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MIRAMARE - P.O.B. 586 - 34100 TRIESTE (ITALY) - TELEPHONES: 224281/2/3/4/5/6 - CABLE: CENTRATOM

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Diffusion Problems with Free Boundaries

M. Frémond

Laboratoire Central
des Ponts et Chaussées
Paris, France

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Diffusion problem with free boundaries.

M. Frémond (*)

Many diffusion problems with free boundaries due to phase changes (Stefan problem) occur in technical problems: melting or solidification of ice or water in porous media, in a reservoir, in a pipe, melting of soils in cold regions, solidification and melting of soils under refrigerated warehouses in warm regions, melting of a metal [18]. Some problems in electricity involve the same equations [19]. These problems have been already studied [3, 4, 5, 9, 11, 12, 14, 15]. We present here an other formulation [6, 7, 8] which uses a new unknown the freezing index which is an important technical parameter. This new formulation is made in terms of variational inequalities. An other formulation is also given in [17].

The programme of the course is the following:

I Thermal phenomenons

a) heat conduction

b) phase change. internal energy

c) heat equation

II Hydraulic phenomenons in porous media

a) Definition of a porous medium

b) Diffusion equation

III Heat diffusion with phase change - The Stefan equations.

a) the equations

b) the freezing index - physical properties.

c) Equations verified by the freezing index

d) Variational formulation

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IV Heat diffusion with hydraulic phenomenons. Coupled Stefan equations

V Numerical results in one dimension. Technical Applications.

I Thermal phenomenons.

a) heat conduction.

Most materials conduct the heat. Experiments show that a good description of the heat conduction is obtained by assuming that the heat flux vector $\vec{q}$ is the gradient of the temperature θ. This is the Fourier law:

$$\vec{q} = -\lambda(x,\theta)\,\mathrm{grad}\,\theta \quad , (1)$$

The thermal conductivity λ can depend on the point x if the material is not homogeneous and of the temperature θ. This is the case for some foodstuffs.

b) Phase change. Internal energy.

The change from one physical state to another occurs at a constant temperature: the phase change temperature. θ_f, (fusion temperature for the change from the solid state to the liquid state). This change needs a certain quantity of energy: the specific phase change latent heat ℓ. The internal energy has a jump or a discontinuity equal to ℓ at the temperature θ_f. In the sequel we deal only with material whose (specific) internal energy depends only on the temperature θ:

$$e = \theta\,C(x,\theta) + \ell\,H(\theta-\theta_f) \quad , (2),$$

where C is the specific heat capacity and H is the step function ($H(y) < 0$ if $y<0$, $H(y)=1$ if $y>0$). Generaly C

is actually a function of θ. For the usual materials C is piecewise linear in the neighbourhood of θ_f. For foodstuffs C varies a lot with θ.

Note. One can remark that the energy is not defined if $\theta = \theta_f$ because H is not defined. We define e as follows: there is a (mass) proportion μ of the first phase (unfrozen material, for instance) and a proportion $1-\mu$ of the second phase (frozen material, for instance); the energy is:

$$e = \theta_f \, C(x, \theta_f) + \mu l \,, \quad (3)$$

c) the heat equation

We neglect all the dynamic effects and we assume that the variations of temperature involve no phase change temperature. We have then:

$$e = \theta \, C(\theta, x) \,.$$

The energy conservation law [10] gives:

$$\rho \frac{de}{dt} + q_{i,i} = r \,.$$

where ρ is the mass by unit volume and r the rate of heat production. By using the relation (1):

$$\rho \frac{d}{dt}\left(\theta C(x,\theta)\right) - \frac{\partial}{\partial x_i}\left(\lambda(x,\theta)\frac{\partial \theta}{\partial x_i}\right) = r \,, \quad (4)$$

or

$$\rho \frac{\partial}{\partial t}\left(\theta C(x,\theta)\right) + \rho \frac{dx_i}{dt} \cdot \frac{\partial}{\partial x_i}\left(\theta C(x,\theta)\right) - \frac{\partial}{\partial x_i}\left(\lambda(x,\theta)\frac{\partial \theta}{\partial x_i}\right) = r \,.$$

The second term of this expression is the heat transmitted by

[c]onvection. As we have neglected the dynamic effects $\left(\frac{dx_i}{dt}=0\right)$, we [ob]tain:

$$\rho \frac{\partial}{\partial t}(\theta C(x,\theta)) - \frac{\partial}{\partial x_i}\left(\lambda(x,\theta)\frac{\partial \theta}{\partial x_i}\right) = r ,$$

[I]f λ and C are constant:

$$\rho C \frac{\partial \theta}{\partial t} - \lambda \Delta \theta = r \quad , (5)$$

which is the <u>heat equation</u>.

<u>Note</u>. In practice the rate of heat production r can be produced by internal electrical heating or by chemical reaction (in young concrete, for instance, there is a large r).

The boundary condition which are related to the heat equation are:

$$-\vec{q}\cdot\vec{n} = g(x,t,\theta) ,$$

where $\vec{n}$ is the outer normal vector and g is a rate by unit area of heat produced by the outside. We obtain by using (1):

$$\lambda \frac{\partial \theta}{\partial n} = g .$$

Several situations can occur:

1/ g is given. A particular case is $g=0$: the boundary is <u>adiabatic</u>.

2/ g is a function of θ. The more simple case and the more useful in practice is:

$$g = -\alpha(\lambda\theta - k) .$$

The heat flux through the boundary is proportional to the fference $\theta - \frac{k}{\lambda}$, where k/λ can be the outside temperature. If $\alpha = 0$, we have the adiabatic boundary. If $\alpha \to \infty$ we obtain by passing to the limit:

$$\theta = \frac{k}{\lambda}.$$

which is the Dirichlet boundary condition. The temperature is then known on the boundary.

Hydraulic phenomenons in porous media.

a) Definition of a porous medium

Porous medium means a medium made of pieces of materials stuck or not to one another and separated by voids. We decide to consider this medium only from the point of view of his average properties and not from the point of view of his local properties which are the properties of the materials and of the void. We use only average parameters and consider a porous medium as a continuous medium where all the parameters are defined at every point. The scale of the average must be defined. For instance, a sand is a porous medium if the scale is one meter but it is not a porous medium if the scale is one millimeter. One must remind that the notion of porous medium is relative.

The porosity of a porous medium is the average value of the ratio of the volume of voids to the total volume:

$$\varepsilon = \frac{V_{voids}}{V_{total}}$$

The porosity can depend on the point x.

The liquids can flow through the porous media. From now on we assume that the porous media we consider are saturated. The experiments show that for those saturated porous media, the speed of the flow is proportional to the gradient of the pressure head h (if the speed is not too large):

$$\vec{U} = - \frac{m(x,h)}{\varepsilon} \operatorname{grad} h \; .$$

The mobility m depends on the medium and on the liquid. It can also depends on the head h. This relation is the Darcy law.

There are two other classical expression of the Darcy law:

1) $\vec{V} = \varepsilon \vec{U} = - \frac{k}{\mu} \operatorname{grad} h = - m \operatorname{grad} h$

where $\vec{V}$ is the filter velocity, k the permeability (k does not depend on the liquid) and μ the viscosity of the liquid,

2) $\vec{V} = \varepsilon \vec{U} = - \tilde{k} \operatorname{grad} \tilde{h}$,

where $\tilde{k}$ is also the permeability and $\tilde{h}$ is the head expressed in meters.

One must be careful when one speaks of permeability which has two meanings.

b) The diffusion equation.

The mass conservation law for a liquid which flows through a saturated porous medium is:

$$\frac{\partial}{\partial t}(\varepsilon \rho_e) + div(\varepsilon \rho_e \overrightarrow{U_e}) = 0$$

where ρ_e is the mass by unit volume of the water and $\overrightarrow{U_e}$ its velocity. Using the Darcy law, we obtain:

$$\frac{\partial}{\partial t}(\varepsilon \rho_e) - \frac{\partial}{\partial x_i}\left(\rho_e m \frac{\partial h}{\partial x_i}\right) = 0 , \quad (6).$$

The water is practically uncompressible. If we assume that the medium is uncompressible ($\frac{\partial \varepsilon}{\partial t} = 0$) we have:

$$-\frac{\partial}{\partial x_i}\left(m \frac{\partial h}{\partial x_i}\right) = 0 ,$$

or $\qquad - \Delta h = 0$ if m is constant.

n fact experiments show that the porous media are slightly ompressible. We assume then that $\rho_e \varepsilon$ is a linear unction of h and we obtain:

$$\nu \frac{\partial h}{\partial t} - \rho_e \frac{\partial}{\partial x_i}\left(m \frac{\partial h}{\partial x_i}\right) = 0 , (7a)$$

r if m is constant:

$$\nu \frac{\partial h}{\partial t} - \rho_e m \, \Delta h = 0 , \quad (7b) ,$$

where ν is a coefficient which describes the compressibility of the orous medium.

The equations (7a) or (7b) are the <u>diffusion equation</u> which have the same properties than the heat equation.

The boundary condition which are related to this equation are :

$$-\varepsilon \rho_e \vec{U_e} \cdot \vec{n} = \hat{g} \quad \text{or} \quad \rho_e m \frac{\partial h}{\partial n} = \hat{g}$$

where $\hat{g}$ is the mass flux through the boundary. Several situations can occur:

1) $\hat{g}$ is given. A particular case is $g = 0$: the boundary is <u>watertight</u>.

2) $\hat{g}$ is a function of h. The more simple case (and) the more useful in practice is:

$$\hat{g} = -\hat{\alpha}\,(h - \bar{h})$$

where $\bar{h}$ is the outside head. This relation expresses the fact that the flow through the boundary is proportional to (the) difference of the heads h and h_0.

3) if $\hat{\alpha} = 0$, we have again the watertight boundary. If $\hat{\alpha} = 0$ we have the Dirichlet boundary condition.

$$h = \bar{h}.$$

The head is known on that part of the boundary.

III Heat diffusion with phase change. The Stefan problem.

We suppose that the temperature of a structure Ω varies around a phase change temperature which is chosen as the origin of the temperature scale. This is for instance the case of a water saturated porous medium which is chilled. The structure occupies an open set Ω of $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$.

At any time t, Ω is divided into three parts that (we) assume o be open:

- $\Omega_1(t)$ the unfrozen part (or the part of phase 1),
- $\Omega_2(t)$ the frozen part (or the part of phase 2),
- $\Omega_3(t)$ the intermediate part or the <u>cloud</u> [18] where

he two phases coexist at the temperature $0°$. We will see ater on situations where the part Ω_3 actually exists.

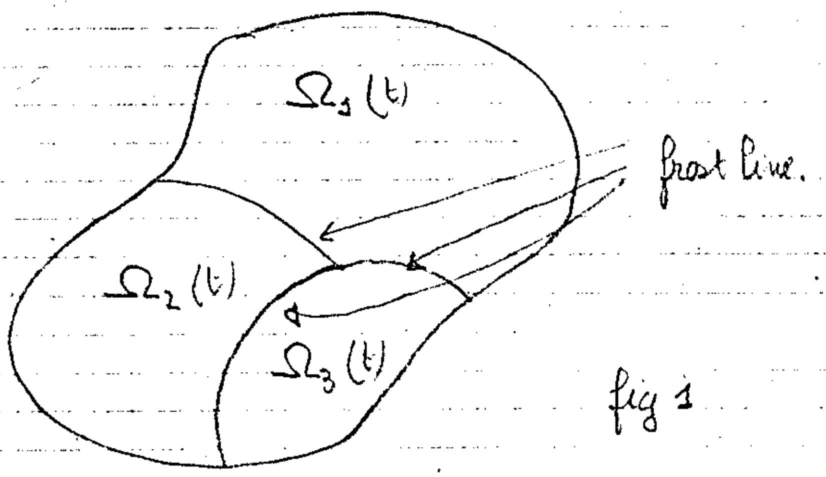


fig 1

Those three parts are separated by the frost line with ocal equation $t - \gamma(x) = 0$ (t is the time when the frost ine is at the point x). In fact it would be more correct to use parametric representation of the frost line.

In the sequel the subscript i refer to quantities defined in $\Omega_i(t)$.

The frost line depends on the time t. This dependance is apriori unknown; this is why we call it a <u>free surface</u>. We will need <u>two transmission conditions</u> on this free surface instead of one on a known boundary. We know already that the temperature is known on the frost line. The second equation will be a consequence of the energy conservation.

a) The equations.

The temperature θ verifies the heat equation in $\Omega_1(t)$ and $\Omega_2(t)$. We assume (in order to simplify) that the physical parameters depends only on θ and that they are constant in each phase.

$$\rho C_1 \frac{\partial \theta_1}{\partial t} - \lambda_1 \Delta \theta_1 = r \quad , \text{ in } \Omega_1(t), \quad (8).$$

$$\rho C_2 \frac{\partial \theta_2}{\partial t} - \lambda_2 \Delta \theta_2 = r \quad , \text{ in } \Omega_2(t), \quad (9),$$

In the cloud $\Omega_3(t)$ we know that $\theta(x,t)=0$ and by using the Fourier law (3) that $q=0$. Following (4) we have:

$$\rho \frac{de}{dt} = r$$

and using the relation (3) and neglecting the dynamic effects we obtain:

$$\rho l \frac{\partial \mu}{\partial t} = r \quad , (10) .$$

On the frost line where the temperature is 0°, the energy conservation law [10] gives a second equation:

$$[\vec{q}] \cdot \vec{N} + \rho v [e] = 0 \quad , (11)$$

where $\vec{N}$ is an unit normal (vector) to the frost line, v the normal velocity of the particles with respect to the frost line. The brackets represent the discontinuity of the quantities which are between them.

