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THEORY OF PLATES

Part IV : An elementary example of buckling for beams

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An elementary example of buckling for beams -

As introduction, we are going to study ~~the~~ example of buckling for beams : we will assume that the beam is simply supported at ends .

$M(0) = M(L) = 0$

(+ vertical displacement $= 0$)

(First, we must complete the (general) equations written before by constitutive relations ; we will consider only the elastic situation.)

One know that linear theories are often adequate in engineering sciences ; from the simplicity of these theories, it results they are very used.

But it happens sometimes that linear theories are not adequate ; it is the case in the study of buckling phenomena. We illustrate this remark, beginning to study ~~the case of~~ " an elementary case of buckling for beams -

We consider a beam ($0 < s < L$) with prescribed end thrust

$(-P\vec{e_1} \,,\, P\vec{e_1})$ $\qquad P\vec{e_1}$ ———————— $-P\vec{e_1}$

$(\longrightarrow \vec{e_1})$

Experimentally, one knows that, when P increases from zero, the beam deforms by shortening : its center line remains straight. But, beyond a critical value $P_{critical} = P_{cr}$, the ~~beam~~ beam bends.

We recall the equations are

$$\begin{cases} \dfrac{d\vec{\mathcal{T}}}{ds} = 0 \\ \dfrac{d\vec{\mathcal{M}}}{ds} + \vec{\tau} \wedge \vec{\mathcal{T}} = 0 \end{cases}$$

Boundary conditions imply $\vec{\mathcal{T}} = \text{cste} = -P\vec{e}_1$

Let be θ the polar angle of unitary tangent vector $\vec{\tau}$;

then $\vec{\tau} \wedge \vec{T} = P \sin\theta \, \vec{e}_3$.

Second equation gives

$$\vec{\mathcal{M}} = M\vec{e}_3$$

$$\boxed{\frac{dM}{ds} + P\sin\theta = 0}$$

Only the stress M (bending moment) is unknow. It is impossible to determinate M by the above equilibrium relation, ~~and~~ we must write a constitutive equation.

For this, first observe that the deformation is caracterized by $\dfrac{d\theta}{ds}$. We take the constitutive equation

$$\boxed{M = EI\frac{d\theta}{ds}}$$

Where E is a Mechanics coefficient (Young modulus), I is a ~~geometrical co~~ moment of inertia.

The equilibrium of the beam, ~~with~~ simply supported at ends is described by the problem

$$\begin{cases} EI\,\theta'' + P\sin\theta = 0 & (0 < s < L) \\ \theta'(0) = 0 & \\ \theta'(L) = 0 & \text{(Result from } M(0) = M(L) = 0 \text{)} \end{cases}$$

... that $\theta = 0$ is a solution for all P.

This problem is non linear.

We can also consider the linearized problem

$$\begin{cases} EI\theta'' + P\theta = 0 & (0 < s < L) \\ \theta'(0) = 0 \\ \theta'(L) = 0 \end{cases}$$

Observe that $\theta = 0$ is solution for all P. But, can exist other solutions for suitable P?

In addition, write the energy functional associated with each problem:

$$W = W(\theta, P) = \int_0^L \left\{ \frac{1}{2} EI\theta'^2 - P(1-\cos\theta) \right\} ds$$

$$W_L = W_L(\theta, P) = \int_0^L \frac{1}{2} \left\{ EI\theta'^2 - P\theta^2 \right\} ds.$$

First. We study the linearized problem.

$\theta = 0$ is solution for all values of P. The question is to know if other solutions can exist.

One put $\lambda = \frac{P}{EI} = \omega^2$ (because $\theta = 0$ is the unique solution if $\lambda < 0$), and we have

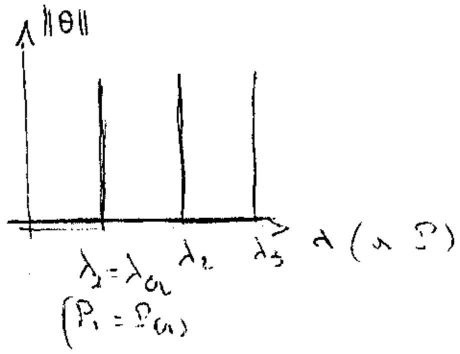
$$\begin{cases} \theta'' + \omega^2 \theta = 0 \\ \theta'(0) = 0 \ , \ \theta'(L) = 0 \end{cases}$$

