text{Const.} |u|_{L^4(\Omega)} |v|_{L^4(\Omega)} \|w\|$$

Thus

$$\|B(u,v)\|_* \leq C |u|_{L^4} |v|_{L^4}$$

If $u,v \in L^2(0,T;V) \cap L^\infty(0,T;H)$ by lemma 3

$$u,v \in L^4(0,T;L^4(\Omega))$$

so that

$$|u|_{L^4(\Omega)} |v|_{L^4(\Omega)} \in L^2(0,T)$$

and

$$B(u,v) \in L^2(0,T;V').$$