The energy jump is:

$$[e] = l \quad \text{from } \Omega_2 \text{ to } \Omega_1,$$

$$[e] = \mu \ell \quad \text{from } \Omega_2 \text{ to } \Omega_3,$$

$$[e] = -(1-\mu)\ell \quad \text{from } \Omega_1 \text{ to } \Omega_3.$$

We have also:

$$v = \vec{U}.\vec{N} - \vec{W}.\vec{N} = -\vec{W}.\vec{N},$$

where $\vec{U}$ is the velocity of the particle (we have assumed that $\vec{U}=0$), $\vec{W}$ is the velocity of the frost line at the point we consider.

The derivative of $t = \gamma(x)$ with respect to t gives:

$$1 = \text{grad}\,\gamma . \frac{dx}{dt} = \text{grad}\,\gamma . \vec{W}.$$

As $\text{grad}\,\gamma$ has the same direction as $\vec{N}$, we have:

$$\text{grad}\,\gamma = \frac{\vec{N}}{\vec{W}.\vec{N}}.$$

The relation (11) gives:

$$\left(\rho\,[e] - [q].\text{grad}\,\gamma\right) v = 0$$

or

$$\rho\,[e] = [q]\,\text{grad}\,\gamma.$$

<u>Note.</u> The case $v=0$ (where the frost line does not move) can be included in this relation by allowing $\text{grad}\,\gamma$ to have the value ∞. This is, for instance, the case when there is a discontinuity of energy inside a part $\Omega_3(t)$: we have $v=0$, $\text{grad}\,\gamma = \infty$, and the line of discontinuity does not move.

By using the Fourier law we obtain:

$(\lambda_1 \operatorname{grad}\theta_1 - \lambda_2 \operatorname{grad}\theta_2)\cdot\operatorname{grad}\gamma = \rho l$ on the boundary between $\Omega_1(t)$ and $\Omega_2(t)$,

$\lambda_2 \operatorname{grad}\theta_2 \cdot \operatorname{grad}\gamma = \mu_3 \rho l$ on the boundary between $\Omega_2(t)$ and $\Omega_3(t)$,

$\lambda_1 \operatorname{grad}\theta_1 \cdot \operatorname{grad}\gamma = -(1-\mu_3)\rho l$ on the boundary between $\Omega_1(t)$ and $\Omega_3(t)$,

we have also,

$$\theta_1(x,\gamma(x)) = \theta_2(x,\gamma(x)) = \theta_3(x,\gamma(x)) = 0 \quad , (12).$$

These equation are the two conditions which allow to determine the frost line.

We give now the boundary condition. We suppose that the boundary Γ of Ω is divided into 5 given parts where we have:

— on Γ_0 $\quad \lambda(\theta)\,\theta = k_0(x,t)$, $(\lambda(\theta)=\lambda_1$ if $\theta>0$, $\lambda(\theta)=\lambda_2$ if $\theta<0)$,

the function k_0 can change sign,

— on Γ_1 $\quad \lambda_1\theta_1 = k_1(x,t) > 0$,

the part Γ_1 is unfrozen,

— on Γ_2 $\quad \lambda_2\theta_2 = k_2(x,t) < 0$,

the par Γ_2 is frozen,

— on Γ_3

$$\lambda(\theta)\frac{\partial\theta}{\partial n} + q\big(\lambda(\theta)\,\theta - k_3(x,t)\big) = 0 ,$$

— on Γ_4 $\quad \lambda(\theta)\dfrac{\partial \theta}{\partial n} = k_4(x,t)$, (13).

The last equations are the initial conditions. The initial temperature $\theta_0(x)$, the initial parts $\Omega_1(0)$, $\Omega_2(0)$, $\Omega_3(0)$, the initial propation μ_0 of unfrozen material are given:

$$\theta(x,0) = \theta_0(x), \quad \mu(x,0) = \mu_0(x), \quad (14).$$

They verify the <u>compatibility equations</u>:

$$\theta_0|\Omega_1(0) \geq 0 \quad , \quad \theta_0|\Omega_2(0) \leq 0 \quad , \quad \theta_0|\Omega_3(0) = 0$$

$$\mu_0|\Omega_1(0) = 1 \quad , \quad \mu_0|\Omega_2(0) = 0 \quad , \quad 0 \leq \mu_0|\Omega_3(0) \leq 1 \quad , (15).$$

The equations that allow to find $\theta(x,t)$ and $\mu(x,t)$ are the ditions (8), (9), (10), (12), (13), (14), (15). This set of equation is a <u>Stefan probl</u>

<u>Note.</u> We have two unknowns $\theta(x,t)$ and $\mu(x,t)$.

The datas are Ω, Γ_j $(j=0,1,2,3,4)$, k_j, C_i, λ_i, ρ, r, θ_0, μ_0, $\Omega_i(0)$.

To solve this Stefan problem, one can introduce a new unknown : <u>the freezing index</u>.

b) <u>The freezing index. Physical properties.</u>

Many engineers who have studied the frost action on civil engineering structures, roads for instance, have remarked that many results can be described in terms of the freezing index:

$$I(t) = \int_0^t \theta(\tau)\, d\tau$$

computed on the surface of the structure. It is possible to estimate the frost penetration in a road by a linear unction of the square root of the freezing index [16]. In our situation it is convenient to define the freezing index at any point of the structure by:

$$I(x,t) = \int_0^t \{\lambda_1 \theta^+(x,\tau) - \lambda_2 \theta^-(x,\tau)\} d\tau ,$$

where $\theta^+ = \mathrm{Sup}\{0, \theta\}$ and $\theta^- = \mathrm{Sup}\{0, -\theta\}$ ($\theta = \theta^+ - \theta^-$).

To simplify the computation we suppose that all the physical constants except ℓ are equal to 1.

The solution of the partial differential equation supposes a certain continuity of the involved functions. The function θ is continuous across the frost line but gradθ is discontinuous (relation (12)). This is a drawback to solve the equation for θ and μ. On the contrary it is easy to check that $I(x,t)$ and grad $I(x,t)$ are continuous across the frost line. We will see later that the freezing has another interesting property: it sums up in one unknown the two unknowns θ and μ.

) Equations verified by the freezing index

To find a set of partial differential equations verified by $I(x,t)$ we will use the transmission condition (12). Our result will depend on the initial situation of the point x and on the situation of this point x at the time t (x in Ω_1, Ω_2 or Ω_3). We will have to deal with $3 \times 3 = 9$ cases.

We will make the computation for only two cases and give the result for the 7 other situations.

Case 1 - $x \in \Omega_1(0) \cap \Omega_2(t)$. A figure in one dimension shows this situation: the abscissa can stand for the depth in a plate chilled on one of his faces.

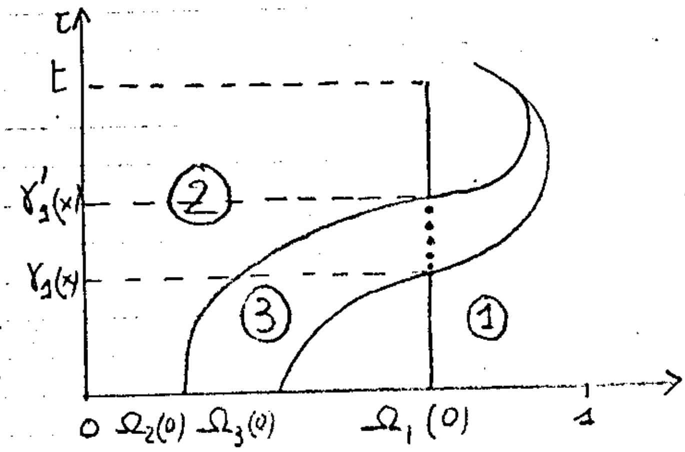


figure 2.

We suppose that the temperature has changed sign an odd number of times: $2n+1$. Let us call $\gamma_j(x)$ the $2n+1$ times when θ reaches the temperature 0 and $\gamma_j'(x)$ the $2n+1$ times when θ leaves the temperature 0. One can have: $\gamma_j(x) < \gamma_j'(x)$ if $x \in \Omega_3(\tau)$ for $\gamma_j(x) < \tau < \gamma_j'(x)$.

Let us compute grad I and ΔI.

We have:

$$\operatorname{grad} I(x,t) = \int_0^{\gamma_1(x)} \operatorname{grad} \theta(x,\tau)\,d\tau + \sum_{j=1}^{2n} \int_{\gamma_j'(x)}^{\gamma_{j+1}(x)} \operatorname{grad} \theta(x,\tau)\,d\tau + \int_{\gamma_{2n+1}'(x)}^{t} \operatorname{grad} \theta(x,\tau)\,d\tau$$

because the function θ is continuous across the front line and equal to 0 in a neighbourhood of x between the times $\gamma_j(x)$ and $\gamma_j'(x)$. The gradient of I is then continuous across the front line.

We have:

$$\Delta I = \int_0^t \Delta\theta(x,\tau)\,d\tau + \sum_{j=0}^{n} \Big[\operatorname{grad}\theta_1(x,\gamma_{2j+1}(x))\cdot\operatorname{grad}\gamma_{2j+1}$$

$$-\operatorname{grad}\theta_2(x,\gamma'_{2j+1}(x))\cdot\operatorname{grad}\gamma'_{2j+1}(x)\Big] - \sum_{j=1}^{n}\Big[\operatorname{grad}\theta_1(x,\gamma'_{2j}(x))\cdot\operatorname{grad}\gamma'_{2j}(x)$$

$$-\operatorname{grad}\theta_2(x,\gamma_{2j}(x))\cdot\operatorname{grad}\gamma_{2j}(x)\Big]$$

and following the relation (12):

$$= \int_0^t \Delta\theta(x,\tau)\,d\tau + \sum_{j=0}^{n}\Big[-\ell\,\mu(x,\gamma'_{2j+1}(x)) - \ell\big(1-\mu(x,\gamma_{2j+1}(x))\big)\Big]$$

$$+\sum_{j=1}^{n}\Big[\ell\big(1-\mu(x,\gamma'_{2j}(x))\big) - \ell\,\mu(x,\gamma_{2j}(x))\Big].$$

But we have by using the equation (10):

$$\ell\big[\mu(x,\gamma'_j(x)) - \mu(x,\gamma_j(x))\big] = \int_{\gamma_j(x)}^{\gamma'_j(x)} r(x,\tau)\,d\tau .$$

Following the equations (8) and (9) we have:

$$\int_0^t \Delta\theta\,d\tau = \int_0^{\gamma_1(x)}\Big\{\frac{\partial\theta}{\partial t} - r\Big\}\,d\tau + \sum_{j=1}^{2n}\int_{\gamma'_j(x)}^{\gamma_{j+1}(x)}\Big\{\frac{\partial\theta}{\partial t} - r\Big\}\,d\tau$$

$$+\int_{\gamma'_{2n+1}}^{t}\Big\{\frac{\partial\theta}{\partial t} - r\Big\}\,d\tau = \int_0^t \frac{\partial\theta}{\partial t}\,d\tau - \int_0^{\gamma_1(x)} r(x,\tau)\,d\tau - \sum_{j=1}^{2n}\int_{\gamma'_j(x)}^{\gamma_{j+1}(x)} r(x,\tau)\,d\tau$$

$$-\int_{\gamma'_{2n+1}}^{t} r(x,\tau)\,d\tau .$$

We obtain then:

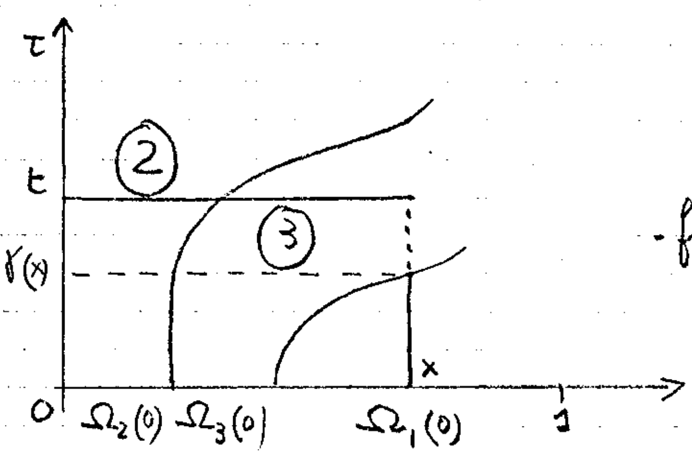
$$\Delta I = \theta(x,t) - \theta_0(x) - \ell - \int_0^t r(x,\tau)\,d\tau \,;$$

but $\theta(x,t) = \frac{\partial I}{\partial t}$ and we obtain:

$$\frac{\partial I}{\partial t} - \Delta I = \theta_0 + R + \ell \quad , \quad \frac{\partial I}{\partial t} \leq 0 \ \text{ in } \Omega_1(0) \cap \Omega_2(t), \ (16),$$

where $R(x,\tau) = \int_0^t r(x,\tau)\,d\tau$.

