INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
MIRAMARE - P.O.B. 586 - 34100 TRIESTE (ITALY) - TELEPHONES: 224281/2/3/4/5/6 - CABLE: CENTRATOM

SMR/23/34

AUTUMN COURSE ON APPLICATIONS OF ANALYSIS TO MECHANICS

22 September - 26 November 1976

THEORY OF PLATES

Part I : General equations for curvilinear media and plates

Part II : Elastic plates

Part III : Green's formulas associated with theories of plates

Claude Do
Université of Nantes (E.N.S.M.)
France

I - General equations for curvilinear media and plates -

To write these general equations, the particular geometric configuration of these media is used; important simplifications of general equations of Mechanics of continua are obtained.

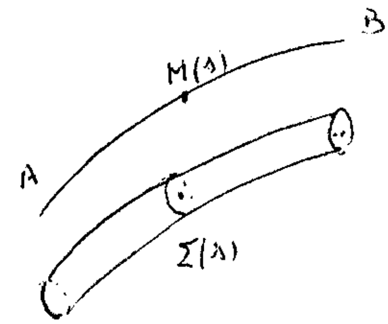
Many theories are possible; for instance, we can deduce the theory from the three dimensional one.

For curvilinear media, we are going directly to write the fundamental law of dynamics, using the particular geometric aspect of the system.

For plates, we use the virtual power principle (which is recalled in this section); we will see below that this principle is very comparable to variational formulations.

1 - Curvilinear media.

The system can be represented by an arc of curve AB; s is the arc length on AB. In fact a point M on this arc, with parameter s is the "center" of a section $\Sigma(s)$ of the three dimensional domain; we suppose that



$\left\{ \begin{array}{l} \text{diameter of } \Sigma(s) \text{ is small with} \\ \text{respect to the length } L \text{ of } AB. \end{array} \right.$

We suppose that the exterior forces are given by only a density $f(s)$; there is no exterior couple.

Now, we consider the part $s_1 < s < s_2$; the body and contact forces on this part have:

- the resultant : $\mathcal{T}(s_2) - \mathcal{T}(s_1) + \int_{s_1}^{s_2} f(s)\, ds$

- the moment (with respect to an origin O) :

$$OM_2 \wedge \mathcal{T}(s_2) + \mathcal{M}(s_2) - OM_1 \wedge \mathcal{T}(s_1) + \mathcal{M}(s_1) + \int_{s_1}^{s_2} OM \wedge f(s)\, ds,$$

where

$\mathcal{T}(s)$ is the resultant of contact actions of the part $[\sigma > s]$ on the part $[\sigma < s]$;

$\mathcal{M}(s)$ is the moment of - - - - - -
- - - -

The fundamental law of static implies that the resultant and the moment of all exterior forces acting on (s_1, s_2) are 0 ; then, we obtain

$$\begin{cases} \mathcal{T}(s_2) - \mathcal{T}(s_1) + \int_{s_1}^{s_2} f(s)\, ds = 0 \\ OM_2 \wedge \mathcal{T}(s_2) + \mathcal{M}(s_2) - OM_1 \wedge \mathcal{T}(s_1) + \mathcal{M}(s_1) + \int_{s_1}^{s_2} OM \wedge f(s)\, ds = 0 \end{cases}$$

If we assume $\mathcal{T}$ and $\mathcal{M}$ are continuously differentiable, we obtain the relations for equilibrium in terms of differential equations :

$$\begin{cases} \dfrac{d\mathcal{T}}{ds} + f = 0 \\ \dfrac{d\mathcal{M}}{ds} + \vec{\tau} \wedge \vec{\mathcal{T}} = 0 \end{cases}$$

where τ is the unit vector of the tangent at AB.

Boundary conditions

If $s(A) = 0$ and $s(B) = L$, we have

$$\begin{cases} \mathcal{T}(0) = -F_A \\ \mathcal{M}(0) = -\mu_A \end{cases}$$

$$\begin{cases} \mathcal{T}(L) = F_B \\ \mathcal{M}(L) = \mu_B \end{cases}$$

F_A, F_B, μ_A, μ_B are exterior forces (and couples), given or not according as situations.

For example, if the origin O is clamped, F_A, μ_A are not given; (F_A, μ_A) is the reaction of the support on the curvilinear media.

At the contrary, if the end B is free, exterior forces are zero, then $F_B = 0$, $\mu_B = 0$.

2 - Virtual power principle.

