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MATHEMATICAL ASPECTS OF FLUID FLOW
(Part 3)

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Part 3. Summary of Existence Theory for Steady Flows

The mathematical problem in question was presented as (2) in Part 2 (p. 14).

The aim now is to explain the principal ideas of the existence theory developed successively by J. Leray, E. Hopf and O. A. Ladyzhenskaya. For further details reference may be made to Ch. 5 of the book by Ladyzhenskaya ("The Mathematical Theory of Viscous Incompressible Flow", 2nd edition 1969) or to §7.4, of the book by D. H. Sattinger ("Topics in Stability and Bifurcation Theory", Springer Lecture Notes No. 309, 1973). Additional results from the application of degree theory to the hydrodynamic problem have been given in a recent paper by T. B. Benjamin ("Applications of Leray-Schauder degree theory to problems of hydrodynamic stability", Math. Proc. Camb. Phil. Soc. (1976), $\mathbf{79}$, pp. 373-392: also "Bifurcation phenomena in steady flows of a viscous fluid. Part 1", Fluid Mechanics Research Institute, Univ. of Essex, Rep. No. 76).

The theory is simplified somewhat if it is assumed that $\mathbf{q} \cdot \mathbf{n} = 0$ everywhere on ∂D (i.e. only a tangential velocity is imposed on the boundary ∂D), although this assumption is not essential. Then $\mathbf{q}$ given on ∂D can readily be extended to the interior of D in the form $\mathbf{q} = \nabla \times \mathbf{r}$, so that, if we write

$$\mathbf{U} = \mathbf{q} + \mathbf{V}, \tag{1}$$

we have automatically

$$\nabla \cdot \mathbf{V} = 0 \tag{2}$$

(since $\operatorname{div} \operatorname{curl} \mathbf{r} = 0$ identically) and

The differential equation to be satisfied by $\mathbf{V}$ is

$$R\{\mathcal{L}\mathbf{V} + N(\mathbf{V},\mathbf{V})\} = -\nabla P + \Delta\mathbf{V} + \mathbf{f}, \tag{4}$$

where $$\mathcal{L}\mathbf{V} = N(\mathbf{q},\mathbf{V}) + N(\mathbf{V},\mathbf{q})$$

is a linear operation on $\mathbf{V}$ (see p. 14 for the definition of the bilinear operator N) and

$$\mathbf{f} = \Delta\mathbf{q} - RN(\mathbf{q},\mathbf{q}).$$

If ∂D and the boundary velocity $\mathbf{q}|_{\partial D}$ are taken to be smooth enough, the extension $\mathbf{q}$ on D and hence $\mathbf{f}$ can be supposed to be smooth.

The next step is to recast the system (2), (3), (4) as a compact-operator equation in a certain Hilbert space $\mathcal{H}$. Accordingly, Leray-Schauder degree theory becomes applicable.

Let $\dot{J}$ denote the collection of C^∞, three-dimensional, solenoidal vector functions of $\mathbf{x}$ with compact support in D. The space $\mathcal{H}$ is defined as the completion of $\dot{J}$ under the norm

$$\|\mathbf{v}\|_1 = (\mathbf{v},\mathbf{v})_1^{\frac{1}{2}}$$

introduced on p. 16.† Thus $\dot{J} \subset \mathcal{H}$, but $\mathcal{H}$ is 'bigger' than $\dot{J}$, including the limits of infinite sequences in $\dot{J}$: such limits need not, of course, be in $\dot{J}$. The inner product for $\mathcal{H}$ is

$$(\mathbf{v},\mathbf{w})_1 = \int_D \frac{\partial v_i}{\partial x_j}\cdot\frac{\partial w_i}{\partial x_j}\,dx \qquad \forall\ \mathbf{v},\mathbf{w} \in \mathcal{H}.$$

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† Note that the suffix 1 was omitted in error from $(\mathbf{v},\mathbf{v})_1$ in the fifth line of the section 'Notation' on p. 16. Note also that the third line should read $\|\mathbf{v}\| = (\mathbf{v},\mathbf{v})^{\frac{1}{2}}$ (Sorry!)