Case 2. $x \in \Omega_1(0) \cap \Omega_3(t)$. We suppose to simplify that the temperature has not change sign between 0 and t (figure 3).



- fig 3.

We have:

$$\operatorname{grad} I(x,t) = \int_0^{\gamma(x)} \operatorname{grad} \theta(x,\tau)\,d\tau \ .$$

By using the relations (8), (12) we obtain:

$$\Delta I = \int_0^{\gamma(x)} \Delta\theta_1\,d\tau + \operatorname{grad}\theta_1(x,\gamma(x)) \cdot \operatorname{grad}\gamma(x)$$

$$= \theta(x,\gamma(x)) - \theta_0(x) - \int_0^{\gamma(x)} r(x,\tau)\,d\tau - \ell(1 - \mu_3(x,\gamma(x))) \,.$$

But the relation (10) gives:

$$\int_{\gamma(x)}^{t} r(x,\tau)\,d\tau = \ell\left(\mu_3(x,t) - \mu_3(x,\gamma(x))\right).$$

By using this relation we obtain:

$$\Delta I = -\theta_0(x) - R(x,t) + \ell\left(\mu_3(x,t) - 1\right), \quad (17)$$

which gives:

$$\theta_0 + R \leqslant -\Delta I \leq \theta_0 + R + \ell \quad , \frac{\partial I}{\partial t} = 0 \quad \text{in } \Omega_1(0) \cap \Omega_3(t), \quad (18).$$

By similar computation we obtain the following results:

$$\frac{\partial I}{\partial t} - \Delta I = R - \ell(1-\mu_0) \quad , \frac{\partial I}{\partial t} > 0, \quad \text{if } x \in \Omega_3(0) \cap \Omega_1(t),$$

$$\frac{\partial I}{\partial t} - \Delta I = \theta_0 + R, \quad \frac{\partial I}{\partial t} > 0 \quad , \quad \text{if } x \in \Omega_1(0) \cap \Omega_1(t),$$

$$\frac{\partial I}{\partial t} - \Delta I = \theta_0 + R - \ell \quad , \frac{\partial I}{\partial t} > 0 \quad , \quad \text{if } x \in \Omega_2(0) \cap \Omega_1(t),$$

$$\frac{\partial I}{\partial t} - \Delta I = \theta_0 + R \quad , \frac{\partial I}{\partial t} \leqslant 0 \quad , \quad \text{if } x \in \Omega_2(0) \cap \Omega_2(t),$$

$$\theta_0 + R - \ell \leqslant -\Delta I \leq \theta_0 + R \quad , \frac{\partial I}{\partial t} = 0 \quad , \quad \text{if } x \in \Omega_2(0) \cap \Omega_3(t),$$

$$\frac{\partial I}{\partial t} - \Delta I = R + \ell\mu_0 \quad , \quad \frac{\partial I}{\partial t} \leqslant 0 \quad , \quad \text{if } x \in \Omega_3(0) \cap \Omega_2(t),$$

$$R + \ell(\mu_0 - 1) \leqslant -\Delta I \leq R + \ell\mu_0 \quad , \frac{\partial I}{\partial t} = 0 \quad , \text{if } x \in \Omega_3(0) \cap \Omega_3(t), \quad (19).$$

We obtain then the boundary conditions for I by using the boundary conditions for θ:

$$I(x,t) = \int_0^t k_0(x,\tau)\,d\tau = K_0(x,t) \quad \text{on } \Sigma_0 = \Gamma_0 \times]0,T[,$$

$$I(x,t) = \int^t k_1(x,\tau)\,d\tau = K_1(x,t) \quad \text{on } \Sigma_1 = \Gamma_1 \times]0,T[,$$

$$I(x,t) = k_2(x,t) \text{ on } \Sigma_2,$$

$$\frac{\partial I}{\partial n} + q(I - k_3) = 0 \text{ on } \Sigma_3,$$

$$\frac{\partial I}{\partial n} = K_4 \text{ on } \Sigma_4, \quad (20).$$

The initial condition is obviously :

$$I(x,0) = 0, \quad (21).$$

The equation to find I are the equations (16), (18), (19), (20) and (21).

<u>Notes.</u> 1) The second part of the heat equation which appears in the equations is a function of $\frac{\partial I}{\partial t}$. This fact expresses the phase change.

2) Starting from two unknowns θ and μ, we have now only one I. Is it possible to find θ and μ if I is known? The answer is given by the following formulae:

in a part where

$\frac{\partial I}{\partial t} > 0$, we have $\theta = \frac{\partial I}{\partial t}$ and $\mu = 1$,

$\frac{\partial I}{\partial t} < 0$, we have $\theta = \frac{\partial I}{\partial t}$ and $\mu = 0$,

$\frac{\partial I}{\partial t} = 0$, we have $\theta = 0$ and by using the relation (17):

$$\mu = \frac{1}{\ell}(\Delta I + \theta_0 + R + \ell)$$

if we are in $\Omega_1(0)$,

by using similar relations we have :

$$\mu = \frac{1}{\ell}\left(\theta_0 + R + \Delta I\right) \text{ if we are in } \Omega_2(0) \text{ and}$$

$$\mu = \frac{1}{\ell}\left(R + \Delta I + \ell\mu_0\right) \text{ if we are in } \Omega_3(0).$$

To solve the previous set of partial differential equation for I we use a variational formulation. It is more convenient for this formulation to have homogeneous Dirichlet boundary conditions. Then we do a technical translation on I and we assume that we know a function $\tilde{u}(x,t)$ such that:

$$\tilde{u}(x,t) = K_j(x,t) \quad \text{on } \sum_j \quad \text{for } j = 0, 1, 2,$$

$$\tilde{u}(x,0) = 0 \quad , \quad \frac{\partial \tilde{u}}{\partial t}(x,0) = \theta_0 .$$

We define the new unknow u by:

$$I = u + \tilde{u} .$$

This new unknown verifies the equations:

$$-\Delta u = \theta_0 + R + \ell - \left(\frac{\partial \tilde{u}}{\partial t} - \Delta \tilde{u}\right), \quad \frac{\partial u}{\partial t} < -\frac{\partial \tilde{u}}{\partial t}, \text{ in } \Omega_1(0) \cap \Omega_2(t),$$

$$+ R + \Delta \tilde{u} \leq -\Delta u \leq \theta_0 + R + \ell + \Delta \tilde{u}, \quad \frac{\partial u}{\partial t} = -\frac{\partial \tilde{u}}{\partial t}, \text{ in } \Omega_1(0) \cap \Omega_3(t),$$

$$-\Delta u = R - \ell(1-\mu_0) - \left(\frac{\partial \tilde{u}}{\partial t} - \Delta \tilde{u}\right), \quad \frac{\partial u}{\partial t} > -\frac{\partial \tilde{u}}{\partial t}, \text{ in } \Omega_3(0) \cap \Omega_1(t),$$

$$-\Delta u = \theta_0 + R - \left(\frac{\partial \tilde{u}}{\partial t} - \Delta \tilde{u}\right), \quad \frac{\partial u}{\partial t} > -\frac{\partial \tilde{u}}{\partial t}, \text{ in } \Omega_1(0) \cap \Omega_1(t),$$

$$-\Delta u = \theta_0 + R - \ell - \left(\frac{\partial \tilde{u}}{\partial t} - \Delta \tilde{u}\right), \quad \frac{\partial u}{\partial t} > -\frac{\partial \tilde{u}}{\partial t}, \text{ in } \Omega_2(0) \cap \Omega_1(t),$$

$$-\Delta u = \theta_0 + R - \left(\frac{\partial \tilde{u}}{\partial t} - \Delta \tilde{u}\right), \quad \frac{\partial u}{\partial t} < -\frac{\partial \tilde{u}}{\partial t}, \text{ in } \Omega_2(0) \cap \Omega_2(t),$$

$$+ R - \ell + \Delta \tilde{u} \leq -\Delta u \leq \theta_0 + R + \Delta \tilde{u}, \quad \frac{\partial u}{\partial t} = -\frac{\partial \tilde{u}}{\partial t}, \text{ in } \Omega_2(0) \cap \Omega_3(t),$$

$$\frac{\partial u}{\partial t} - \Delta u = R + \ell \mu_0 - \left(\frac{\partial \tilde{u}}{\partial t} - \Delta \tilde{u}\right), \quad \frac{\partial u}{\partial t} < -\frac{\partial \tilde{u}}{\partial t}, \text{ in } \Omega_3(0) \cap \Omega_2(t),$$

$$R + \ell(\mu_0 - 1) + \Delta \tilde{u} \leq -\Delta u \leq R + \ell \mu_0 + \Delta \tilde{u}, \quad \frac{\partial u}{\partial t} = -\frac{\partial \tilde{u}}{\partial t}, \text{ in } \Omega_3(0) \cap \Omega_3(t).$$

The boundary conditions are:

$$u = 0 \quad \text{on } \Sigma_j, \quad j = 0, 1, 2,$$

$$\frac{\partial u}{\partial n} + \alpha(u - k_3) = -\left(\frac{\partial \tilde{u}}{\partial n} + \alpha \tilde{u}\right) \quad \text{on } \Sigma_3,$$

$$\frac{\partial u}{\partial n} = k_4 - \frac{\partial \tilde{u}}{\partial n} \quad \text{on } \Sigma_4.$$

The initial condition is:

$$u(x,0) = 0, \qquad (22).$$

d) Variational formulation.

We now introduce some notations:

— $V = \{ v \mid v \in H^1(\Omega) ;\ v|_{\Gamma_j} = 0 ,\ j = 0, 1, 2 \}$

with the usual Sobolev norm.

— $a(u, v) = \int_\Omega \operatorname{grad} u . \operatorname{grad} v \, d\Omega + \alpha \int_{\Gamma_3} u v \, d\Gamma \quad , \forall v, w \in V ,$

— $(u, v) = \int_\Omega v w \, d\Omega \quad , \quad |v| = \sqrt{(v,v)} \quad , \forall v, w \in L^2(\Omega),$

— $L(t, v) = - a(\tilde{u}(t), v) + \alpha \int_{\Gamma_3} K_3(t) \, v \, d\Gamma + \int_{\Gamma_4} K_4(t) \, v \, d\Gamma + \int_\Omega K \, v \, d\Omega ,$

— $\Phi_1(v) = \ell(\mu_0, v^-) \quad , \quad \Phi_2(v) = \ell(1 - \mu_0, v^+) ,$

$\Phi = \Phi_1 + \Phi_2$

We will show that to solve the equations (22) is formally equivalent to find a function u which verifies:

$$u \in C([0, T] ; V) \quad , \quad \frac{du}{dt} \in L^2(0, T ; V) ,$$

$$u(0) = 0$$

$$\forall t \in [0, T] , \ \forall v \in V ,$$

$$\left(\frac{du}{dt} + \frac{d\tilde{u}}{dt} , v - \frac{du}{dt} \right) + a\left(u, v - \frac{du}{dt}\right) + \Phi\left(v + \frac{d\tilde{u}}{dt}\right) - \phi\left(\frac{du}{dt} + \frac{d\tilde{u}}{dt}\right) \geq$$

$$L\left(t, v - \frac{du}{dt}\right) + \left(\theta_0 , v - \frac{du}{dt}\right) \qquad , \quad (23) .$$

Note. "formally" means that if u is a solution of the problem (23) and is smooth enough, then u is a solution of the problem (22). Conversely if u is a solution of the equations (22) it is a olution of the problem (23).

