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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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TOPICS IN LINEAR ELASTICITY

Chapter I - Preliminaries

Chapter II - Kinematics

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INTRODUCTION

These lectures aim at convincing the reader, through the choice of the topics, that firstly the classical infinitesimal theory of elasticity is still an interesting subject of study for the applied mathematician, and, secondly, that it is too soon to declare its death by consumption. On the contrary, this theory has been recently enriched of new elegant results, and offers still a long list of important open problems.

The fact that these lectures are largely based on the article on linear elasticity masterly written by M.E. GURTIN (*) will be immediately clear to the readers of that tract, so that it will be not necessary to quote it again and again later.

(*) M.E. GURTIN, The Linear Theory of Elasticity, in Handbuch der Physik, Vol. VIa/2, S. Flügge Ed. Springer-Verlag: Berlin-Heidelberg-New York 1972.

I - PRELIMINARIES

Both to establish our notation and to give an idea of the minimal background we assume in the audience, we collect hereinafter some miscellaneous results from multidimensional algebra and analysis.

ⓐ - $\mathcal{E}$ 3-dimensional Euclidean point space, $x, y, z, \dots \in \mathcal{E}$

$\mathcal{V}$ vector space associated with $\mathcal{E}$, $u, v, w, \dots \in \mathcal{V}$

- $u \cdot v$ denotes inner product of vectors $u, v \in \mathcal{V}$
- $\{0; e_1, e_2, e_3\}$ cartesian coordinate frame, i.e., a distinguished element of $\mathcal{E}$, the origin 0, plus an orthonormal basis for $\mathcal{V}$, $\{e_1, e_2, e_3\}$ fixed once and for all. Then

  $p = x - 0$, position-vector field on $\mathcal{E}$;

  $x_i = (x - 0) \cdot e_i$, i-th (cartesian) coordinate of point x

  $u_i = u \cdot e_i$, i-th (cartesian) component of vector u

$\mathrm{Lin}(\mathcal{V}, \mathcal{V})$ space of linear transformations from $\mathcal{V}$ into $\mathcal{V}$, $A, B, C, \dots \in \mathrm{Lin}(\mathcal{V}, \mathcal{V})$.

- Currently denoted by Lin. Thus, if $A \in \mathrm{Lin}$, A maps a vector u into a vector $v = Au$. Lin is identified with the space of 2nd-order tensors.
- $A_{ij} = e_i \cdot A e_j$, ij-th component of tensor A
- 1, the identity of Lin, s.that $1v = v$, $\forall v \in \mathcal{V}$. 1 is the element of Lin neutral with respect to the product by composition: if $A, B \in \mathrm{Lin}$, AB is the element
- of Lin s.that $(AB)v = A(Bv)$, $\forall v \in \mathcal{V}$; $(AB)_{ij} = A_{ik}B_{kj}$.
- $\underline{0}$, the null tensor, s.that $\underline{0}v = 0$, $\forall v \in \mathcal{V}$. $\underline{0}$ is the element of Lin neutral w.r.t. summation: $A + \underline{0} = A$, $\forall A \in \mathrm{Lin}$.
- A^T, the transpose of A, is the unique member of Lin s.th. $u \cdot Av = v \cdot A^T u$, $\forall u, v \in \mathcal{V}$. A is symmetric if $A = A^T$; skew-(symmetric) if $A = -A^T$.

Sym (Skw) subspaces of Lin of symmetric (skew) tensors

- Skw is isomorphic to $\mathcal{V}$: given any $W \in \mathrm{Skw}$, $\exists!\ w \in \mathcal{V}$ s.that where e_{ijk} denotes the Ricci alternator).