So $\mathcal{H}$ is a Hilbert space of solenoidal vector functions defined a.e. on D, for which $\|\mathbf{V}\|_1 < \infty$, and which vanish 'weakly' on ∂D. (Note, incidentally, that the completion of $\dot{J}$ under the L_2 norm, i.e. $\|\mathbf{V}\|_0$, is a space which includes $\mathcal{H}$ [see p. 17 for reason why] but is evidently bigger than $\mathcal{H}$. The elements of this space are vector functions whose normal components vanish on ∂D, but whose tangential components are not necessarily zero.)

Now, multiply equation (4) by any $\boldsymbol{\psi} \in \dot{J}$ and integrate over D, assuming each term of the equation to be a smooth function. The pressure term makes no contribution because of the identity noted at the middle of p. 17. Hence, integrating by parts and again using (2) and the boundary condition (3), we obtain

$$\int_D \left[\frac{\partial V_i}{\partial x_j}\cdot\frac{\partial \psi_i}{\partial x_j} + R\left\{ V_j \frac{\partial q_i}{\partial x_j}\psi_i - V_i q_j \frac{\partial \psi_i}{\partial x_j} - V_i V_j \frac{\partial \psi_i}{\partial x_j} \right\}\right] dx$$

$$= \int_D f_i \psi_i \, dx. \qquad (5)$$

The function $\mathbf{V}$ is said to be a <u>'weak'</u> (<u>or 'generalized'</u>) <u>solution</u> of our problem if $\mathbf{V} \in \mathcal{H}$ and (5) is satisfied for all $\boldsymbol{\psi} \in \mathcal{H}$. (<u>Exercise</u>. Confirm (5) in the case that $\mathbf{U} = \mathbf{q} + \mathbf{V}$ is a classical, twice continuously differentiable solution of the original problem and $\mathbf{q}$ is correspondingly smooth.)

We consider the various terms in (5) as follows:

(i) The first integral on the left-hand side is just $(\mathbf{V}, \boldsymbol{\psi})_1$.

(ii) The term on the right-hand side is also a linear functional and then the Poincaré inequality (see p. 17)

$$F(\underset{\sim}{\psi}) = (\underset{\sim}{f}, \underset{\sim}{\psi})_0 \leq \|\underset{\sim}{f}\|_0 \cdot \|\underset{\sim}{\psi}\|_0 \leq c\,\|\underset{\sim}{f}\|_0 \cdot \|\underset{\sim}{\psi}\|_1 .$$

Thus, if $\underset{\sim}{f} \in L_2(D \to \mathbb{R}^3)$, F is a bounded (continuous) linear functional on $\mathcal{H}$. Hence, according to the Riesz representation theorem, there is an element $\underset{\sim}{f}' \in \mathcal{H}$ such that

$$F(\underset{\sim}{\psi}) = (\underset{\sim}{f}', \underset{\sim}{\psi})_1 .$$

(iii) Regarding the remaining terms on the left-hand side of (5), we note in general that

$$\left| \int_D v_i\, w_j\, \frac{\partial \psi_i}{\partial x_j}\, dx \right| \leq \|\underset{\sim}{v}\|_{L_4} \cdot \|\underset{\sim}{w}\|_{L_4} \cdot \|\underset{\sim}{\psi}\|_1 ,$$

where

$$\|\underset{\sim}{v}\|_{L_4} = \left(\int_D |\underset{\sim}{v}|^4\, dx \right)^{\frac{1}{4}} .$$

(<u>Exercise</u>. Verify this result using Hölder's inequality.)

We now use the following lemma (see Ladyzhenskaya, Ch. 1, §2):

<u>Lemma</u>. For any three-dimensional vector function $\underset{\sim}{v}$ on D which vanishes on ∂D, $\|\underset{\sim}{v}\|_{L_4} \leq a\,\|v\|_1$, where <u>a</u> is a finite constant depending only on D. Moreover, a weakly convergent sequence of functions in $\mathcal{H}$ converges strongly in L_4. †

† i.e. if $\{\underset{\sim}{v}_n\}$ $(n = 1,2,3,\dots)$ converges weakly to $\underset{\sim}{v}$ in $\mathcal{H}$, then $\|\underset{\sim}{v}_n - \underset{\sim}{v}\|_{L_4} \to 0$ as $n \to \infty$.