We show first that (22) implies (23). We multiply all the equations of (22) by $\left(v-\frac{du}{dt}\right)$ and all the inequalities by $\left(v-\frac{du}{dt}\right)^+$ and by $-\left(v-\frac{du}{dt}\right)^-$. We add then the 12 equalities and inequalities. We conventionaly integrate by parts and we obtain:

$$\left(\frac{du}{dt}+\frac{d\tilde{u}}{dt}, v-\frac{du}{dt}\right)+a\left(u, v-\frac{du}{dt}\right)-L\left(t, v-\frac{du}{dt}\right)-\left(\Theta_0, v-\frac{du}{dt}\right) \geqslant$$

$$\ell\int_{\Omega_2(0)\cap\Omega_1(t)}\left(v-\frac{du}{dt}\right)d\Omega+\ell\int_{\Omega_1(0)\cap\Omega_2(t)}\left(v-\frac{du}{dt}\right)d\Omega-\ell\int_{\Omega_3(0)\cap\Omega_1(t)}(1-\mu_0)\left(v-\frac{du}{dt}\right)d\Omega$$

$$\ell\int_{\Omega_3(0)\cap\Omega_2(t)}\mu_0\left(v-\frac{du}{dt}\right)d\Omega-\ell\int_{\Omega_3(t)\cap\Omega_1(0)}\left(v-\frac{du}{dt}\right)^-d\Omega-\ell\int_{\Omega_3(t)\cap\Omega_2(0)}\left(v-\frac{du}{dt}\right)^+d\Omega$$

$$\ell\int_{\Omega_3(t)\cap\Omega_3(0)}\left\{(\mu_0-1)\left(v-\frac{du}{dt}\right)^+-\mu_0\left(v-\frac{du}{dt}\right)^-\right\}d\Omega=\ell\int_{\Omega_2(t)}\mu_0\left(v-\frac{du}{dt}\right)d\Omega-\ell\int_{\Omega_1(t)}(1-\mu_0)\left(v-\frac{du}{dt}\right)$$

$$+\ell\int_{\Omega_3(t)}\left\{(\mu_0-1)\left(v-\frac{du}{dt}\right)^+-\mu_0\left(v-\frac{du}{dt}\right)^-\right\}d\Omega=\delta.$$

We have:

$$\ell\int_{\Omega_2(t)}\mu_0\left(v-\frac{du}{dt}\right)d\Omega=\ell\int_{\Omega_2(t)}\left\{\mu_0\left(v+\frac{d\tilde{u}}{dt}\right)^+-\mu_0\left(v+\frac{d\tilde{u}}{dt}\right)^-\right\}d\Omega+\ldots$$

$$-\ell\int_{\Omega_2(t)}\left\{\mu_0\left(\frac{d\tilde{u}}{dt}+\frac{du}{dt}\right)^+-\mu_0\left(\frac{d\tilde{u}}{dt}+\frac{du}{dt}\right)^-\right\}d\Omega\geqslant-\ell\int_{\Omega_2(t)}\mu_0\left(v+\frac{d\tilde{u}}{dt}\right)^-d\Omega+\ell\int_{\Omega_2(t)}\mu_0\left(\frac{d\tilde{u}}{dt}+\frac{du}{dt}\right)^-$$

, (24), because $\left(\frac{d\tilde{u}}{dt}+\frac{du}{dt}\right)^+=0$ in $\Omega_2(t)$.

We have then:

$$-\ell\int_{\Omega_3(t)}\mu_0\left(v-\frac{du}{dt}\right)^- d\Omega = -\ell\int_{\Omega_3(t)}\mu_0\left(v+\frac{d\tilde{u}}{dt}\right)^- d\Omega + \ell\int_{\Omega_3(t)}\mu_0\left(\frac{du}{dt}+\frac{d\tilde{u}}{dt}\right)^- d\Omega, \quad (25)$$

because $\frac{d\tilde{u}}{dt}+\frac{du}{dt}=0$ in $\Omega_3(t)$.

By adding the relations (24) and (25):

$$\ell\int_{\Omega_2(t)}\mu_0\left(v-\frac{du}{dt}\right)d\Omega - \ell\int_{\Omega_3(t)}\mu_0\left(v-\frac{du}{dt}\right)^- d\Omega \geqslant -\ell\int_{\Omega_3(t)\cup\Omega_2(t)}\mu_0\left(v+\frac{d\tilde{u}}{dt}\right)^- d\Omega$$

$$+\ell\int_{\Omega_3(t)\cup\Omega_2(t)}\mu_0\left(\frac{du}{dt}+\frac{d\tilde{u}}{dt}\right)^- d\Omega \geqslant -\ell\int_{\Omega}\mu_0\left(v+\frac{d\tilde{u}}{dt}\right)^- d\Omega + \ell\int_{\Omega}\mu_0\left(\frac{du}{dt}+\frac{d\tilde{u}}{dt}\right)d\Omega =$$

$$= -\Phi_1\left(v+\frac{d\tilde{u}}{dt}\right) + \Phi_1\left(\frac{du}{dt}+\frac{d\tilde{u}}{dt}\right), \quad (26).$$

We can also show that:

$$\ell\int_{\Omega_1(t)}(1-\mu_0)\left(v-\frac{du}{dt}\right)d\Omega - \ell\int_{\Omega_3(t)}(1-\mu_0)\left(v-\frac{du}{dt}\right)^+ d\Omega \geqslant -\Phi_2\left(v+\frac{d\tilde{u}}{dt}\right)+\Phi_2\left(\frac{du}{dt}+\frac{d\tilde{u}}{dt}\right), (27).$$

By adding (26) and (27), we obtain.

$$\delta \geqslant -\Phi\left(v+\frac{d\tilde{u}}{dt}\right)+\Phi\left(\frac{du}{dt}+\frac{d\tilde{u}}{dt}\right),$$

which gives the inequality of the problem (23). As we have assumed that u and the $\Omega_j(t)$ were smooth (enough) to perform the computation the other conditions of problem (23) are verified.

We show now that (23) implies (22). Let us suppose that u is smooth and compute $\theta(x,t)$ and $\mu(x,t)$ by the formulas already given. We define (then) the open set $\Omega_i(t)$ and the frost line ($\Omega_1(t)$ is the set of the points x such that $\theta(x,t)>0$).

We prove two of the equation of the set (22). Let us choose, for instance, φ with compact support in $\Omega_1(0)\cap\Omega_2(t)$ such that $\pm\varphi + \frac{du}{dt}+\frac{d\tilde u}{dt}\leq 0$. This is possible because $\frac{du}{dt}+\frac{d\tilde u}{dt}<0$.

In the inequality (23) let $v = \pm\varphi + \frac{du}{dt}$. We obtain easily:

$$\left(\frac{du}{dt}+\frac{d\tilde u}{dt}, \pm\varphi\right) + a(u,\pm\varphi) - \ell\int_{\Omega_1(0)\cap\Omega_2(t)} \pm\varphi \, d\Omega \geq (\theta_0, \pm\varphi) + L(t,\pm\varphi).$$

As we can choose any φ such that $\pm\varphi+\frac{du}{dt}+\frac{d\tilde u}{dt}<0$, the inequality in this relation becomes an equality. Integrating by parts, we obtain:

$$\int_\Omega \left\{\frac{du}{dt}+\frac{d\hat u}{dt} - \Delta(u+\tilde u) - R - \theta_0 - \ell\right\}\varphi \, d\Omega = 0 \quad \text{for any } \varphi.$$

This relation is equivalent to the first equation of the set (22).

Let us choose now $\varphi = v - \frac{du}{dt}$ with compact support in $\Omega_3(t)\cap\Omega_1(0)$ the inequality (23) gives:

$$a(u+\tilde u,\varphi) + \ell\int_{\Omega_3(t)\cap\Omega_1(0)}\left(\varphi+\frac{du}{dt}+\frac{d\tilde u}{dt}\right)^- d\Omega - \ell\int_{\Omega_3(t)\cap\Omega_1(0)}\left(\frac{du}{dt}+\frac{d\tilde u}{dt}\right)^- d\Omega \geq \int_{\Omega_3(t)\cap\Omega_1(0)}\varphi(R+\theta_0)\,d\Omega$$

but we know that $\frac{du}{dt}+\frac{d\tilde u}{dt}=0$ in $\Omega_3(t)$, so:

$$a(u+\tilde u,\varphi) + \ell\int_{\Omega_3(t)\cap\Omega_1(0)}\varphi^- \, d\Omega \geq \int_{\Omega_3(t)\cap}(R+\theta_0)\varphi \, d\Omega$$

By integrating by parts, choosing first $\varphi\geq 0$ and then $\varphi\leq 0$, we obtain :

$$-\Delta(u+\tilde{u}) \geqslant R + \theta_0 \quad \text{and then} \quad -\Delta(u+\tilde{u}) \leqslant R + \theta_0 + \ell$$

which are the second inequalities of set (22). The other equalities and inequalities of (22) are obtained by the same manner. The boundary condition are conventionaly obtained.

e) <u>the main results</u>

We can now state an existence and uniqueness theorem for the function u:

<u>Theorem 1</u>. If the function $\tilde{u}$ is such that:

$$\tilde{u} \in H^2([0,T]; H^1(\Omega)) \; ; \; \frac{d^3\tilde{u}}{dt^3} \in L^2(Q) \qquad (Q = \Omega \times]0,T[)$$

$$\tilde{u}(0) = 0, \quad \frac{d\tilde{u}}{dt}(0) = \theta_0 ,$$

if θ_0, μ_0 verify the compatibility conditions

$$\theta_0|\Omega_1(0) \geqslant 0 \qquad \theta_0|\Omega_2(0) \leqslant 0 , \qquad \theta_0|\Omega_3(0) = 0,$$

$$\mu_0|\Omega_1(0) = 1 \quad , \mu_0|\Omega_2(0) = 0, \quad 0 \leqslant \mu_0|\Omega_3(0) \leqslant 1 ,$$

if $\mu_0 \in L^2(\Omega)$, $R \in H^2(0,T; L^2(\Omega))$, $k_j \in H^2(0,T; L^2(\Gamma))$,

there exists a unique function u such that :

$$u \in C([0,T]; V) \; ; \; \frac{du}{dt} \in L^\infty(0,T; V) \; ; \; \frac{d^2u}{dt^2} \in L^2(Q)$$

which is solution of the problem (22). The freezing index $I = u + \tilde{u}$ is also unique.

The case where the heat capacities c_1 and c_2 are equal to 0 is practically useful because those heat capacities are often small and the most important term in the energy is the jump due to the phase change. A good approximation is then to chose $c_1 = c_2 = 0$.

The solution u_c of the problem (22) where $c_1 = c_2 = c \neq 1$ is solution of the problem (23) where the inequality has been replaced by:

$$c\left(\frac{du}{dt} + \frac{d\tilde{u}}{dt}, v - \frac{du}{dt}\right) + a\left(u, v - \frac{du}{dt}\right) + \phi\left(v + \frac{d\tilde{u}}{dt}\right) - \phi\left(\frac{du}{dt} + \frac{d\tilde{u}}{dt}\right) \geq$$
$$L\left(t, v - \frac{du}{dt}\right) + c\left(\theta_0, v - \frac{du}{dt}\right), \quad (23\text{ bis})$$

(Of course, the problem with $c_1 \neq c_2$ can be solved [8]).

Concerning the solution $\bar{u}$ of the problem (23 bis) with $c=0$ we have the following theorem:

<u>Theorem 2.</u> If the hypothesis of theorem I are verified, there exists a unique function $\bar{u}$ such that:

$$\bar{u} \in C([0,T]; V), \quad \frac{d\bar{u}}{dt} \in L^\infty([0,T]);$$

and a.e. on $(0,T)$, $\forall v \in V$:

$$a\left(\bar{u}, v - \frac{d\bar{u}}{dt}\right) + \phi\left(v + \frac{d\tilde{u}}{dt}\right) - \phi\left(\frac{d\bar{u}}{dt} + \frac{d\tilde{u}}{dt}\right) \geq L\left(t, v - \frac{d\bar{u}}{dt}\right).$$

The solution u_c of the problem (23 bis) (resp. $\frac{du_c}{dt}$) tends to $\bar{u}$ (resp. $\frac{d\bar{u}}{dt}$) in $L^\infty(0,T; V)$ (resp. $L^\infty(0,T;V)$ weak $*$) when $c \to 0$.

The freezing index $I = u + \tilde{u}$ is also unique.

We give now proof outlines of theorems I and II.

We solve first a regularized problem in a finite dimension space. We obtain next a priori estimates which allow to conclude.

a) Regularized problem. Let $\eta > 0$ and

$$\varphi_\eta(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ -\dfrac{y^4}{2\eta^3} + \dfrac{y^3}{\eta^2}, & \text{if } 0 \leq y \leq \eta, \\ y - \dfrac{\eta}{2}, & \text{if } y \geq \eta. \end{cases}$$

The function φ_η is convex and is a C^2-function. We define also:

$$\Phi_\eta(v) = \ell \int_\Omega \mu_0 \varphi_\eta(-v)\, d\Omega + \ell \int_\Omega (1-\mu_0)\, \varphi_\eta(v)\, d\Omega .$$

Let $[w_1, ---, w_n]$ be a basis of $V_n \subset V$ such that $\bigcup_n V_n$ is dense in V. Let $D^1\Phi_\eta(v)$, $D^2\Phi_\eta(v)$ the first derivatives of Φ_η.

We seek $u_n = \sum_{i=1}^n y_i(t)\, w_i$, solution of the ordinary differential equations:

$$u_n(0) = 0,$$

$$\left(\frac{du_n}{dt} + \frac{d\tilde{u}}{dt}, w_j\right) + a(u_n, w_j) + \left(D^1\Phi_\eta\left(\frac{du_n}{dt} + \frac{d\tilde{u}}{dt}\right), w_j\right) = L(t, w_j) + (\theta_0, w_j)$$

$, j = 1, ---- n, \quad (24).$

One can observe that $\frac{du_n}{dt}(0) = 0$ for $D^1\Phi_\eta\left(\frac{d\tilde{u}}{dt}(0)\right) = 0$. The function $\frac{du_n}{dt}(t)$ is then solution of the ordinary differential equations obtained by differentiation of (24):

$$\frac{du_n}{dt}(0) = 0$$

$$\left(\frac{d^2u_n}{dt^2} + \frac{d^2\tilde{u}}{dt^2} + D^2\Phi_\eta\left(\frac{du_n}{dt} + \frac{d\tilde{u}}{dt}\right)\left(\frac{d^2u_n}{dt^2} + \frac{d^2\tilde{u}}{dt^2}\right), w_j\right) + a\left(\frac{du_n}{dt}, w_j\right) = \frac{\partial L}{\partial t}(t, w_j),$$

$j = 1, \ldots n, \quad (25).$

e can show by using the properties of Φ_η that these
fferential equations have solutions on $(0, t^n)$. The following
priori estimates will show that $t^n = T$.