- $\mathrm{Lin} = \mathrm{Sym} \oplus \mathrm{Skw}$.

det : Lin $\to \mathbb{R}$, the determinant fnal, $\det A = Ae_i \cdot Ae_j \times Ae_k$

- $\det A = \det A^T$; $\det(AB) = \det A \cdot \det B$
- let $A \in \mathrm{Lin}$ be s.th. $\det A \neq 0$. Then $\exists!\ A^{-1} \in \mathrm{Lin}$, s.th. $A^{-1}A = AA^{-1} = 1$, $A^{-1}_{ij} = \dfrac{\mathrm{cof}\, A_{ji}}{\det A}$

Dya subset of Lin of dyads, i.e., if a, b are any elements of $\mathcal{V}$, members $A = a \otimes b$ of Lin s.th. $(a \otimes b)u = (b \cdot u)a$, $\forall u \in \mathcal{V}$. Clearly, $a \otimes b : \mathcal{V} \to \mathrm{span}\ a$ (thus, if $A \in \mathrm{Dya}$, $\det A = 0$). Dya is closed under scalar multiplication and under product by composition: $(a \otimes b)(c \otimes d) = (b \cdot c)\, a \otimes d$. Dya is not a subspace of Lin.

tr : Lin $\to \mathbb{R}$, the trace fnal, $\mathrm{tr}\, A = \sum_i e_i \cdot Ae_i$

- tr is the unique fnal s.th. $\mathrm{tr}(a \otimes b) = a \cdot b$, $\forall a, b \in \mathcal{V}$
- tr is a linear fnal on Lin
- $\mathrm{tr}\, A = \mathrm{tr}\, A^T$ (thus, $\mathrm{tr}\, W = 0$, $\forall W \in \mathrm{Skw}$)
- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
- the trace fnal can be used to endow Lin with a natural inner product: $A \cdot B = \mathrm{tr}(AB^T)$. Accordingly, A and B are orthogonal if $A \cdot B = 0$. Also, $\mathrm{tr}\, A = 1 \cdot A$; $|A| = (A \cdot A)^{1/2}$ is called the magnitude of A. It is easily seen that $\mathrm{Sym} = (\mathrm{Skw})^{\perp}$. Moreover, $\{e_i \otimes e_j$, for $i,j = 1,2,3\}$ is a natural orthonormal basis for Lin, and, for any $A \in \mathrm{Lin}$, $A = \sum_{i,j} A_{ij}\, e_i \otimes e_j$, with components A_{ij} as before.

Sph subspace of Lin of spherical tensors, i.e., if α is any real number, elements $A = \alpha 1$ of Lin. Clearly, Sph is isomorphic to $\mathbb{R}$ and is a subspace of Sym.

Dev subspace of Sym of deviatoric tensors, i.e., elements $A \in \mathrm{Sym}$ s.th. $\mathrm{tr}\, A = 0$.

- $\mathrm{Sym} = \mathrm{Sph} \oplus \mathrm{Dev}$; $\mathrm{Sph} = (\mathrm{Dev})^{\perp}$

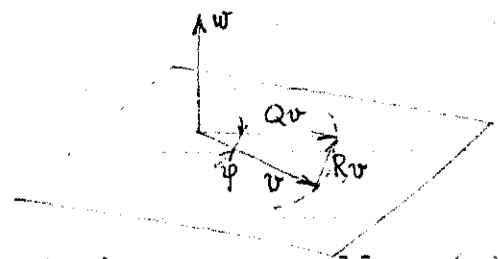
Orth , group of <u>orthogonal tensors</u> , i.e., elements $Q \in \mathrm{Lin}$ s.that $Qu \cdot Qv = u \cdot v$, $\forall\, u, v \in \mathcal{V}$.

- $Q \in \mathrm{Orth} \Leftrightarrow Q^{-1} = Q^T$
- $Q \in \mathrm{Orth} \Rightarrow |\det Q| = 1$
- Let Orth^+ be the subgroup of orthogonal tensors with positive determinant (Orth^+ is called the <u>proper</u> orth. g.). Then

  $Q = 1 + R(\varphi)$, with

  $R(\varphi) = \sin\varphi\, W - (1 - \cos\varphi)(1 - w \otimes w)$

  where φ is the angle and w, the unit vector associated with the skew tensor W, is the axis of the rotation def. by Q; clearly, $R(\varphi) = \varphi W + o(\varphi)$, as $\varphi \to 0$.