This principle is often used to describe the forces acting on a systems S, and especially for describing internal forces -

Let be a material system S, Σ an other system ...
The virtual power principle is ~~founded~~ based on the following remark:

{ We will know the actions of system Σ on the system S
if we know the "power" of these actions for a set
of motions which is sufficiently rich.

More precisely, we consider a vector space $\mathcal{V}$ of virtual motions defined on S (with a suitable topology).

Definition. The power of forces of system Σ on S is a continuous linear functional Π on $\mathcal{V}$.; if $V \in \mathcal{V}$,

$$\mathcal{P} = \Pi(V)$$

is the power of forces of system Σ on S in the virtual motion V.

Example. When S is reduced to one material point M, $\mathcal{V}$ is a three-dimensional space. Then $\Pi(V)$ has the form

$$\Pi(V) = f \cdot V \; .$$

The vector f ~~can be~~ is the exterior force acting on M.

The virtual power principle is constituted by two axioms, valid for all system S.

Axiom 1. The power of internal forces is zero for all rigid motion.

Axiom 2. The virtual power of all forces acting on S is zero (in a Galilean frame)

The idea that the description of these forces is better

Consequence of Axiom 1. First Gradient theory.

Let be a Continuous System Ω and S a part of Ω.
$V = \{V(M), M \in \Omega\}$ a virtual motion on S.
We call $\mathcal{P}_{(i)}(S)$ the virtual power of internal forces, in this motion :

$$\mathcal{P}_{(i)}(S) = \Pi(V).$$

The first gradient theory consists to assume that Π is a linear functional depending only of V and gradient V;
, writing $V_{i,j} = D_{ij} + \Omega_{ij}$
with

$$D_{ij} = \frac{1}{2}(V_{i,j} + V_{j,i}) \quad \text{(symetric part of } V_{i,j})$$

$$\Omega_{ij} = \frac{1}{2}(V_{i,j} - V_{j,i}) \quad \text{(antisymetric part of } V_{i,j})$$

we obtain, in this theory

Definition.

$$\boxed{\mathcal{P}_{(i)}(S) = -\int_S (b_i V_i + \mathcal{E}_{ij}\Omega_{ij} + \sigma_{ij} D_{ij})\,dx}$$

Then, Axiom 1 implies $\boxed{\begin{matrix} b_i = 0 \\ \mathcal{E}_{ij} = 0 \end{matrix}}$

And, we have only

$$\mathcal{P}_{(i)}(S) = -\int_S \sigma_{ij} D_{ij}\,dx \qquad (*)$$

which is the usual form of virtual power of internal forces.
From this expression, we can deduce the definition of stresses tensor in the continua.

(*) Therefore, in this theory, interior forces are caracterized by 6 functions σ_{ij} (6 because we can assume that σ_{ij} are symetric)

In the following, we write Axiom 2 in the form

$$\mathcal{P}_{(i)} + \mathcal{P}_{(d)} + \mathcal{P}_{(c)} = 0 \qquad \text{for all } V \in \mathcal{V};$$

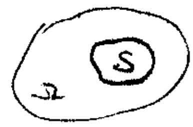
we have put

$\mathcal{P}_{(i)}$ = virtual power of internal forces (as before)

$\mathcal{P}_{(d)}$ = virtual power of exterior body forces

$\mathcal{P}_{(c)}$ = virtual power of boundary forces

In general, exterior body forces are <u>given</u>.
At the contrary, boundary forces can be contact forces, when S is a part of Ω



3. General equations for plates.

We consider a three dimensional domain $\bar{S}$ which is a "plate"; then $\bar{S} = S \times \left(-\frac{h}{2}, \frac{h}{2}\right)$, where S is a two dimensional domain (of the plane of the plate) and h is the thickness of the plate. We assume that h is small with respect to the diameter of S. (It is possible to consider that h is not a constant).

A plate theory consists to write equations only on the domain S, and therefore for functions of $x = (x_\alpha)$ only (1); we use the notations:

$$\bar{x} \in \bar{S} \quad , \quad \bar{x} = (x, x_3) \quad , \quad x = (x_\alpha) \in S, \quad |x_3| < \frac{h}{2}.$$

Equations of plates are derived from the virtual power principle.