β) A priori estimates. From relation (25) we obtain:

$$\left(\frac{d^2 u_n}{dt^2} + \frac{d^2\tilde{u}}{dt^2}\right) + D^2\Phi_\eta\left(\frac{du_n}{dt} + \frac{d\tilde{u}}{dt}\right)\left(\frac{d^2u_n}{dt^2} + \frac{d^2\tilde{u}}{dt^2}\right), \frac{d^2u_n}{dt^2} + \frac{d^2\gamma}{dt^2}\Big) + a\left(\frac{du_n}{dt}, \frac{d^2u_n}{dt^2}\right)$$

$$\frac{d}{dt}\left\{\left(\frac{du_n}{dt} + \frac{d\tilde{u}}{dt}\right) + D^1\Phi_\eta\left(\frac{du_n}{dt} + \frac{d\tilde{u}}{dt}\right), \frac{d^2\tilde{u}}{dt^2}\right) + \frac{d}{dt}\left(\frac{\partial L}{\partial t}\left(t, \frac{du_n}{dt}\right) - \frac{\partial^2 L}{\partial t^2}\left(t, \frac{du_n}{dt}\right)\right).$$

integrate this relation from 0 to t $(t \leq t^n)$ and use the
owing relations where A is now and later on a strictly positive
stant independant of n and η:

$$\left(D^2\Phi_\eta(w)v, v\right) \geq 0 \quad , \quad \left|D^1\Phi_\eta(w)v\right| \leq A|v| \quad , \quad A\|v\|_V^2 \leq a(v,v)$$

$$|a(w,v)| \leq C\|w\|_V\|v\|_V \quad .$$

fter some computations, we obtain:

$$A\int_0^t \left|\frac{d^2u_n}{dt^2}\right|^2 d\tau + A\left\|\frac{du_n}{dt}(t)\right\|_V^2 \leq C + C\left\|\frac{du_n}{dt}(t)\right\|_V + \int_0^t \xi(\tau)\left\|\frac{du_n}{dt}(\tau)\right\|_V d\tau$$

where $\xi \in L^2(0,T)$. It follows that $t^n = T$ and that:

$$A\left\|\frac{du_n}{dt}(t)\right\|_V^2 \leq A + \int_0^t \xi(\tau)\left\|\frac{du_n}{dt}(\tau)\right\| d\tau .$$

By using the gronwall's lemma, we obtain:

$$\forall t \in [0,T] \quad , \left\|\frac{du_n}{dt}(t)\right\|_V \leq A, \quad \|u_n(t)\|_V \leq A, \quad \left\|\frac{d^2u_n}{dt^2}\right\|_{L^2(Q)} \leq A, \quad (26).$$

γ) Passage to the limit. Uniqueness. We now let $n \to \infty$ and $\eta \to 0$. From the estimates (26) we obtain that there exists a function u and a subsequence (u_n) such that:

$$u_n \to u \quad \text{in} \quad L^2(0,T;V),\ L^\infty(0,T;V) \text{ strong},$$

$$\frac{du_n}{dt} \to \frac{du}{dt} \quad \text{in} \quad L^2(0,T;V) \text{ weak},\ L^\infty(0,T;V) \text{ weak } *,$$

$$\frac{d^2u_n}{dt^2} \to \frac{d^2u}{dt^2} \quad \text{in} \quad L^2(Q) \text{ weak}.$$

We know [13] that we can choose $u \in C([0,T];V)$ and $\frac{du}{dt} \in C([0,T];L^2(\Omega))$. By letting $n \to \infty$ and $\eta \to 0$ in the relation (24) (and by using the fact that Φ_2 is convex), we obtain that u is a solution of the problem (23). The proof of the uniqueness is straightforward.

δ). Theorem II. The a priori estimates which have been obtained for u are true for u_c solution of the problem (23 bis). But the constants in the relations (26) can depends on c; in fact we have:

$$\| u_c(t) \|_V \leq A \quad , \quad \left\| \frac{du_c}{dt}(t) \right\|_V \leq A \quad , \qquad (27)$$

where A does not depend on t. (We now let $c \to 0$.) From the a priori estimates (27) we deduce that there exists a function $\bar{u}$ and a subsequence (u_c) such that:

$$u_c \to \bar{u} \quad \text{in} \quad L^2(0,T;V) \text{ and } L^\infty(0,T;V),$$

$$\frac{du_c}{dt} \to \frac{d\bar{u}}{dt} \quad \text{in} \quad L^2(0,T;V) \text{ (weak) and } L^\infty(0,T;V) \text{ weak } *.$$

It is also possible to choose $\bar{u} \in C([0,T]; V)$. We prove also that $\bar{u}$ is a solution of the problem (23bis). The proof of the uniqueness is straightforward.

f) Closed form solutions.

α). Example 1. Let there be a piece of ice at the temperature 0° in an adiabatic cell. We suppose that there is a rate of heat production $r(x,t) = a > 0$ inside the ice. This heat production can be produced by micro-waves or by a net of electric wire (think of the memory of a computer). In this case one must use the idea of the porous media to assume that the rate of heat production is continuous with respect to x (II, a). The initial condition are:

$$\Omega = \Omega_2(0) \ , \quad \mu_0(x) = 0 \ , \quad \theta_0(x) = 0 .$$

We assume that all the physical constants are equal to 1 except the latent heat. It is easy to check that the solution $u = I$ (we have $\bar{u} = 0$ because we have no Dirichlet boundary condition) of the problem (23) is:

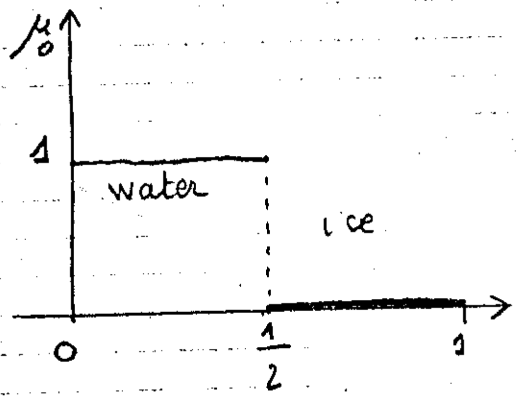
• between the times $t = 0$ and $t = \frac{\ell}{a}$:

$$I = u = 0 \ , \quad \theta(x,t) = 0 \ , \quad \mu(x,t) = \frac{a}{\ell} t .$$

We have also $\Omega = \Omega_3(t)$.

• after the time $t = \frac{\ell}{a}$, we have $\mu(x,t) = 1$ (we have water) and the temperature increases.

This example shows the possibility of existence of clouds $\Omega_3(t)$. We think that the corresponding experiments can be (β) Example 2. This example is in one dimension the initial situation is given on the figure. The domain Ω is $]0,1[$.



fig

The initial temperature is $\theta_0(x) = 0$.

The solution θ, μ (the rate of heat production r) between the times $t=0$ and $t=\frac{1}{2}$ are:

$0 < x < t$, $\mu = 0$, ice, $\theta_2 = x - t$, $r = -1$, $\Omega_2(t)$,

$t < x < \frac{1}{2}$, $\mu = 1$, water, $\theta_3 = 0$, $r = 0$

$\frac{1}{2} < x < 1-t$, $\mu = 0$, ice, $\theta_3 = 0$, $r = 0$ $\Big\}$ $\Omega_3(t)$,

$1-t < x < 1$, $\mu = 1$, water, $\theta_1 = x + t - 1$, $r = 1$ $\Omega_1(t)$.

It is easy to check that the heat equation is verified inside Ω and that the transmission conditions are also verified on the two frost lines ($x = t$ and $x = 1-t$). One can also note that the energy discontinuity point ($x = \frac{1}{2}$) inside $\Omega_3(t)$ does not move as it has been shown (note p. 9).

At the time $t = \frac{1}{2}$ the cloud $\Omega_3(t)$ desappears, one can imagine any evolution for the temperature θ for $t > \frac{1}{2}$.

The boundary conditions have not been defined; one can choose any boundary condition which fits with the values of the temperature $\theta(x,t)$.

Other closed form solutions can be given in one dimension.

I Diffusion of the heat with hydraulic phenomenons. Coupled Stefan problem.

a) <u>the equations.</u> A water saturated porous medium, for instance a water saturated soil, freezes when it is chilled. Experiments show that a depression appears on the frost line [1]. Water is thus sucked in from the watertable through the unfrozen part. This sucked in water freezes when it reaches the frost line. This accumulation of ice induced by the frost results in an heaving of the structure.

This phenomenon is important for road maintenance in cold weather. It is known that ice accumulation and frost heaving results in a decrease of bearing capacity during thaw.

The unknowns of the problem are the temperature $\Theta(x,t)$ and the pressure head $h(x,t)$ at any point x of the porous medium Ω and any time t of the $[0,T]$ period during which this phenomenon is being investigated.

We assume that the temperature of water circulating in the unfrozen part is equal to the temperature of the porous medium skeleton. We assume also that the speed of this water is low, so we can neglect the heat transmitted by convection comparatively to the heat transmitted by conduction. Of course, we take into account the latent heat of fusion of this circulating water. For simplicity, we assume also that there is no part $\Omega_3(t)$ and that there is no rate of heat production ($r = 0$).

the equation are:

in $\Omega_1(t)$, the unfrozen part:

$$\rho c_1 \frac{\partial \theta_1}{\partial t} - \lambda_1 \Delta \theta_1 = 0,$$

$$\nu \frac{\partial h}{\partial t} - m \Delta h = 0,$$

in $\Omega_2(t)$, the frozen part:

$$\rho c_2 \frac{\partial \theta_2}{\partial t} - \lambda_2 \Delta \theta_2 = 0 .$$

On the first line the energy conservation law, the Fourier and Darcy laws give:

$$-\ell = (\lambda_1 \operatorname{grad} \theta_1 - \lambda_2 \operatorname{grad} \theta_2 + \ell' m \operatorname{grad} h) . \operatorname{grad} \gamma$$

where ℓ and ℓ' are the latent heat of fusion by unit volume of the water-saturated porous medium and of the water. We also have:

$$\theta_1(x, \gamma(x)) = \theta_2(x, \gamma(x)) = 0 ,$$

$$h(x, \gamma(x)) = -d \qquad , \qquad ,$$

where $-d$ $(d>0)$ is a constant which measures the depression due to the frost. Moreover, experiments have shown that:

$$\text{in } \Omega_1(t) , \; h(x,t) \geq -d .$$

The pressure head which is a priori defined in $\Omega_1(t)$ only, is extended to $\Omega_2(t)$ by $h = -d$.

We assume that the boundary of Ω is divided into four parts Γ_j where:

— on Γ_1 which is in general unfrozen, the temperature and the head are given:

$$\theta_1(x,t) = \bar{\theta}(x,t) < 0 \qquad h = \bar{h}(x,t) \geq -d .$$

— on Γ_2 which is in general frozen, the temperature and the head are given:

$$\theta(x,t) = \bar{\theta}_2(x,t) \leq 0 \quad , \quad h(x,t) = -d,$$

— on Γ_3 the flows are proportional to the difference between inside and outside:

$$\frac{\partial h}{\partial n} + \alpha(h+d) = 0 \quad , \quad \lambda_i \frac{\partial \theta}{\partial n} + \alpha(\lambda_i \theta - s) = 0 \quad \text{on } \Gamma_3 \cap \bar{\Omega}_i(t),$$

— on Γ_4 which is watertight the heat flow $\omega(x,t)$ is given:

$$\frac{\partial h}{\partial n} = 0 \quad , \quad \lambda_i \frac{\partial \theta}{\partial n} = \omega \quad \text{on } \Gamma_4 \cap \bar{\Omega}_i(t).$$

The initial temperature and pressure head are given as well as the parts $\Omega_1(0)$ and $\Omega_2(0)$, they verify the compatibility condition:

$$\theta_0|\Omega_1(0) \geq 0 \quad , \quad h_0|\Omega_1(0) \geq -d \quad , \quad \theta_0|\Omega_2(0) \leq 0 \quad , \quad h_0|\Omega_2(0) = -d, \quad (28)$$

c) The freezing index and the equation it verifies.

We now suppose that all the physical constants are equal to 1 except ℓ. We define the freezing index by:

$$I(x,t) = \int_0^t \{\theta(x,\tau) + h(x,\tau) + d\}\, d\tau .$$

The equation verified by I are obtained by the same computations already made in the part III. We obtain that I verifies the equation (19) where θ_0 is replaced by $\theta_0 + h_0 + d$ and $R = 0$. For this computation when the physical constants are not equal to 1 we need that $c_1 = \lambda_1 \nu$. This assumption is not too restrictive because the usual numerical values are close. It has also been shown that those two terms are not important for the phenomenon (The important terms are l and l').

