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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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MATHEMATICAL RESULTS FOR BINGHAM FLUIDS
(Part III)

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1. Regularisation.

We shall approximate j by a family of convex differentiable functionals $j_\varepsilon(v)$ defined by

$$j_\varepsilon(v) = \frac{2}{1+\varepsilon} \int_\Omega \left(D_{II}(v) \right)^{\frac{1+\varepsilon}{2}} dx \qquad \varepsilon > 0$$

We pass to the problem:

$$\begin{cases} (u_\varepsilon', v-u_\varepsilon) + \mu\, a(u_\varepsilon, v-u_\varepsilon) + b(u_\varepsilon, u_\varepsilon, v-u_\varepsilon) + g j_\varepsilon(v) - g j_\varepsilon(u_\varepsilon) \geq \\ \qquad \geq (f, v-u_\varepsilon) \qquad \forall v \in V \\ u_\varepsilon \in L^2(0,T; V) \cap L^\infty(0,T; H) \\ u_\varepsilon(0) = 0 \\ f \text{ given in } L^2(0,T; V) \end{cases}$$

On the other hand

$a(u,u) = \|u\|^2$ so that A is the canonical isomorphisme from V to V' and $A^{-1} \in \mathcal{L}(V';V)$.

The injection $V \subset H$ is compact so that A^{-1} is a compact symmetric linear operator in H.

By Riesz's theory (see Jacqueline Boyot's lectures) A^{-1} admits an orthonormal family of vectors $w_1, \dots, w_m, \dots$ such that:

$$\begin{cases} (w_i, w_j) = \delta_{ij} \\ A^{-1} w_i = \mu_i w_i \qquad \mu_i > 0 \qquad \mu_n \to 0 \end{cases}$$

It follows that

$$A w_i = \lambda_i w_i \qquad w_i \in V$$

and $\lambda_i > 0$, $\lambda_n \to +\infty$.

We can now calculate the norms in V, H & V':

$$(5.3) \begin{cases} \|u\|^2 = \sum_{i=1}^{\infty} \lambda_i \, |(u, w_i)|^2 & \text{if } u \in V \\ |u|^2 = \sum_{i=1}^{\infty} |(u, w_i)|^2 & \text{if } u \in H \\ \|u\|_*^2 = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} |(u, w_i)|^2 & \text{if } u \in V' \end{cases}$$

The subspace spanned by w_i is dense in V, H & V'.

We shall use the basis $\{w_i\}_{i=1}^{\infty}$ as a Galerkin's basis

3. Finite dimensional problem

We start with the following problem:

$$(5.4) \Big\{ \left(\frac{du_\varepsilon^m}{dt}, w_i \right) + \mu \, a(u_\varepsilon^m, w_i) + b(u_\varepsilon^m, u_\varepsilon^m, w_i) + g(j'_\varepsilon(u_\varepsilon^m), w_i) \dots$$

We shall solve this problem and then we shall pass to limit with $m \to +\infty$.

We look for solutions of (5.4) of the form:

$$u_\varepsilon^m(t) = \sum_{i=1}^{m} c_{\varepsilon i}^m(t)\, w_i(x)$$

so that we can write (5.4) as follows:

$$(5.5)\quad \begin{cases} c_{\varepsilon i}^{m\,\prime}(t) + \mu \lambda_i c_{\varepsilon i}^m(t) + \sum_{j,k}' c_{\varepsilon j}^m c_{\varepsilon k}^m\, b(w_j, w_k, w_i) + \\ \qquad + F_i(c_{\varepsilon 1}^m, \dots c_{\varepsilon m}^m) = (f, w_i) \qquad i = 1, \dots, m \end{cases}$$

$$\left(F(c_{\varepsilon 1}^m, \dots, c_{\varepsilon m}^m) = g(j_\varepsilon'(u_\varepsilon^m), w_i)\right)$$

which is a system of ordinary differential equations. By Cauchy-Lipschitz theorem it has a solution on an interval $[0, t^m[$.

4. <u>A priori estimates</u>.

We have to prouve that $t^m = T$ and then to obtain estimates for u_ε^m independently of m. For prouving that $t^m = T$ it is sufficient to find estimates independent of m. We shall multiply (5.5) by $c_{\varepsilon i}^m$ and than we add in i (from 1 to m), We get:

$$(5.6)\quad \begin{cases} \left(\frac{du_\varepsilon^m}{dt}, u_\varepsilon^m\right) + \mu\, a(u_\varepsilon^m, u_\varepsilon^m) + b(u_\varepsilon^m, u_\varepsilon^m, u_\varepsilon^m) + \\ \qquad + g(j_\varepsilon'(u_\varepsilon^m), u_\varepsilon^m) = (f, u_\varepsilon^m) \end{cases}$$

But $b(u_\varepsilon^m, u_\varepsilon^m, u_\varepsilon^m) = 0$ (lemma 4 § 4) and

$(j_\varepsilon'(u_\varepsilon^m), u_\varepsilon^m) \geq 0$.