Lin (Lin) space of <u>linear transformations</u> from Lin into Lin, $\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots \in \mathrm{Lin}(\mathrm{Lin})$

- $\mathbb{C}_{ijhk} = (e_i \otimes e_j) \cdot \mathbb{C}(e_h \otimes e_k)$; using components, $S = \mathbb{C}E$ becomes $S_{ij} = \mathbb{C}_{ijhk} E_{hk}$.
- $\mathrm{sym} : \mathrm{Lin} \longrightarrow \mathrm{Sym}$, $\mathrm{sym}\, A = \frac{1}{2}(A + A^T)$
- $\mathrm{skw} : \mathrm{Lin} \longrightarrow \mathrm{Skw}$, $\mathrm{skw}\, A = \frac{1}{2}(A - A^T)$
- $\mathrm{sph} : \mathrm{Lin} \longrightarrow \mathrm{Sph}$, $\mathrm{sph}\, A = \frac{1}{3}(1 \cdot A)\, 1$
- $\mathrm{dev} : \mathrm{Sym} \longrightarrow \mathrm{Dev}$, $\mathrm{dev}\, A = A - \frac{1}{3}(1 \cdot A)\, 1$
- $\mathbb{1}$, the identity transformation of Lin (Lin)
- $1 \otimes 1$, the transformation from Lin into Sph defined by

$$(1 \otimes 1) A = (1 \cdot A)\, 1$$

ⓑ - Let $\mathcal{S}(1)$ be the unit sphere in $\mathcal{V}$: $\mathcal{S}(1) = \{ v \in \mathcal{V} \mid v \cdot v = 1 \}$. Let further $A \in \mathrm{Sym}$. We call $\lambda \in \mathbb{R}$ a <u>proper</u> (or <u>principal</u>, or <u>characteristic</u>) <u>value</u> of A, and $u \in \mathcal{S}(1)$ a <u>proper vector</u> of A if the pair (λ, u) solves the equation

$$A u = \lambda u \quad ; \qquad (*)$$

we also call the space of all vectors u that satisfy $(*)$ the <u>proper</u>

(<u>Spectral Theorem</u>) $\exists$ 3 pairs (λ_i, u_i) ; $\lambda_i \neq \lambda_k \Rightarrow u_i \cdot u_k = 0$. Moreover,

(i) if $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$ $\quad A = \sum_i \lambda_i\, u_i \otimes u_i$;

(ii) if $\lambda_1 \neq \lambda_2 = \lambda_3$ $\quad A = \lambda_2 \mathbf{1} + (\lambda_1 - \lambda_2)\, u_1 \otimes u_1$;

(iii) if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ $\quad A = \lambda \mathbf{1}$.

Let $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$ - If $A \in \mathrm{Sym}$, and $n \in \mathcal{S}(1)$, we call <u>normal component</u> of A for the direction n, and write $\gamma_N(A;n)$, the scalar

$$\gamma_N(A;n) = n \cdot An .$$

In particular, $A_{ii} = \gamma_N(A; e_i)$. It is easily shown that

$$\lambda_1 \geqslant \gamma_N(A;n) \geqslant \lambda_3 \quad , \forall\, n \in \mathcal{S}(1) ;$$

moreover, $\max \gamma_N = \lambda_1$ is attained for the proper direction u_1 (likewise, $\min \gamma_N = \lambda_3$ is attained for the proper direction u_3).

Now let $A \in \mathrm{Sym}$; $n, m \in \mathcal{S}(1)$, with $n \cdot m = 0$. We call <u>shear component</u> of A for the directions n, m, and write $\gamma_S(A;n,m)$, the scalar

$$\gamma_S(A;n,m) = n \cdot Am .$$

Clearly, $\gamma_S(A;n,m) = \gamma_S(A;m,n)$; also, $A_{ij} = \gamma_S(A; e_i, e_j)$. It is not difficult to show that

$$\tfrac{1}{2}(\lambda_1 - \lambda_3) \geqslant \gamma_S(A;n,m) \geqslant \tfrac{1}{2}(\lambda_3 - \lambda_1) \quad , \forall\, n, m \in \mathcal{S}(1) \text{ s.th. } n \cdot m = 0 ;$$

moreover, $\max \gamma_S = \frac{1}{2}(\lambda_1 - \lambda_3)$ for the directions $u = \frac{1}{\sqrt{2}}(u_1 + u_3)$, $v = \frac{1}{\sqrt{2}}(u_1 - u_3)$.