We can use the theorem I (or II) to obtain the functions I and $\frac{\partial I}{\partial t} = \theta + h + d$. We know the sum $\theta + h$. We find h by the following set of equation. It is easy to see that:

$$-d \leq h \leq \left(\frac{dI}{dt}\right)^{+} - d = \psi, \quad (29).$$

It is also easy to check that $\frac{\partial h}{\partial t} - \Delta h$ is a non-positive distribution whose support is the frost line. This relation means that the frost line is a water sink. We have then:

$$\forall v \leq \psi \qquad (v - h)\left(\frac{\partial h}{\partial t} - \Delta h\right) \geq 0,$$

$$(\psi - h)\left(\frac{\partial h}{\partial t} - \Delta h\right) = 0, \quad (30).$$

These relations (30), the inequalities (29) and the boundary and initial conditions allow to find h.

To have homogeneous boundary conditions on $\Gamma_1 \cup \Gamma_2$, we perform a translation. Let $\tilde{h}$ such that:

$$\tilde{h} = h_1 \text{ on } \Gamma_1, \quad \tilde{h} = -d \text{ on } \Gamma_2, \quad \frac{\partial \tilde{h}}{\partial n} + \alpha(\tilde{h} + d) = 0 \text{ on } \Gamma_3, \quad \frac{\partial \tilde{h}}{\partial n} = 0 \text{ on } \Gamma_4,$$

$$\tilde{h}(x,0) = h_0(x), \quad \frac{\partial \tilde{h}}{\partial t} - \Delta \tilde{h} = 0 \text{ in } \Omega.$$

We define p by $h = p + \tilde{h}$. The equations verified by p are:

$$d-\tilde{h} = \Psi_1 \leq p \leq \Psi_2 = \Psi - \tilde{h}$$

$$\forall v \leq \Psi_2 \ , \ (v-p)\left(\frac{\partial p}{\partial t} - \Delta p\right) \geq 0 \ ,$$
$$(\Psi_2 - p)\left(\frac{\partial p}{\partial t} - \Delta p\right) = 0 \ ,$$

$=0$ on $\Gamma_1 \cup \Gamma_2$, $\frac{\partial p}{\partial n} + p = 0$ Γ_3 , $\frac{\partial p}{\partial n} = 0$ on Γ_4 , $p(x,0) = 0$, (31).

te The function $\tilde{h}$ is the head when there is no freezing.

To give a variational formulation of the equation (31) we introduce:

$$W = \{ v \mid v \in L^2(0,T;V) ; \frac{\partial v}{\partial t} \in L^2(0,T;V') ; v(0)=0 \}$$

where V' is the dual space of V,

$$K(t) = \{ v \mid v \in V ; \ v \leq \Psi_2(t) \} \ ,$$

$$\mathcal{K} = \{ v \mid v \in W ; \ v(t) \leq \Psi_2(t) \} \ .$$

One can show that the equation (31) are formally equivalent to find the function p which verify:

$$p(0) = 0 \ , \quad p(t) \in K(t) \quad \text{a.e. in } t,$$

$\forall v \in \mathcal{K}$,

$$\int_0^T \left\{ \left(\frac{dv}{dt}, v-p\right) + a(p, v-p) \right\} d\tau \geq \frac{1}{2} \left| v(T) - p(T) \right|^2 , \quad (32) .$$

ote. The condition $p \geq \Psi_1$ is not part of the definition of $\mathcal{K}$ and K.
Ve will see later on that the solution of problem (32) verifies it.

We have:

Theorem III. If the hypotheses of theorem I are verified and if $\tilde{h} \in L^2(0,T; H^1(\Omega))$, $\frac{d\tilde{h}}{dt} \in L^2(Q)$, there exists a unique solution p of problem ( ) which verifies:

$$p \in L^\infty(0,T; V) \cap C([0,T]; L^2(\Omega)) \quad , \; p \leq 0, \quad p \geq \varphi_1 \, .$$

The head of water h is also unique.

The proof of theorem III uses the penalization of the projection on the convex $K(t)$ which depends on time.

α) Penalized problem. Let $\eta > 0$; according to [13] there exists $p_\eta \in W$ such that:

$$\forall v \in V \; , \quad \left(\frac{dp_\eta}{dt}, v\right) + a(p_\eta, v) + \frac{1}{\eta}\left((p_\eta - \psi_2)^+, v\right) = 0 \quad , \; (33).$$

By letting $v = -(p_\eta - \psi_1)^-$ and then p_η^+ in relation ( ), one can sucessively show that $(p_\eta - \psi_1)^- = 0$ and that $p_\eta^+ = 0$. We have then: $\quad p_\eta \geq \psi_1$ and $p_\eta \leq 0$, $\quad (34)$.

β) A priori estimates. From the properties of I and $\tilde{h}$ we have: $\quad \psi_2^- \in L^2(0,T; V)$ and $\frac{d\psi_2^-}{dt} \in L^2(Q)$. By letting $v = p_\eta + \psi_2^-$ in relation ( ), we obtain:

$$\forall t \in [0,T] \qquad |p_\eta(t)| \leq A \quad , \; \|p_\eta\|_{L^2(0,T;V)} \leq A$$

and $\quad \frac{1}{\eta}\int_0^T |(p_\eta - \psi_2)^+|^2 d\tau \leq A \, .$

γ) Passage to the limit. We now let $\eta \to 0$. From the previous estimates, we obtain that there exists a function p and a subsequence (p_η) such that :

$$p_\eta \to p \quad \text{in } L^2(0,T;V) \text{ weak}, \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak}*,$$

$$p \in K(t) \text{ a.e. in } t.$$

We now have for any $v \in \mathcal{K}$:

$$\int_0^T \left\{ \left(\frac{dv}{dt}, v - p_\eta\right) + a(p_\eta, v - p_\eta) \right\} d\tau = \int_0^T \left\{ \left(\frac{dv}{dt} - \frac{dp_\eta}{dt}, v - p_\eta\right) - \cdots \right.$$

$$\left. - \frac{1}{\eta}\left(\left(p_\eta - \psi_2\right)^+, v - p_\eta\right) \right\} d\tau \geqslant \frac{1}{2}\left|\left(v(T) - p_\eta(T)\right)\right|^2, \quad (35),$$

for $-\frac{1}{\eta}\int_0^T \left(\left(p_\eta - \psi_2\right)^+, v - p_\eta\right) d\tau \geqslant 0$ because of $v \in \mathcal{K}$.

By letting $\eta \to 0$ in relation (35), we obtain that p is a solution of problem (32). The relations (34) show that $p \leq 0$ and $p \geqslant \psi_1$. The uniqueness and the continuity of p can be shown by using a method introduced in [13].

Note 1) The relation $p \leq 0$ shows that $h \leq \tilde{h}$ and means that the head is lower in the case of freezing than in the case of no freezing. It means also that in a neighbourhood of Γ_1 where $\psi_2 \geqslant 0$ the medium is not frozen.

2) It is also possible to consider the quasi-static situation $(v = 0)$.

V Numerical results in one dimension. Technical applications.

For application to road problems, a one dimensional program has been developped. The abscissa stands for the depth inside a multilayered pavement. In this program it is assumed that there is no cloud $\Omega_3(t)$. The first line is a set of points (up to seven). The computer drawn pictures which follow represent the results which are used by the engineers. Developments concerning applications can be found in [1,2,16,18]. An axi-symetric program has also been developped for cavities which are dug in the ground to be filled with liquefied gaz. The equation are solved by using finite differences with variable size mesh.

RAPPEL DES DONNEES 2008

TEMPERATURE DE SURFACE

DE	11.0	A	-12.0	DEGRES EN	20.0	JOURS
DE	-12.0	A	-15.0	DEGRES EN	25.0	JOURS
DE	-15.0	A	8.0	DEGRES EN	2.5	JOURS
DE	8.0	A	8.0	DEGRES EN	2.5	JOURS
DE	8.0	A	-10.0	DEGRES EN	10.0	JOURS
DE	-10.0	A	2.0	DEGRES EN	20.0	JOURS
DE	2.0	A	14.0	DEGRES EN	20.0	JOURS

COUCHE NO 2

HAUTEUR	=	.15000E 01	
DENSITE SECHE	=	.20600E 04	
TENEUR EN EAU	=	.54500E 01	
CHAL. SPECIFI.	=	.21914E 07	NON GELE
		.19568E 07	GELE
CONDUCTIBILITE	=	.20000E 01	NON GELE
		.20000E 01	GELE
PERMEABILITE	=	.10000E-13	
DEPRESSION	=	.10000E 05	

COUCHE NO 1

HAUTEUR	=	.39000E 02	
DENSITE SECHE	=	.20600E 04	
TENEUR EN EAU	=	.54500E 01	
CHAL. SPECIFI.	=	.21914E 07	NON GELE
		.19568E 07	GELE
CONDUCTIBILITE	=	.20000E 01	NON GELE
		.20000E 01	GELE
PERMEABILITE	=	.10000E-13	
DEPRESSION	=	.10000E 05	

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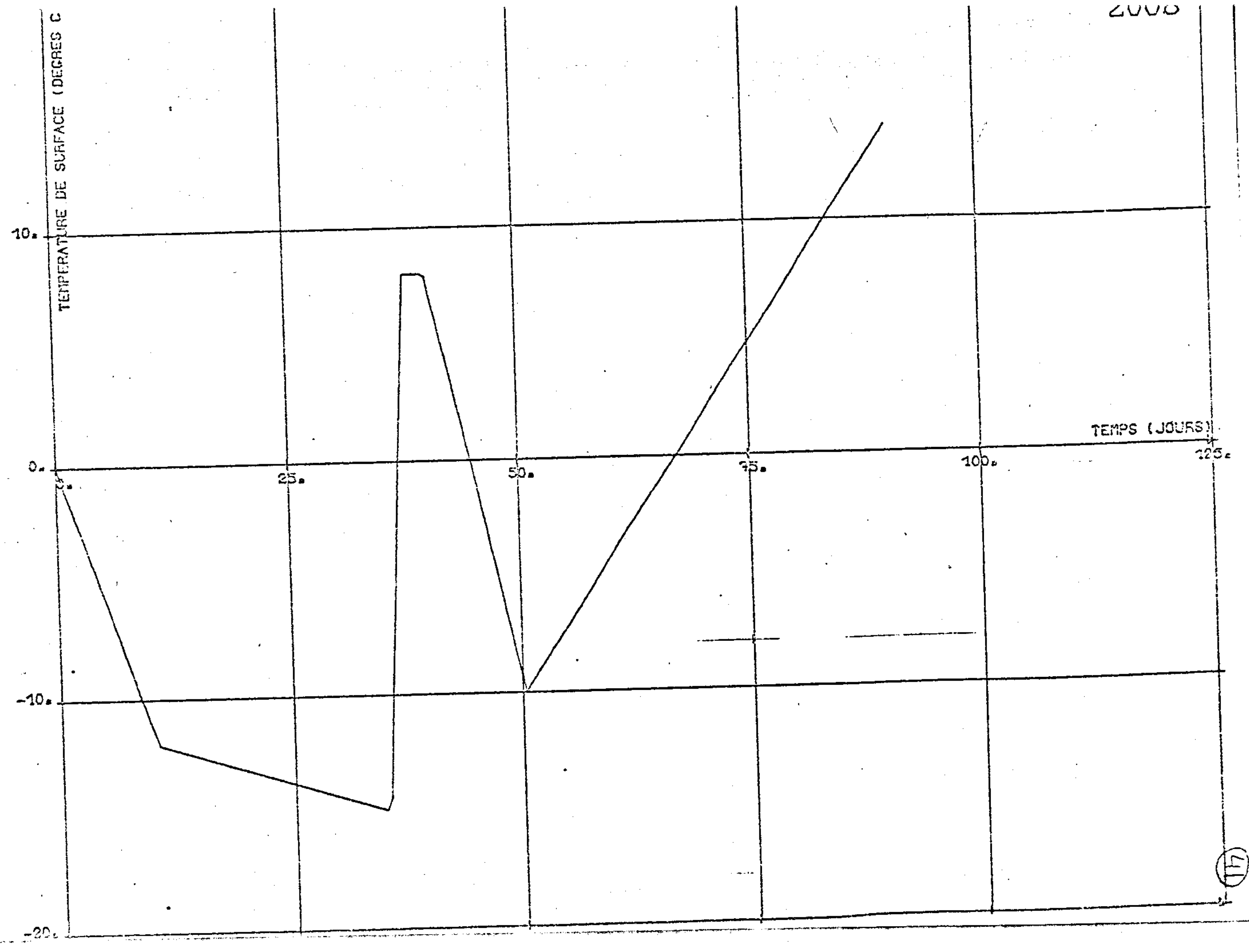
HAUTEUR D EAU INITIALE = .40500E 02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-20.
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75.
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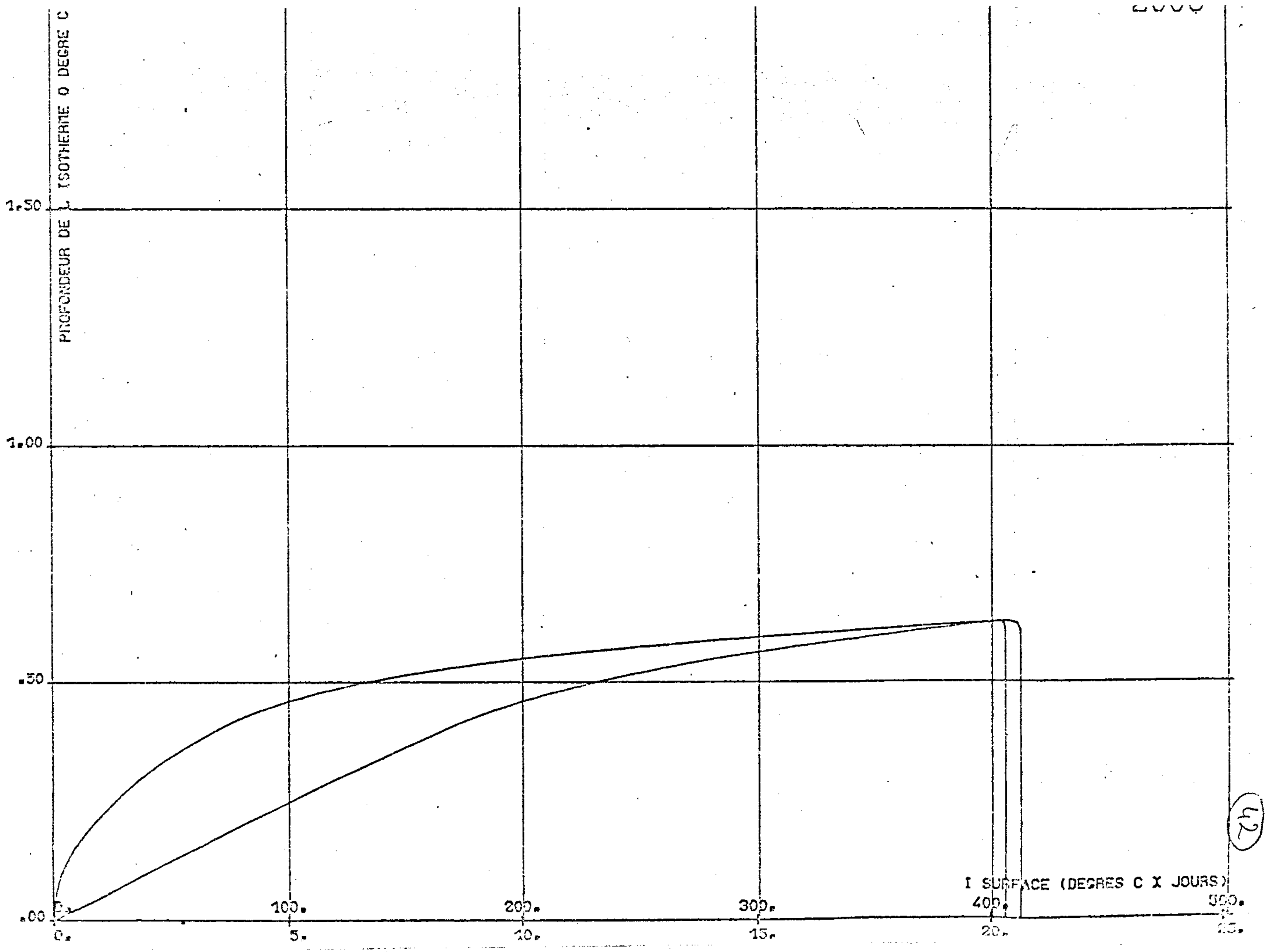