The general solution of this differential equation is $\theta = A \cos \omega s + B \sin \omega s$. We have $\theta' = \omega(-A \sin \omega s + B \cos \omega s)$ and the first boundary condition furnishes $B = 0$. The second one gives $A \sin \omega L = 0$ or (to obtain non trivial solutions) $\omega = \frac{n\pi}{L}$ $n \in \mathbb{Z}$, $n \neq 0$, and

$$\boxed{\lambda = \lambda_n = \frac{n^2 \pi^2}{L^2} \quad n \neq 0, \ n \in \mathbb{Z}}$$

Conclusions. (i) $\theta = 0$ is the unique solution, ~~except~~ if $\lambda \neq \lambda_n$

(ii) if $\lambda = \lambda_n$, there exist non trivial solutions, which are indeterminate



Physically, the beam remains straight except for exceptional thrusts $P_n = EI \frac{n^2\pi^2}{L^2}$. This result is not acceptable: when P increases, we have, only for P_{cr}, solutions $\neq 0$, which are indeterminate. Beyond P_1, $\theta = 0$ is again the unique solution, until P_2 !

Third conclusion. The linearized problem (hence a linear problem) cannot describe buckling phenomena.

To study buckling phenomena, it is absolutely necessary

Now, we study the non linear problem

$$(1)\quad \begin{cases} \theta'' + \lambda \sin\theta = 0 & (0 < s < L) \\ \theta'(0) = 0 \\ \theta'(L) = 0 \end{cases}$$

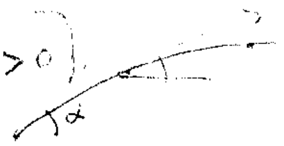
Observe that $\theta = 0$ is solution for all λ in $\mathbb{R}$ (i.e. for each thrust P)

We are going to search a non trivial solution $\theta(s)$ with following properties:

(2) θ is decreasing $\quad (\theta' \leq 0)$

(3) $\begin{cases} \text{symetry with regard to } s = \frac{L}{2} : \\ \theta\left(\frac{L}{2} + \sigma\right) = -\,\theta\left(\frac{L}{2} - \sigma\right) \end{cases}$

Therefore, $\boxed{\theta\left(\frac{L}{2}\right) = 0}$ and $\theta(0) = \alpha > 0$



we obtain, by multiplying (1) by θ':

$$\frac{d\theta}{ds} = -\sqrt{\lambda}\,\sqrt{2(\cos\theta - \cos\alpha)}$$

where α is a suitable integration constant: $\alpha = \theta(0)$ (which is supposed > 0)

We obtain

$$(4)\quad s = -\frac{1}{\sqrt{\lambda}} \int_0^{\theta} \frac{d\varphi}{\sqrt{2(\cos\varphi - \cos\alpha)}} + \frac{L}{2}$$

But $s = 0$ for $\theta = \alpha$, hence

$$\boxed{0 = -\frac{1}{\sqrt{\lambda}} \int_0^{\alpha} \frac{d\varphi}{\sqrt{2(\cos\varphi - \cos\alpha)}} + \frac{L}{2}}$$

which is an equation for α. From a solution α of this equation, we obtain, via (4), a non trivial solution of the problem (1)

More precisely, introduce a new unknown ψ by

$$\begin{cases} \sin\frac{\alpha}{2} = k \\ \sin\frac{\theta}{2} = k\sin\psi \end{cases}$$

Then :

(i) $\frac{1}{2}\cos\frac{\theta}{2}\,\theta' = k\cos\psi\,\psi'$

and $\boxed{\theta'^2 = \dfrac{4k^2\cos^2\psi}{1-k^2\sin^2\psi}\,\psi'^2}$

(ii) $\cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1 - 2k^2\sin^2\psi$

$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} = 1 - 2k^2$

$\Rightarrow 2\lambda(\cos\theta - \cos\alpha) = 4\lambda k^2(1-\sin^2\psi) = 4\lambda k^2\cos^2\psi$