On accounts of the small thickness of the plate, we suppose that a motion (defined on the three dimensional domain) is well described by the field $V(x, x_3)$ such that:

$$(1) \quad \begin{cases} V_\alpha(x, x_3) = v_\alpha(x) + x_3 \, \ell_\alpha(x), \\ V_3(x, x_3) = w(x) \end{cases}$$

Therefore, a virtual motion is caracterized by only five scalar functions $(v_\alpha, \ell_\alpha, w)$ defined on S. The field of virtual rate deformations tensor is given by

$$(2) \quad \begin{cases} D_{\alpha\beta} = d_{\alpha\beta} + x_3 \, K_{\alpha\beta} \\ 2 D_{\alpha 3} = \ell_\alpha + w_{,\alpha} = b_\alpha \\ D_{33} = 0 \end{cases}$$

with

$$(3) \quad \begin{cases} d_{\alpha\beta} = \frac{1}{2}\left(v_{\alpha,\beta} + v_{\beta,\alpha}\right) \\ K_{\alpha\beta} = \frac{1}{2}\left(\ell_{\alpha,\beta} + \ell_{\beta,\alpha}\right) \end{cases}$$

(1) represents a solid motion if, and only if,

$D_{ij} = 0$, or $d_{\alpha\beta} = 0$, $K_{\alpha\beta} = 0$, $b_\alpha = 0$.

Now, let be $\mathcal{D}$ a part of S, $\bar{\mathcal{D}} = \mathcal{D} \times (-\frac{h}{2}, \frac{h}{2})$.

In the <u>first gradient theory</u>, we must choose

$$\mathcal{P}_{(i)}(\mathcal{D}) = -\int_{\mathcal{D}} (N_{\alpha\beta}\, d_{\alpha\beta} + M_{\alpha\beta} K_{\alpha\beta} + Q_\alpha b_\alpha)\, dx$$

for the definition of virtual power of internal forces.

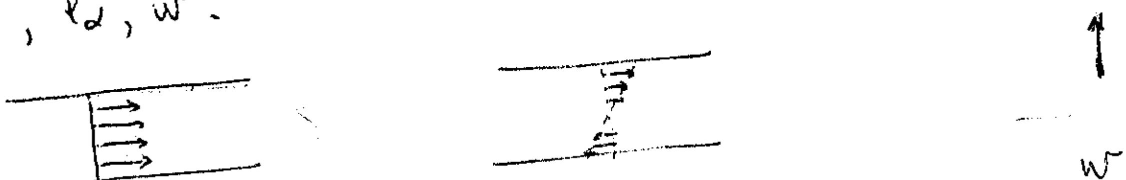
$N_{\alpha\beta}$ is the stress tensor of tensions (in the plane of the plate);

$M_{\alpha\beta}$ is the bending stress tensor;

Q_α is the shearing stress vector.

We obtain interpretations, looking the meaning of

v_α, l_α, w.



$v_\alpha\ (\to d_{\alpha\beta})$ $\qquad$ $x_3 l_\alpha\ (\Rightarrow K_{\alpha\beta})$ $\qquad$ w

In this theory, we must choose following expressions as virtual power of exterior forces:

$$\mathcal{P}_{(d)} = \int_{\mathcal{D}} (p_\alpha v_\alpha + m_\alpha l_\alpha + \omega w)\, dx$$

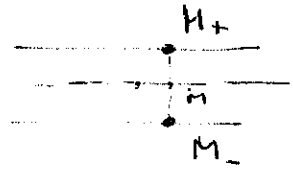
$$\mathcal{P}_{(c)} = \int_{\partial\mathcal{D}} (T_\alpha v_\alpha + C_\alpha l_\alpha + T w)\, d\sigma$$

$\mathcal{P}_{(d)}$ is the virtual power of body forces;

$\mathcal{P}_{(c)}$ " " " " " boundary forces (boundary of $\mathcal{D}$).

Love-Kirchhoff theory

But we are going to take a simpler theory. First, observe that in previous considered motions, each segment M_-M_+ which is normal to the plate before deformation, has a rigid motion.



(because, for fixed $x \in S$,

we have $\quad V(P) = V(M) + \omega \wedge MP$

where $\quad \vec{MP} = x_3 \vec{e}_3 \quad (P \in M_- M_+)$

and $\quad \vec{\omega} = - \ell_2 \vec{e}_1 + \ell_1 \vec{e}_2$)

In the following (Love-Kirchhoff theory), we assume that

<u>Hypothesis</u>. each segment $M_- M_+$ remains normal to the plate, in the deformation.