) - Let $\mathcal{D}$ be an open set in $\mathcal{E}$. We consider <u>scalar, vector or tensor fields on $\mathcal{D}$</u>, i.e., functions Ψ which assign to each point $x \in \mathcal{D}$, respectively, a scalar, vector or tensor $\Psi(x)$. We say that $\Psi : \mathcal{D} \longrightarrow \mathcal{W}$ (where $\mathcal{W}$ may be $\mathbb{R}$, $\mathcal{V}$, Lin, ...) is <u>differentiable</u> at a point $x \in \mathcal{D}$ if there exists a <u>linear</u> transformation $\nabla\Psi(x)[\cdot]$ (the <u>gradient</u> of Ψ at x) from $\mathcal{V}$ into $\mathcal{W}$, s. that

$$\Psi(y) - \Psi(x) = \nabla\Psi(x)[y-x] + o(|y-x|) \qquad \text{as } y \to x .$$

If $\nabla\Psi(x)[\cdot]$ exists, it is unique and can be computed by the formula

$$\nabla\Psi(x)[v] = \frac{d}{d\alpha}\Psi(x + \alpha v)\Big|_{\alpha=0} \quad , \quad \forall\, v \in \mathcal{V} .$$

$$\varphi(y) - \varphi(x) = g \cdot (y - x) + o(|y - x|) \qquad \text{as} \quad y \to x,$$

and we have $(\nabla\varphi(x))_i = \nabla\varphi(x) \cdot e_i = \dfrac{\partial \varphi}{\partial x_i} = \varphi_{,i}$.

Also, let $\Psi \equiv u$, a vector field . Then , $\nabla u(x)$ is the 2nd-order <u>tensor</u> G such that

$$u(y) - u(x) = G\,(y - x) + o(|y - x|) \qquad \text{as} \quad y \to x,$$

and we have $(\nabla u(x))_{ij} = e_i \cdot \nabla u(x)\, e_j = \dfrac{\partial u_i}{\partial x_j} = u_{i,j}$.

Beside the gradient, we will occasionally consider other well-known differential operators, e.g., the <u>divergence</u> of a vector field u

$$\operatorname{div} u = 1 \cdot \nabla u = u_{i,i} \quad ,$$

and of a tensor field S

$$(\operatorname{div} S) \cdot a = \operatorname{div}(S^T a) \quad , \forall \text{ fixed } a \in \mathcal{V} \quad ((\operatorname{div} S)_i = S_{ij,j}) ;$$

the <u>curl</u> of a vector field u

$$(\operatorname{curl} u) \times a = 2\,(\operatorname{skw} \nabla u)\, a \quad , \forall \text{ fixed } a \in \mathcal{V} \quad ((\operatorname{curl} u)_i = e_{ijk}\, u_{k,j}) ;$$

the <u>laplacian</u> of a scalar field φ

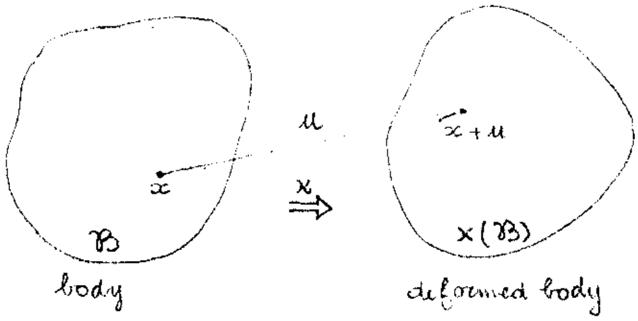
$$\Delta\varphi = \operatorname{div} \nabla\varphi = \varphi_{,ii} \quad ,$$

and of a vector field u

$$\Delta u = \operatorname{div} \nabla u \qquad ((\Delta u)_i = u_{i,jj}) .$$

II _ KINEMATICS

1 — We consider a *body* $\mathcal{B}$, i.e., a compact regular region of $\mathcal{E}$. A *deformation* of $\mathcal{B}$ is a smooth homeomorphism x of $\mathcal{B}$ onto $x(\mathcal{B}) \subset \mathcal{E}$, s.th.