The equation $\theta'^2 = 2\lambda(\cos\theta - \cos\alpha)$ gives

$$\boxed{\psi'^2 = \lambda(1 - k^2\sin^2\psi)}$$

If we assume $0<\alpha\leq\pi$, we have $0\leq\frac{\theta}{2}\leq\frac{\alpha}{2}\leq\frac{\pi}{2}$ on $0<s<\frac{L}{2}$; and ψ (as θ) is decreasing, hence $\psi'<0$

and

$$\psi' = -\sqrt{\lambda}\,\sqrt{1-k^2\sin^2\psi}$$

$$ds = -\frac{1}{\sqrt{\lambda}}\,\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

Boundary conditions are

$$\theta = 0 \Rightarrow \begin{cases} s = \frac{L}{2} \\ \psi = 0 \end{cases} \qquad \theta = \alpha \Rightarrow \begin{cases} s = 0 \\ \psi = \frac{\pi}{2} \end{cases}$$

We obtain

$$s = -\frac{1}{\sqrt{\lambda}}\int_0^{\psi}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} + \frac{L}{2}$$

and, for $\psi = \frac{\pi}{2}$:

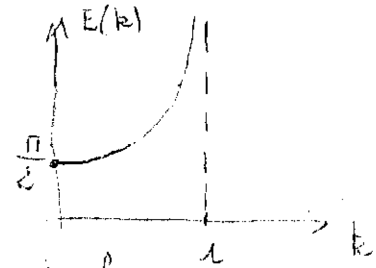
$$0 = -\frac{1}{\sqrt{\lambda}} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} + \frac{L}{2}$$

or, it is the same :

$$(5) \qquad \boxed{\frac{L\sqrt{\lambda}}{2} = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = E(k)}$$

Which is an equation for k, hence for α.

It is very simple to observe that the graph of $E(k)$ as the following form $\longrightarrow$



Therefore, if $\frac{L\sqrt{\lambda}}{2} > \frac{\pi}{2}$, there exists a solution k for this relation

Consequence

If $\lambda > \frac{\pi^2}{L^2} = \lambda_{cr}$ $\left(\text{or if } P > \frac{EJ\pi^2}{L^2} = P_{cr}\right)$,

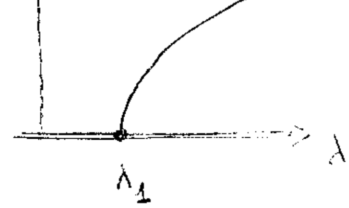
there exists a non trivial solution of the (non linear) problem,

given by

$$s = \frac{L}{2} - \frac{1}{\sqrt{\lambda}} \int_0^{\psi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

With k is given by (5).

Such a solution can be schematized on the picture :



In addition, observe that the critical thrust P_{cr} is the first "eigenvalue" of the linearized

Physically, the exhibited solution of non linear problem is acceptable, contrary to the (the solution of) linearized problem.

Now, we have (for $\lambda > \lambda_{cr}$) 2 solutions: the trivial one and the non trivial one. We are going to compare the value of energy for these solutions:

We have $W(0,\lambda) = 0$.

$W(\theta,\lambda)$ is proportional to

$$\widehat{W}(\theta,\lambda) = \frac{1}{2}\int_0^L \{\theta'^2 + 2\lambda(\cos\theta - 1)\}\, ds = \int_0^{\frac{L}{2}} \{\theta'^2 + 2\lambda(\cos\theta - 1)\}\, ds$$

but, for the solution, we have

$$\theta'^2 + 2\lambda(\cos\theta - 1) = 4\lambda k^2 \cos^2\psi$$

with previous notation; and

$$ds = -\frac{1}{\sqrt{\lambda}} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

We obtain

$$W(\theta,\lambda) = 4\sqrt{\lambda}\, k^2 \int_0^{\pi/2} \frac{\cos 2\psi \, d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

It is not difficult to observe that

$$\boxed{W(\theta,\lambda) < 0 = W(0,\lambda)}$$

This inequality shows that the non trivial solution is more stable that the trivial one, from an energy point of view.