It follows that

$$\left| \int_D v_i \, w_j \, \frac{\partial \psi_i}{\partial x_j} \, dx \right| \quad \leq \quad \# \, a^2 \, \|\mathbf{v}\|_1 \cdot \|\mathbf{w}\|_1 \cdot \|\boldsymbol{\psi}\|_1 \; ;$$

and hence, by virtue of the Riesz theorem, there is an element $N(\mathbf{v}, \mathbf{w}) \in \mathcal{H}$ such that

$$\int_D v_i \, w_j \, \frac{\partial \psi_i}{\partial x_j} \, dx \quad = \quad (N(\mathbf{v}, \mathbf{w}), \boldsymbol{\psi})_1 \, .$$

Here N is evidently a bilinear operator $\mathcal{H} \times \mathcal{H} \to \mathcal{H}$.

In particular, considering the last term on the left-hand side of (5), we conclude that

$$\int_D V_i \, V_j \, \frac{\partial \psi_i}{\partial x_j} \, dx \quad = \quad (C\mathbf{V}, \boldsymbol{\psi})_1 \, ,$$

where $C\mathbf{V} = N(\mathbf{V}, \mathbf{V})$ is a quadratic operation on $\mathbf{V} \in \mathcal{H}$. It next needs to be shown that C is a completely continuous operator.

Let $\{\mathbf{v}_n\}$ be a weakly convergent sequence in $\mathcal{H}$. From the definition of C, we have

$$\begin{aligned} |(C\mathbf{v}_n, \boldsymbol{\psi})_1 - (C\mathbf{v}_m, \boldsymbol{\psi})_1| &= |(N(\mathbf{v}_n - \mathbf{v}_m, \mathbf{v}_n) - N(\mathbf{v}_m, \mathbf{v}_m - \mathbf{v}_n), \boldsymbol{\psi})_1| \\ &\leq \|\mathbf{v}_n - \mathbf{v}_m\|_{L_4} \cdot (\|\mathbf{v}_n\|_1 + \|\mathbf{v}_m\|_1) \cdot \|\boldsymbol{\psi}\|_1 \, . \end{aligned}$$

This holds for all $\boldsymbol{\psi} \in \mathcal{H}$, so that we may put $\boldsymbol{\psi} = C\mathbf{v}_n - C\mathbf{v}_m$ and hence obtain

$$\|C\mathbf{v}_n - C\mathbf{v}_m\|_1 \quad \leq \quad \|\mathbf{v}_n - \mathbf{v}_m\|_{L_4} \cdot (\|\mathbf{v}_n\|_1 + \|\mathbf{v}_m\|_1).$$

But, by the lemma given on p. 22, $\|\mathbf{v}_n - \mathbf{v}_m\|_{L_4} \to 0$ as

Thus, $\{C\underset{\sim}{v}_n\}$ is a Cauchy sequence in $\mathcal{H}$, and C is shown to be completely continuous (so compact).

(vi) In a similar way, it can be shown that

$$\int_D \left\{ - V_j \frac{\partial q_i}{\partial x_j} \psi_i + V_i q_j \frac{\partial \psi_i}{\partial x_j} \right\} dx = (B_q \underset{\sim}{V}, \underset{\sim}{\psi})_1 ,$$

where $B_q : \mathcal{H} \to \mathcal{H}$ is a completely continuous linear operator depending on $\underset{\sim}{q}$.

Collecting together the results, we have that (5) is equivalent to

$$(\underset{\sim}{V} - R\{B_q \underset{\sim}{V} + C\underset{\sim}{V}\} - \underset{\sim}{f}', \underset{\sim}{\psi})_1 = 0.$$

And since $\underset{\sim}{\psi}$ is an arbitrary element of $\mathcal{H}$, this implies

$$\underset{\sim}{V} = R(B_q \underset{\sim}{V} + C\underset{\sim}{V}) + \underset{\sim}{f}', \qquad (6)$$

which is the required operator equation in $\mathcal{H}$.