Then, there is no deformation in directions (x_α, x_3) (no shearing deformation) and $b_\alpha = 0$. Therefore

$$K_{\alpha\beta} = - w_{,\alpha\beta}$$

Then, in this theory, we have the following expression for $\mathcal{P}_{(i)}$:

$$\mathcal{P}_{(i)}(\mathcal{D}) = - \int_{\mathcal{D}} \left(N_{\alpha\beta}\, d_{\alpha\beta} + M_{\alpha\beta} K_{\alpha\beta} \right) dx.$$

In addition, we can assume that N and M are <u>symetric tensors</u> (because d and K have the same property).

We integrate by parts (two times in the term $K_{\alpha\beta} = - w_{,\alpha\beta}$), and we obtain:

$$\begin{cases} \mathcal{P}_{(1)}(\mathcal{D}) = \int_{\mathcal{D}} \left(M_{\alpha\beta,\alpha\beta}\, w + N_{\alpha\beta,\beta}\, v_\alpha \right) dx \; + \\ \quad + \int_{\partial\mathcal{D}} \left\{ M_n \dfrac{\partial w}{\partial n} - \left(\dfrac{dM_\tau}{ds} + Q \right) w - N_{\alpha\beta}\nu_\beta v_\alpha \right\} d\sigma \\ \quad - \sum_i \llbracket M_\tau \rrbracket\, w . \end{cases}$$

where ν is the unitary outward normal to $\partial\mathcal{D}$, $\vec{\tau} = \vec{e}_3 \wedge \vec{\nu}$, and

$$Q = M_{\alpha\beta,\beta}\,\nu_\alpha$$

$$M_\tau = M_{\alpha\beta}\,\nu_\beta \tau_\alpha$$

$$M_n = M_{\alpha\beta}\,\nu_\alpha \nu_\beta$$

$\llbracket\ \rrbracket$ is the discontinuity at corners of the boundary, and the sum $\sum_i$ is on all the corners.

Computation of $\mathcal{P}_{(1)}$:

$$-\int_{\mathcal{D}} N_{\alpha\beta}\, d_{\alpha\beta}\, dx = \int_{\mathcal{D}} N_{\alpha\beta,\beta}\, v_\alpha\, dx - \int_{\partial\mathcal{D}} N_{\alpha\beta} v_\alpha \nu_\beta\, d\sigma$$

$$-\int_{\mathcal{D}} M_{\alpha\beta} K_{\alpha\beta}\, dx = \int_{\mathcal{D}} M_{\alpha\beta,\alpha\beta}\, w\, dx - \int_{\partial\mathcal{D}} M_{\alpha\beta,\beta}\,\nu_\alpha\, w\, d\sigma + \int_{\partial\mathcal{D}} M_{\alpha\beta}\nu_\beta\, w_{,\alpha}\, d\sigma$$

On the last term, we can do an integration by parts, writing

$$w_{,\alpha} = \frac{\partial w}{\partial n}\nu_\alpha + \frac{\partial w}{\partial s}\tau_\alpha \text{, and}$$

$$M_{\alpha\beta}\nu_\beta w_{,\alpha} = M_n \frac{\partial w}{\partial n} + M_\tau \frac{\partial w}{\partial \tau}$$

Then $\displaystyle\int_{\partial\mathcal{D}} M_\tau \frac{\partial w}{\partial \tau}\, ds = -\int_{\partial\mathcal{D}} \frac{\partial M_\tau}{\partial \tau}\, w\, ds + \sum_i \llbracket M_\tau \rrbracket\, w$,

and we obtain announced formula.

To choose the form of $\mathcal{P}_{(d)}$ and $\mathcal{P}_{(c)}$, we use the before formula for $\mathcal{P}_{(i)}$, and we take:

$$\mathcal{P}_{(d)} = \int_{\mathcal{D}} (p_\alpha v_\alpha + \omega w)\, dx ,$$

$$\mathcal{P}_{(c)} = \int_{\partial\mathcal{D}} \left(T_\alpha v_\alpha - \Gamma \frac{dw}{dn} + T w\right) ds + \sum_i R_i w_i$$

Via the virtual power principle, we write

$$\mathcal{P}_{(i)} + \mathcal{P}_{(d)} + \mathcal{P}_{(c)} = 0 \qquad \text{for all } v, w ,$$

and we obtain

the equilibrium relations,

the boundary conditions.