42
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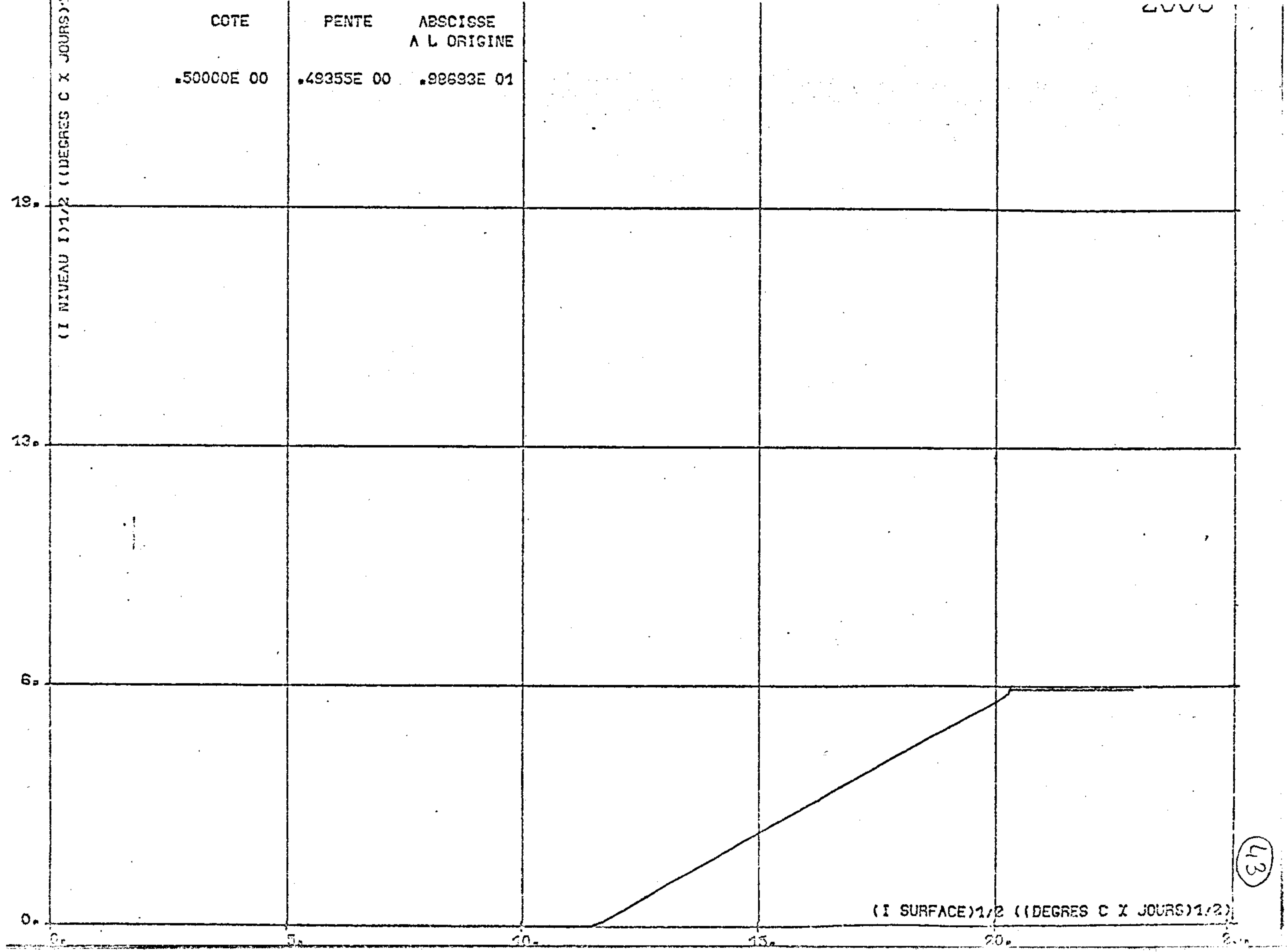
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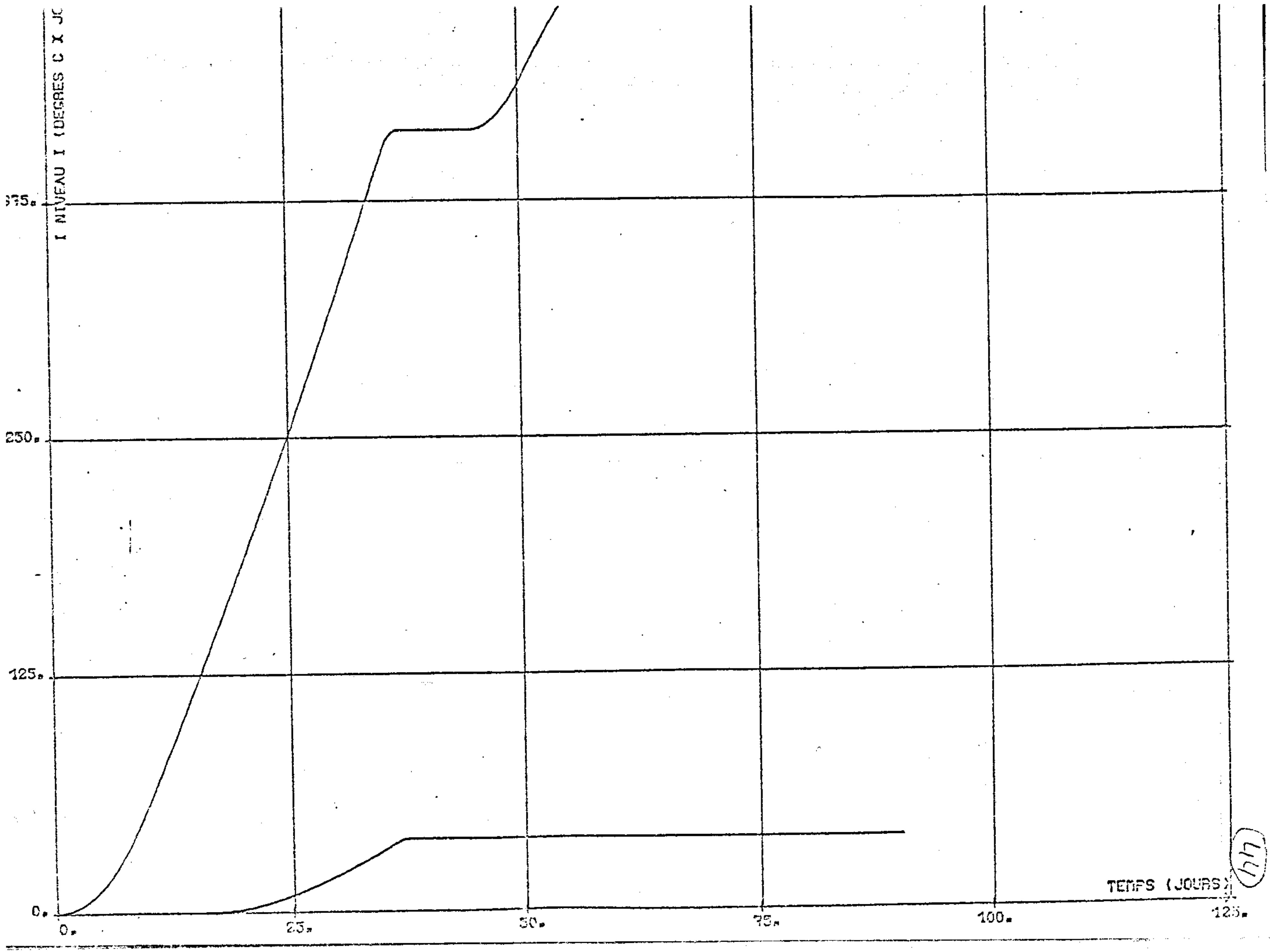
43