The next stage of the argument is to establish an <u>a priori</u> bound for solutions of (6). Take the inner product of (6) with $\underset{\sim}{V}$ (in effect, put $\underset{\sim}{\psi} = \underset{\sim}{V}$ in the preceding equation). It has already been shown in Part 2 (p. 16) that

$$(\underset{\sim}{V}, C\underset{\sim}{V})_1 = 0$$

if $\underset{\sim}{V}$ is smooth (C^∞, solenoidal, vanishing on ∂D), and this result extends by continuity to $\underset{\sim}{V} \in \mathcal{H}$. Hence we are left with

$$\|\underset{\sim}{V}\|_1^2 = R(\underset{\sim}{V}, B_q \underset{\sim}{V})_1 + (\underset{\sim}{V}, \underset{\sim}{f}')_1 ; \qquad (7)$$

and, recalling the result given in the fifth line of p.16, we have

$$(\underset{\sim}{V}, B_{\underset{\sim}{\psi}} \underset{\sim}{V}) \quad = \quad \int_D V_i V_j \frac{\partial \psi_i}{\partial x_j}\, dx .$$

To estimate this term, the following lemma due to E. Hopf is used (see Sattinger, pp. 160–163: the proof of the lemma is fairly complicated and so will not be attempted here).

<u>Lemma.</u> For any $\varepsilon > 0$ however small, the extension of $\underset{\sim}{\psi}$ to the interior of D can be chosen so that

$$\left| \int_D V_i V_j \frac{\partial \psi_i}{\partial x_j}\, dx \right| \quad \leq \quad \varepsilon \, \| \underset{\sim}{V} \|_1^2$$

It hence follows that (7) will give

$$\| \underset{\sim}{V} \|_1^2 \quad \leq \quad \varepsilon R \, \| \underset{\sim}{V} \|_1^2 \; + \; |(\underset{\sim}{f}', \underset{\sim}{V})_1|$$

$$\leq \quad \varepsilon R \, \| \underset{\sim}{V} \|_1^2 \; + \; \| \underset{\sim}{f}' \|_1 \cdot \| \underset{\sim}{V} \|_1 ,$$

that is, on division by $\| \underset{\sim}{V} \|_1 > 0$,

$$\| \underset{\sim}{V} \|_1 \quad \leq \quad \varepsilon R \, \| \underset{\sim}{V} \|_1 \; + \; \| \underset{\sim}{f} \|_1 . \qquad (8)$$

Now, choose ε — and accordingly $\underset{\sim}{\psi}$ — such that $\varepsilon R < 1$. Then (8) implies that

$$\| \underset{\sim}{V} \|_1 \quad \leq \quad \frac{\| \underset{\sim}{f}' \|_1}{1 - \varepsilon R} ,$$

which is our <u>a priori</u> bound for solutions of (6).

The conclusions established by Leray-Schauder degree theory (see Appendix) can now be applied to complete an

26.

existence theorem. Take $\Omega = \{ \underset{\sim}{v} \in \mathcal{H} : \| \underset{\sim}{v} \|_1 < r \}$, where

28.

It can easily be shown that B is the Fréchet derivative, at the zero point θ of $\mathcal{H}$, of the full compact operator B + C appearing on the right-hand side of (10). Hence, in the case that 1 is not an eigenvalue of B, the index of the zero solution of (10) (which is the same as the index of the solution $\underset{\sim}{V}$ of (6) that gives the particular steady flow $\underset{\sim}{U} = \underset{\sim}{q}_1 + \underset{\sim}{V}$) is

$$i_1 = \deg (I - RB, \mathcal{B}_1),$$

where $\mathcal{B}_1 = \{ \underset{\sim}{v} \in \mathcal{H} : \| \underset{\sim}{v} \| < 1 \}$ is the unit ball in $\mathcal{H}$.

[illegible] degree (Appendix.