First, we choose v, w with compact support in $\mathcal{D}$; we obtain equilibrium relations:

$$\boxed{\begin{cases} N_{\alpha\beta,\beta} + p_\alpha = 0 \\ M_{\alpha\beta,\alpha\beta} + \omega = 0 \end{cases} \quad \text{in } S}$$

Afterwards, v, w, $\theta = \frac{\partial w}{\partial n}$ are arbitrary, and we deduce (from $\int_{\partial\mathcal{D}} = 0$) that

$$\boxed{\left.\begin{aligned} T_\alpha &= N_{\alpha\beta} \nu_\beta \\ \Gamma &= M_n \\ T &= \frac{dM_t}{ds} + Q \end{aligned}\right\} \quad \text{on } \partial\mathcal{D}}$$

Then, on the boundary ∂S of the plate, we have the boundary conditions

$$\begin{cases} T_\alpha = \overline{T}_\alpha \\ \Gamma = \overline{\Gamma} \\ T = \overline{T} \end{cases} \qquad \text{on } \partial S$$

$$[\![M_t]\!]_i = \overline{R}_i$$

We summarize :

① $N_{\alpha\beta}$ is the stress tensor of tensions (or membrane stresses) in the plane of the plate

$$\begin{cases} T_\alpha = N_{\alpha\beta}\nu_\beta & \text{is the tension (in the plane of the plate) on } \partial S \\ \overline{T}_\alpha & \text{are the exterior forces at the boundary, in the plane of the plate} \end{cases}$$

p_α are exterior body forces, in the plane of the plate.

Equations $\quad N_{\alpha\beta,\beta} + p_\alpha = 0$

Boundary cond. $\quad T_\alpha = \overline{T}_\alpha$

($\overline{T}_\alpha$ given or not according as the situations)

| ~~pb. of plane beam~~ pb. of continua, in the plane. -

② $M_{\alpha\beta}$ is the bending stress tensor

$$\begin{cases} T = \dfrac{dM_\tau}{ds} + Q & \text{is the vertical force on } \partial\mathcal{D} \\ \overline{T} & \text{is the vertical (exterior) force at the boundary } \partial S \end{cases}$$

$$\begin{cases} M_n & \text{is the bending moment on } \partial\mathcal{D} \\ \overline{\Gamma} & \text{is the bending moment at the boundary } \partial S \end{cases}$$

ω is the vertical exterior (body) forces.

Equations $\quad M_{\alpha\beta,\alpha\beta} + \omega = 0$

Boundary conditions. $\quad T = \overline{T}$

$\quad M_n = \overline{\Gamma}$

Examples

1°/ The plate is clamped on the part $\Gamma_1 \subset \partial S$: 
on this part, vertical displacements and rotation $\frac{\partial y}{\partial n}$
are zero. $\overline{T}$ (vertical exterior forces) and $\overline{\Gamma}$ (exterior bending
moment) are the reactions of supports on the plate;
they are unknown.

2°/ The plate is simply supported on the part $\Gamma_2 \subset \partial S$:
on this part, vertical displacement y is zero 
and the vertical exterior forces $\overline{T}$ (reaction
of the support) is unknown. But the
rotation is free and the corresponding force $\overline{M}$,
bending moment applied by the support, is zero.

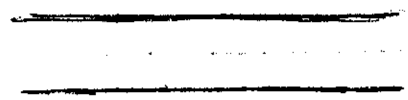
3°/ The plate is free on the part $\Gamma_3 \subset \partial S$: on this
part, the vertical displacement y and the rotation $\frac{\partial y}{\partial n}$
are free; exterior forces on this part, $\overline{T}$ and $\overline{M}$ are
zero.

Other more complicated boundary conditions will be
described in the following.

General equations for plates (continued).

Perhaps, the virtual power principle used to write general equations for plates was a little bit difficult. I am going very briefly to indicate an other method to obtain these equations.

We consider the three-dimensional domain $\bar{S} = S \times \left]-\frac{h}{2}, \frac{h}{2}\right[$.



Let be $\sigma_{ij} = \sigma_{ij}(x, x_3)$ the classical stress tensor, defined in $\bar{S}$; equilibrium relations in $\bar{S}$ are