I NIVEAU I (DEGRES C X JO
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TEMPS (JOURS)
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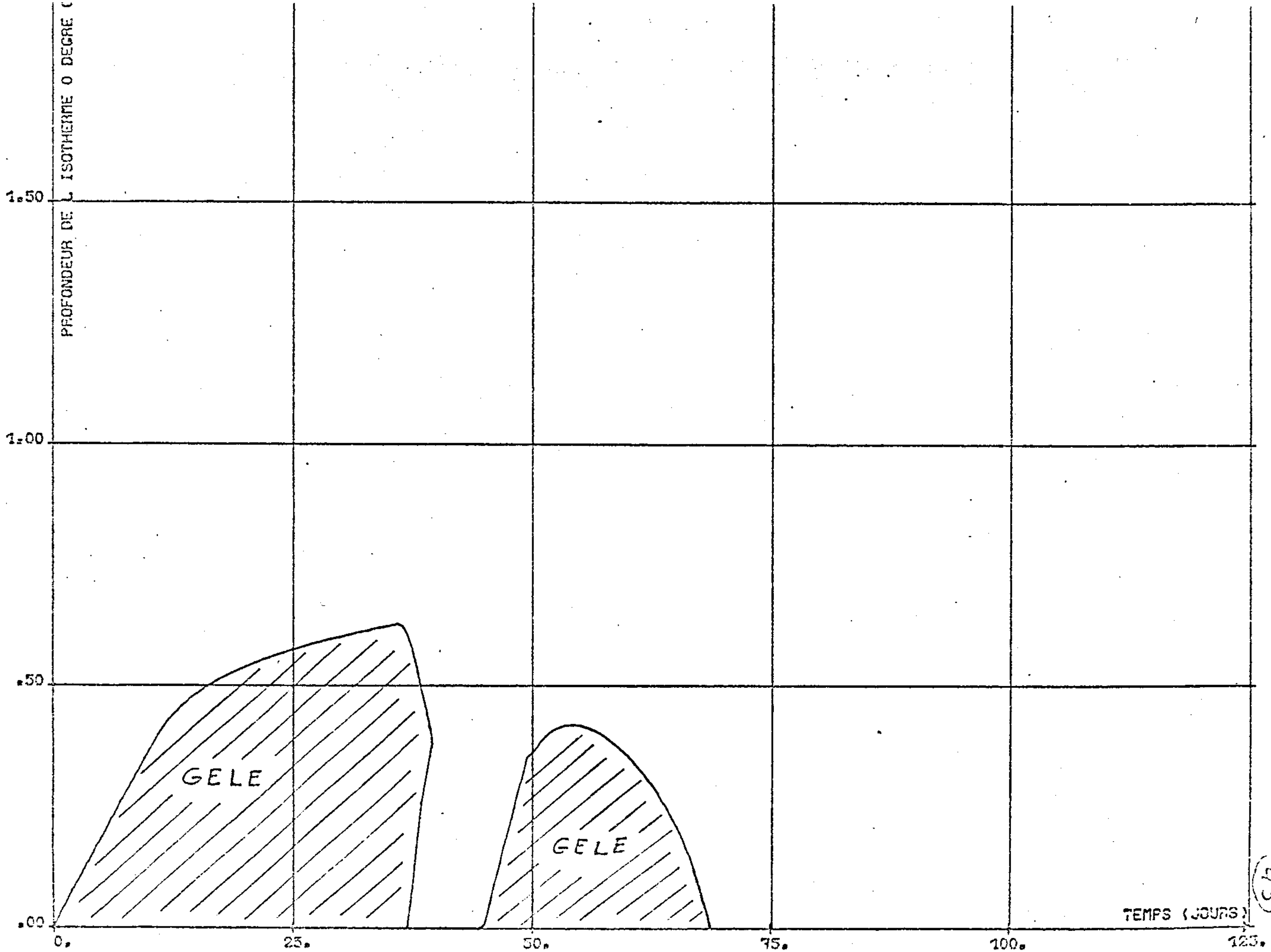
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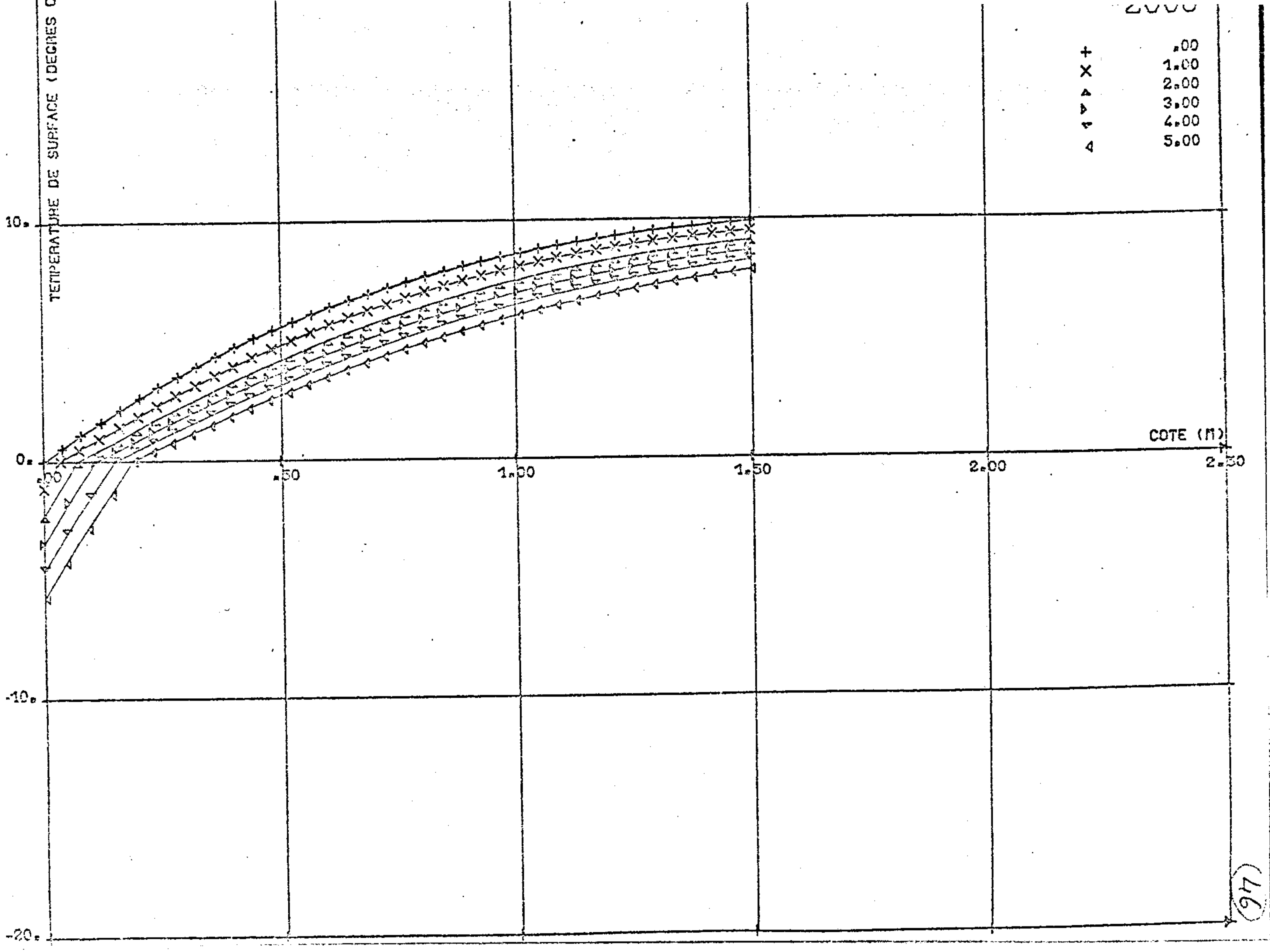
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COTE (M)
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3.00
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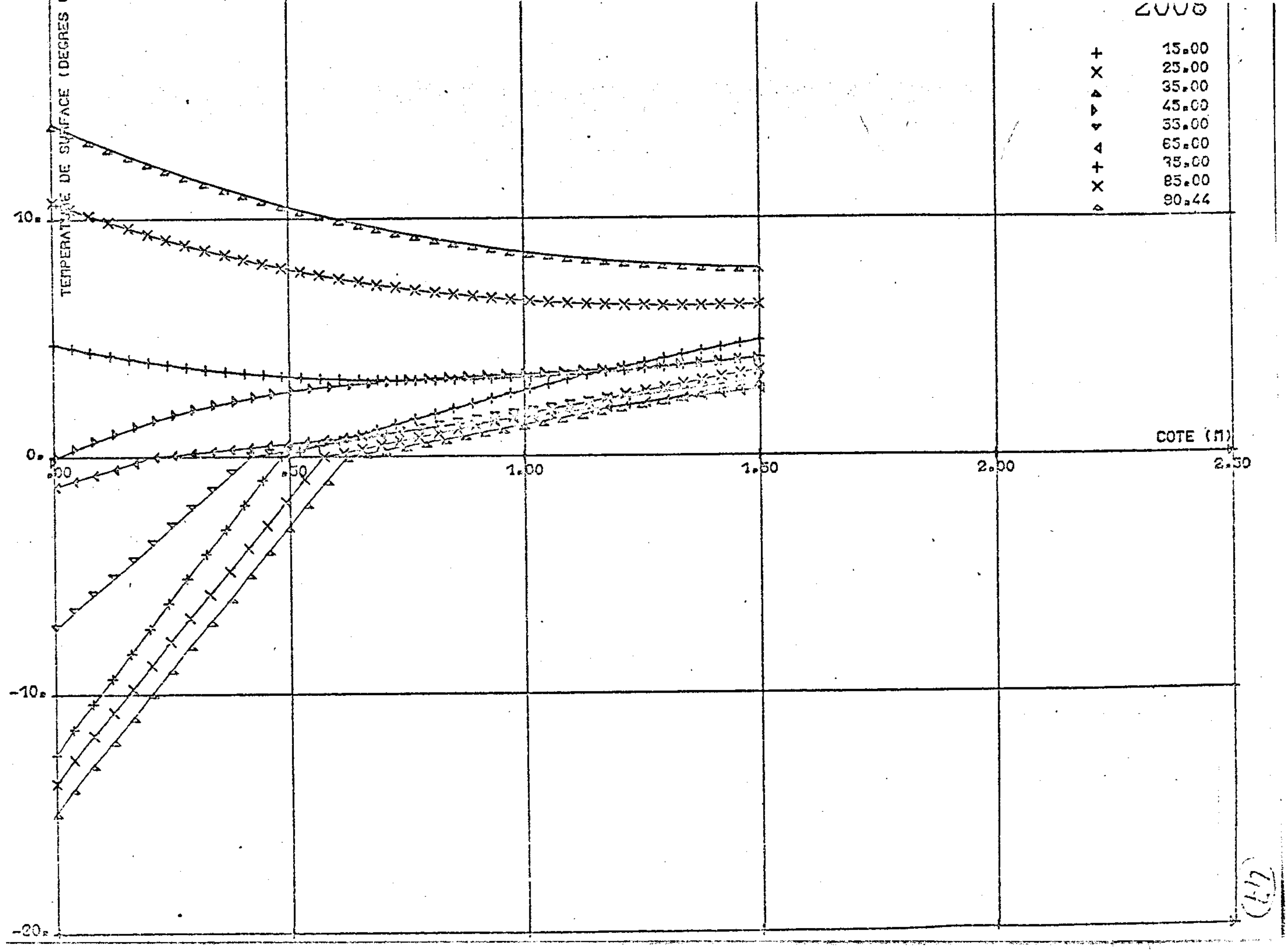
5.00



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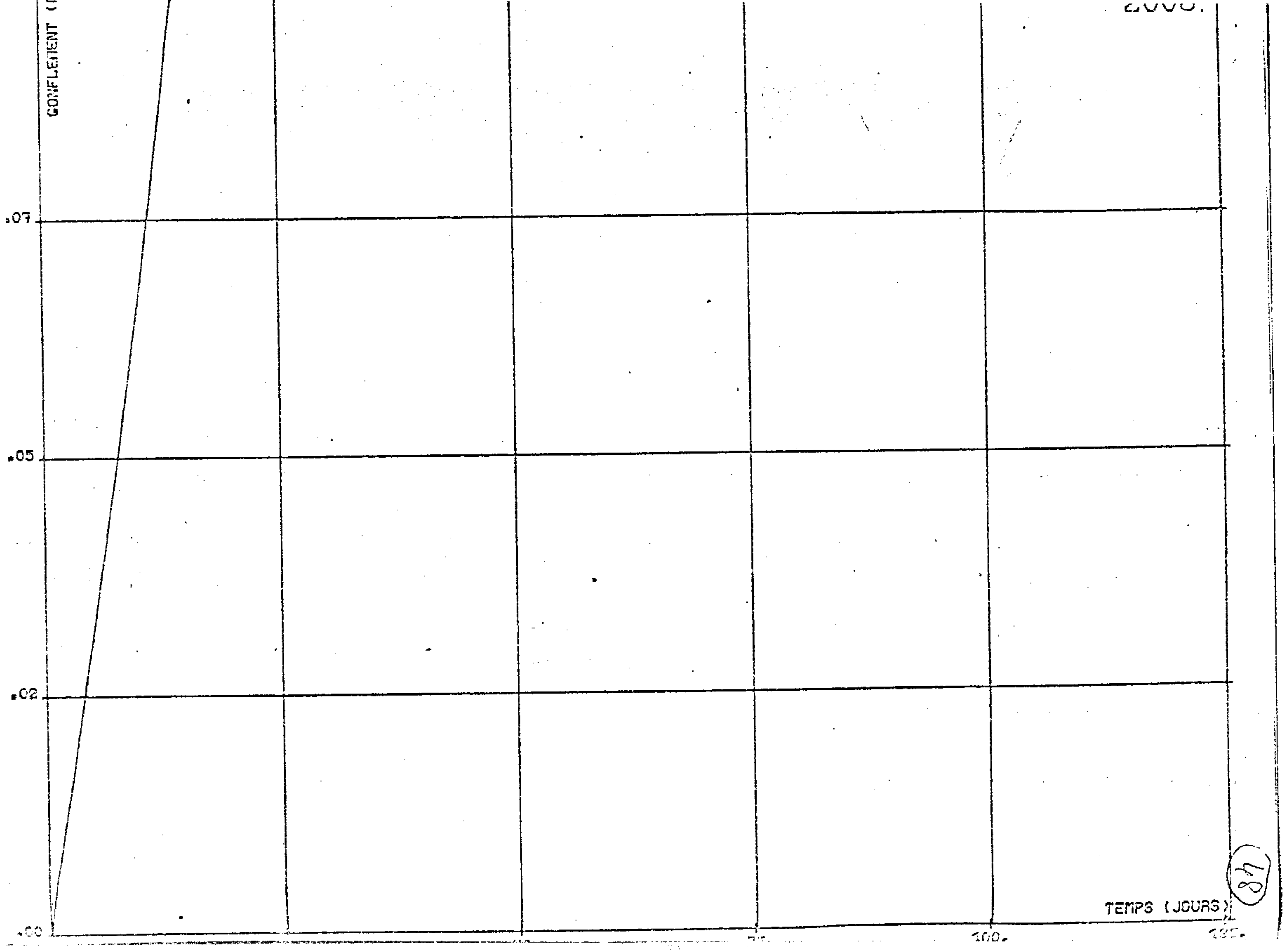
GONFLEMENT (
.07
.05
.02
.00
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TEMPS (JOURS)
48

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