$$\det \nabla x > 0 \quad \text{a.e. in } \mathcal{B}. \tag{1}$$

We call

$$u(x) = x(x) - x \tag{2}$$

the *displacement*;

$$F(x) = \nabla x(x) \tag{3}$$

the *deformation gradient*;

$$\nabla u(x) = F - 1 \tag{4}$$

the *displacement gradient* at x. In order to make precise our intuitive notion of *strained* deformed body, as different from the notion of rigidly deformed body, we introduce the *finite strain tensor* D defined by

$$D = \frac{1}{2}(F^T F - 1) = \operatorname{sym} \nabla u + \frac{1}{2} \nabla u^T \nabla u \tag{5}$$

and the *infinitesimal strain tensor* E

$$E = \operatorname{sym} \nabla u \ . \tag{6}$$

The linear theory models physical situations in which the displacement u and the displacement gradient ∇u are small. If we put $\varepsilon = |\nabla u|$, it follows easily from (5) and (6) that

$$D = E + O(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0 , \tag{7}$$

i.e., E is the linear approximation of D. With a similar procedure, we can derive each relevant formula in linear kinematics from the corresponding formula of the finite theory. As an example, consider the *volume change* δV due to x:

$$\delta V = \int_{x(\mathcal{B})} dV - \int_{\mathcal{B}} dV = \int_{\mathcal{B}} (\det \nabla x - 1)\, dV \ .$$

The volume change per unit volume is then

$$\det \nabla x - 1 = 1 \cdot E + O(\varepsilon^2) \quad \text{as } \varepsilon \to 0 , \tag{8}$$

a formula which accounts for the fact that the requirement (1) is in general dropped in the linear theory. and also shows a first example of the rôle of (b)- To find further grounds for selecting E as a measure of infinitesimal strain, consider an <u>infinitesimal rigid displacement field</u>, i.e., a displacement v of the form

$$v(x) = v_0 + W_0(x - x_0), \tag{9}$$

where x_0, v_0 and W_0 are fixed elements of $\mathcal{E}$, $\mathcal{V}$, and Skw, respectively.
Then, $\nabla v = W_0$, so that $E(v) = 0$ identically in B. To see that the converse also holds, we first prove the following

<u>Lemma 1</u>. v is an infinitesimal rigid displacement field if and only if v has the <u>projection property</u> on $\mathcal{B}$, namely,

$$(v(x) - v(y)) \cdot (x - y) = 0 \quad , \forall\, x, y \in \mathcal{B}. \tag{$*$}$$

<u>Pf.</u> $\Rightarrow$: obvious.

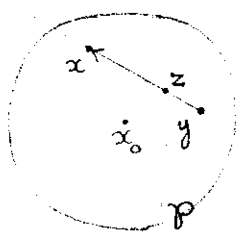
$\Leftarrow$: by differentiating $(*)$ w.r.t. x and then y yields

$$\nabla v(x)^T + \nabla v(y) = 0 \quad , \forall\, x, y \in \mathcal{B}.$$

Thus, ∇v is constant and skew, i.e., v has the form (9). ∎

<u>Theorem 1</u>[1]. Let $\mathcal{B}$ be connected. If $v \in C^1(\mathcal{B})$ & $\nabla v(x) \in$ Skw at each $x \in \mathcal{B}$, then v is an i.r.d.f.

<u>Pf.</u> In view of Lemma 1, and since $\mathcal{B}$ is open & connected, it suffices to show that, given any point $x_0 \in \mathcal{B}$, ∇v is constant in some open ball $\mathcal{P}$ centered at x_0.