(1) $$\boxed{\sigma_{ij,j} + \bar{f}_i = 0}$$

where $\bar{f}_i(x, x_3)$ is the volume body forces. In addition, there are no forces on the upper and lower sides of $\bar{S}$, therefore we have

$$\sigma_{ij} n_j = 0 \quad \text{on the faces } x_3 = \pm \frac{h}{2}.$$

but $n_\alpha = 0$ and $n_3 = 1$, and we obtain

(2) $$\boxed{\sigma_{i3}\left(x, \pm \frac{h}{2}\right) = 0}$$

Now, we introduce the following notations:

$$N_{ij}(x) = \int_{-\frac{h}{2}}^{h/2} \sigma_{ij}(x, x_3)\, dx_3,$$

$$M_{ij}(x) = \int_{-\frac{h}{2}}^{h/2} x_3\, \sigma_{ij}(x, x_3)\, dx_3,$$

$$f_i(x) = \int_{-\frac{h}{2}}^{h/2} \bar{f}_i(x, x_3)\, dx_3,$$

$$m_i(x) = \int^{\frac{h}{2}} x_3\, \bar{f}_i(x, x_3)\, dx_3.$$

First, we integrate (1) on the thickness, to obtain

(3) $$\boxed{N_{\alpha\beta,\beta} + f_\alpha = 0}$$

(4) $$N_{3\beta,\beta} + f_3 = 0$$

(because $\int_{-\frac{h}{2}}^{h/2} \sigma_{i3,3}(x, x_3)\, dx_3 = 0$, via (2)).

After this, we multiply (1) by x_3 and we integrate on the thickness, to obtain :

(5) $$M_{\alpha\beta,\beta} - N_{\alpha 3} + m_\alpha = 0$$

(because $\int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\, \sigma_{\alpha 3,3}\, dx_3 = - N_{\alpha 3}$)

With (4), we can obtain an equation for only $M_{\alpha\beta}$:

(6) $$\boxed{M_{\alpha\beta,\alpha\beta} + \omega = 0}$$

where $\omega = f_3 + m_{\alpha,\alpha}$.

Equations (3) and (6) are exactly the same obtained via virtual power principle.

These theory is more easy than the virtual power principle; but this principle will furnish Green's formula and variational formulation.

II-1 Classical (linear) theory of (elastic) plates

We assume that the field of displacements has the form

$$\begin{cases} u = (u_\alpha) = v(x) \\ y = Y(x) \end{cases}$$

u is the horizontal displacement,
y is the vertical displacement.

Deformations are caracterized by the two tensors fields

$$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}(u) = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right),$$

$$K_{\alpha\beta} = - Y_{,\alpha\beta}.$$

The structure of these deformation tensors is the same as for rate deformation tensors previously considered. Here,

$\varepsilon_{\alpha\beta}$ caracterizes the deformation in the plane of the plate;
$K_{\alpha\beta}$ ———— the curvature deformations.

In a linear theory, the general constitutive relations for elastic plates are

$$\begin{cases} N_{\alpha\beta} = a_{\alpha\beta\gamma\delta}\,\varepsilon_{\gamma\delta} + c_{\alpha\beta\gamma\delta}\,K_{\gamma\delta}, \\ M_{\alpha\beta} = c_{\alpha\beta\gamma\delta}\,\varepsilon_{\gamma\delta} + b_{\alpha\beta\gamma\delta}\,K_{\gamma\delta} \end{cases} \quad (1).$$

But, a very simple case (and very important in applications) is that where horizontal stresses and bending stresses are uncoupled (then $c_{\alpha\beta\gamma\delta} = 0$) and where the continuum is isotropic; in this case, we have

---

(1) The same $c_{\alpha\beta\gamma\delta}$, because these relations can be derived from an energy $W(\varepsilon, K)$ which is a quadratic form in ε and K:

$$\begin{cases} N_{\alpha\beta} = \lambda_1 \varepsilon_{\gamma\gamma} \delta_{\alpha\beta} + 2\mu_1 \varepsilon_{\alpha\beta} \\ M_{\alpha\beta} = \lambda_2 K_{\gamma\gamma} \delta_{\alpha\beta} + 2\mu_2 K_{\alpha\beta} \end{cases}$$

For the plane problem of tensions, equilibrium relations are

$$\begin{cases} N_{\alpha\beta,\beta} + p_\alpha = 0 \\ N_{\alpha\beta} = a_{\alpha\beta\gamma\delta}\, \varepsilon_{\gamma\delta}(u) \end{cases}$$

which is a classical linear problem of elasticity in the plane of the plate (compare with :

$\sigma_{ij,j} + f_i = 0 , \qquad \sigma_{ij} = a_{ijkh}\, \varepsilon_{kh}$)

For the bending problem, equilibrium relations are

$M_{\alpha\beta,\alpha\beta} + \omega = 0$; but $M_{\alpha\beta,\alpha\beta} = -(\lambda_2 + 2\mu_2)\, \Delta^2 \zeta$.

In addition, by comparison with the three dimensional problem, we ~~must~~ ~~choose~~ take (if the thickness h is constant) :

$$\lambda_2 = D\nu \qquad 2\mu_2 = D(1-\nu)$$

With $D = \dfrac{E h^3}{12(1-\nu^2)}$ ~~is~~ the flexural rigidity constant.

(E : Young's modulus ; ν is the Poisson's ratio, for the material constituting the plate). Then the equilibrium relation is reduced to

$$\boxed{D\, \Delta^2 \zeta = \omega}$$

Using the constitutive relations, we can write the vertical force and the bending moment at the boundary as functions of ζ;

We change our notations, and we write on the boundary $\partial\Omega$

$$M = M_n \quad \left(= M_{\alpha\beta}\, n_\alpha n_\beta \right)$$

$$F = \frac{dM_\tau}{ds} + Q \quad \left(= \frac{d}{ds}\left(M_{\alpha\beta}\, n_\alpha \tau_\beta\right) + M_{\alpha\beta,\beta}\, n_\beta \right)$$

(n is the outward unitary normal at the boundary)

We obtain :

$$\boxed{\begin{aligned} M &= -D\left\{ \nu \Delta\zeta + (1-\nu)\, \zeta_{,\alpha\beta}\, n_\alpha n_\beta \right\} \\ F &= -D\left\{ (1-\nu) \frac{\partial}{\partial s}\left(\zeta_{,\alpha\beta}\, \tau_\alpha n_\beta\right) + \frac{\partial}{\partial n} \Delta\zeta \right\} \end{aligned}}$$

II - 2 Non linear von Karman's theory for elastic plates.

When the bending is not weak, the linear theory is not valid; the non linear von Karman's theory is often used to study the strong bending (but not very strong) of plates, and so, the buckling phenomena.

We will not establish the von Karman's equations; it is some difficulties; it is necessary to use two configurations: the natural one, and a configuration after deformation; in these conditions, we must use a Piola Kirchhoff stress tensor. These equations are established in many books; for instance, to see the book of Landau & Lipschitz, Elasticity (VII).

The general idea is as following:

$$\text{Displacements} \xrightarrow[\text{non linear}]{} \text{Deformations} \xrightarrow[\text{linear (Hooke's Law)}]{} \text{Stresses}$$

(therefore, it is used geometrical non linearities)

Recall that $u = (u_\alpha)$ is the horizontal displacement and ζ the vertical one.

Deformations:
$$\begin{cases} \tau_{\alpha\beta} = \varepsilon_{\alpha\beta}(u) + \frac{1}{2}\zeta_{,\alpha}\zeta_{,\beta} \\ \varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \end{cases}$$

Equations
$$\left.\begin{cases} D\Delta^2\zeta - h(\sigma_{\alpha\beta}\zeta_{,\beta})_{,\alpha} = f \\ \sigma_{\alpha\beta,\beta} + p_\alpha = 0 \end{cases}\right\}$$
Here, bending and tension effects are coupled.

Constitutive relations $\sigma_{\alpha\beta} = a_{\alpha\beta\gamma\delta}\,\tau_{\gamma\delta}$.

(Here, we have written $\sigma_{\alpha\beta}$ in place of $N_{\alpha\beta}$)

III - Green's formulas for plates.

For instance, consider the partial differential equation

$$-\Delta u = f \,, \quad \text{in } \Omega$$

We obtain a Green's formula by multiplying this equation with a test function v and by integrating by parts - For instance, we obtain:

$$\int_{\Omega} \operatorname{grad} u \operatorname{grad} v \, dx = \int_{\Omega} f v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, d\sigma$$

It is possible to choose the same way to obtain Green's formulas for elastic plates - Meanwhile, we are going to deduce these formulas from the virtual work principle (or virtual power principle): it is an important idea to consider Green's formulas as the same ~~that~~ Virtual Work principle -

Because we are concerned by static problems, the test functions are now called "virtual displacements" in place of "virtual velocities"; "work" replaces "power".

y is the real vertical displacement;

z is a virtual " " " .

Linear case. Because tension effects and bending ones are uncoupled in the classical theory of plates, we consider only the bending problem, since previously, it has been observed that the tension problem is an elasticity problem in the plane.