Let $x, y \in \mathcal{P}$ and let s be the line segment from y to x. Then, as

$$v(x) - v(y) = \int_s \nabla v(z)\, dz$$

we have

$$(v(x) - v(y)) \cdot (x - y) = \int_s (x - y) \cdot \nabla v(z)\, dz$$

so that, since dz is parallel to $(x - y)$ and $\nabla v(z) \in$ Skw, $(*)$ follows. ∎

(c)- An i.r.d.f. is a special <u>homogeneous d.f.</u>, i.e., a d.f. of the form

$$u(x) = u_0 + A(x - x_0), \tag{10}$$

where x_0, u_0 and A are independent of x. If we put $E = \text{sym}\, A$, we deduce from (10) the form of what is called a pure strain from x_0 :

$$u(x) = E(x - x_0) \quad . \tag{11}$$

Interesting examples of pure strains are :

(i) simple extension (of amount ν in the direction n)

$u = \nu (n \otimes n)(x - x_0)$

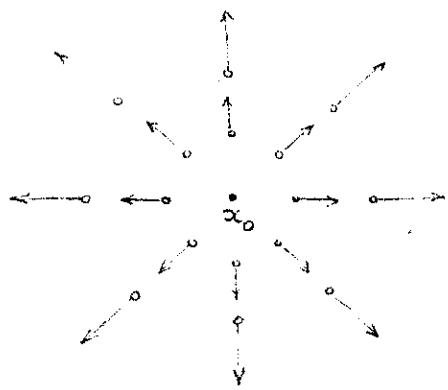
W.r.t. the o.n. basis $\{n, e_2, e_3\}$

we have

$$[E] = \begin{bmatrix} \nu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ;$$

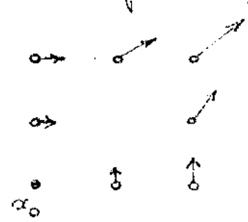
(ii) uniform dilatation (of amount δ)

$u = \delta \, 1 \, (x - x_0)$



(iii). simple shear (of amount σ w.r.t. the direction pair n, m with $n \cdot m = 0$)

$u = 2\sigma \, \mathrm{sym}(n \otimes m)(x - x_0)$



W.r.t. the o.n. basis $\{n, m, e_3\}$

we have

$$[E] = \begin{bmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If we recall the spectral thm, and the direct decomposition of Sym (Sym = Sph ⊕ Dev), we easily arrive at the following decomposition theorem for pure strains.

Theorem 2 - Let u be a pure strain from x_0. Then, u admits the fllng two additive decompositions:

(i) three simple extensions in three mutually orthogonal directions from x_0 ;

(ii) a uniform dilatation from x_0 plus an isochoric pure strain from x_0.

Remark 1 - The above decompositions also apply locally when the strain field does depend on x.

d) - The following result is also of interest, due to its purely kinematical nature.

$$\overline{E}(\mathcal{B}) = \frac{1}{\mathrm{vol}(\mathcal{B})} \int_{\mathcal{B}} E \, dV \, . \tag{12}$$

Theorem 3 - Let $u \in C^2(\mathcal{B})$ and $u, E \in C(\overline{\mathcal{B}})$. Then, the mean strain depends only on the boundary values of u:

$$\overline{E}(\mathcal{B}) = \frac{1}{\mathrm{vol}(\mathcal{B})} \int_{\partial\mathcal{B}} \mathrm{sym}(u \otimes n) \, dS \qquad (n \text{ outer unit normal to } \partial\mathcal{B}).$$

Pf. Follows immediately from the divergence thm

$$\int_{\partial\mathcal{B}} u \otimes n \, dS = \int_{\mathcal{B}} \nabla u \, dV \, . \quad \blacksquare$$

Remark 2 - As an application of the above thm, consider the f'llng one:

$$\delta V = \int_{\mathcal{B}} 1 \cdot E \, dV = \mathrm{vol}(\mathcal{B}) \, 1 \cdot \overline{E}(\mathcal{B}) = \int_{\partial\mathcal{B}} u \cdot n \, dS \, .$$