Virtual work principle can be written ~~the~~

$$(1)\quad \underbrace{\int_\Omega M_{\alpha\beta}\, z_{,\alpha\beta}\, dx}_{\mathcal{C}_{(i)}} + \underbrace{\int_\Omega f z\, dx}_{\mathcal{C}_{(d)}} + \underbrace{\int_{\partial\Omega}\left(-M\frac{\partial z}{\partial n} + F z\right) d\sigma}_{\mathcal{C}_{(c)}} = 0 \qquad \forall z$$

$\mathcal{C}_{(i)}$: work of internal forces $\mathcal{C}_{(d)}$: work of body forces $\mathcal{C}_{(c)}$: work of boundary forces

Recall that :
- f are the vertical body (surfacic) forces inside the plate ;
- M is the bending moment at the boundary ;
- F is the vertical forces ______ ;

Using constitutive relations, we obtain directly the Green's formula from (1).

We have

$$M_{\alpha\beta} = -D\left\{\nu\, y_{,\gamma\gamma}\,\delta_{\alpha\beta} + (1-\nu)\, y_{,\alpha\beta}\right\}$$

In the following, we put

$$a(y,z) = D\int_\Omega \left\{\nu\, y_{,\gamma\gamma} + (1-\nu)\, y_{,\alpha\beta}\right\} z_{,\alpha\beta}\, dx ,$$

which is a bilinear form ~~into~~ with respect of y and z.

The Green's formula is

$$\boxed{a(y,z) = (f,z) + (F,z)_\Gamma - \left(M, \frac{\partial z}{\partial n}\right)_\Gamma \qquad \forall z ,}$$

where $(\ ,\ )$ and $(\ ,\ )_\Gamma$ are classical scalar product on Ω or $\Gamma = \partial\Omega$

<u>Remark</u>. We have also

$$M_{\alpha\beta} z_{,\alpha\beta} = -D\left\{\nu\,(y_{,11}+y_{,22})(z_{,11}+z_{,22}) + (1-\nu)\, y_{,\alpha\beta}\, y_{,\alpha\beta}\right\}$$

$$= -D\left\{(y_{,11}\, z_{,11} + y_{,22}\, z_{,22}) + \nu\,(y_{,11}\, z_{,22} + y_{,22}\, z_{,11}) + 2(1-\nu)\, y_{,12}\, z_{,12}\right\}$$

which is more classical in the litterature.

Non linear case. In this case, tension effects and bending ones are coupled. In addition to the previous case, we introduce the notations:

$u = (u_\alpha)$ is the real/horizontal displacement;

$v = (v_\alpha)$ is a virtual " " ".

Green's formula can be obtained by classical method, as recalled in introduction of this section: we multiply the first von Karman's equation by z and we integrate on Ω: the term $\int_\Omega D\Delta^2 y\, z\, dx$ is the same that one obtained in the linear case; after two integrations by parts, we have

$$\int_\Omega D\Delta^2 y\, z\, dx = a(y,z) - \int_\Gamma \left(M \frac{\partial z}{\partial n} - F z\right) d\sigma$$

(these integrations by parts have been done in the computation of $P(i)$; it is sufficient now to use the constitutive relation).

For the second term, we have

$$-h\int_\Omega (\sigma_{\alpha\beta}\, y_{,\beta})_{,\alpha}\, z\, dx = h\int_\Omega \sigma_{\alpha\beta}\, y_{,\beta}\, z_{,\alpha}\, dx - h\int_{\partial\Omega} \sigma_{\alpha\beta}\, y_{,\beta}\, n_\alpha\, z\, d\sigma$$

and the Green's formula can be written:

$$\boxed{a(y,z) + h\int_\Omega \sigma_{\alpha\beta}\, y_{,\beta}\, z_{,\alpha}\, dx = (f,z) - \left(M, \frac{\partial z}{\partial n}\right)_\Gamma + (F,z)_\Gamma + \underbrace{h\int_\Gamma \sigma_{\alpha\beta}\, y_{,\beta}\, n_\alpha\, z\, d\sigma}_{\lambda h \int_\Gamma t_\alpha n_\alpha z\, d\sigma} \qquad \forall z}$$

Obviously, we have, from the second von Karman's equation (or for the tension problem)

$$\boxed{\int_\Omega \sigma_{\alpha\beta}\, \varepsilon_{\alpha\beta}(v)\, dx = \int_{\Sigma_1} \sigma_{\alpha\beta}\, v_\alpha\, n_\beta\, d\sigma \qquad \forall v}$$

$$\left(+\int_\Omega v_\alpha\, p_\alpha\, dx\right)$$