We obtain hence:

$$\frac{1}{2}\frac{d}{dt}|u_\varepsilon^m|^2 + \mu\|u_\varepsilon^m\|^2 \le |f|_* \, \|u_\varepsilon^m\| \le$$
$$\le c_1 |f|_*^2 + \frac{\mu}{2}\|u_\varepsilon^m\|^2$$

so that

$$\frac{1}{2}\frac{d}{dt}|u_\varepsilon^m|^2 + \frac{\mu}{2}\|u_\varepsilon^m\|^2 \le c_1 |f|_*^2$$

It follows, if integrate from 0 to t that

$$(5.7)\quad \left\{ |u_\varepsilon^m(t)|^2 + \mu\int_0^t \|u_\varepsilon^m\|^2 dt \le c_1 |f|_{L^2(0,T;V')} + |u_{0\varepsilon}^m|^2 \right.$$

Or $|u_{0\varepsilon}^m|^2 \le |u_0|^2$ so that the right term of (5.7) is independent from m.

We have thus prouved that $t^m = T$ and moreover, that:

$$(5.8)\quad \begin{array}{l} u_\varepsilon^m \in \text{bounded set of } L^2(0,T;V) \\ \quad\;\; \in \text{bounded set of } L^\infty(0,T;H) \end{array}$$

We need now another estimates – (5.8) are not sufficient to permit to pass at limit in non linear terms.

Let denote by

$$h_{\varepsilon m} = \mu A(u_\varepsilon^m) + B(u_\varepsilon^m, u_\varepsilon^m) + g j_\varepsilon'(u_\varepsilon^m) - f$$

then from (5.6) it follows that

$$\left(\frac{du_\varepsilon^m}{dt}, w_i\right) = -(h_{\varepsilon m}, w_i) \qquad i = 1, \dots, m$$

so $\dfrac{du_\varepsilon^m}{dt}$ is the orthogonal projection (in V, H or V') of h ... on the subspace spanned by ...

Using (5.8) it follows that

$$A(u_\varepsilon^m) \in L^2(0,T;V')$$

and

$$B(u_\varepsilon^m, u_\varepsilon^m) \in L^2(0,T;V') \quad \text{(see lemma 5 §4)}$$

Moreover we can estimate $j_\varepsilon'(u_\varepsilon^m)$:

$$(5.9) \quad \| j_\varepsilon'(u_\varepsilon^m) \|_* \leq c \left[\int_\Omega (D_{II}(u_\varepsilon^m))^\varepsilon \right]^{1/2}$$

so that

$$j_\varepsilon'(u_\varepsilon^m) \in L^2(0,T;V')$$

It follows that

$$h_{\varepsilon m} \in L^2(0,T;V').$$

Using now the definition of the basis $\{w_i\}_{i=1}^\infty$ we have, by (5.3):

$$\left\| \frac{du_\varepsilon^m}{dt} \right\|_*^2 = \sum_{i=1}^m \frac{1}{\lambda_i} \left| \left(\frac{du_\varepsilon^m}{dt}, w_i \right) \right|^2 =$$

$$= \sum_{i=1}^m \frac{1}{\lambda_i} |(h_{\varepsilon m}, w_i)|^2 \leq \sum_{i=1}^\infty \frac{1}{\lambda_i} |(h_{\varepsilon m}, w_i)|^2 =$$

$$= \| h_{\varepsilon m} \|_*^2 \qquad \text{i.e} \quad \left\| \frac{du_\varepsilon^m}{dt} \right\|_* \leq \text{Constant (indep. of } m)$$

$\Rightarrow$

$$(5.10) \quad \frac{du_\varepsilon^m}{dt} \in \text{bounded set of } L^2(0,T;V')$$

<u>Remark</u>. It is here where we used the fact that $\{w_i\}$ is an orthogonal basis in V, H & V' and where the choice of this basis appears as an essential on

5. Extraction of a subsequence weakly convergent (in several spaces.) Use of the compactness theorem.

We shall use the following compactness theorem (see the last chapter for the proof)

Theorem: Let $1 < p_1, p_2 < +\infty$. Let B_1, B_2 and B_3 be three Banach spaces such that

$$B_1 \subset B_2 \subset B_3$$

with compact injection from B_1 to B_2 and continuous from B_2 to B_3. If $\{u_n\}$ is such that

$$u_n \in \text{bounded set of } L^{p_1}(0,T;\, B_1) \text{ and}$$

$$\frac{du_n}{dt} \in \text{bounded set of } L^{p_2}(0,T;\, B_3) \quad (\text{with } T<+\infty)$$

$\Rightarrow$ we can extract from u_n a subsequence strongly convergent in $L^{p_1}(0,T;\, B_2)$

Back to the estimates (5.8) and (5.10). We can apply the compactness theorem as $V \subset H \subset V'$ verify its assumptions and as

$$u_\varepsilon^m \in \text{bounded set of } L^2(0,T;\, V)$$

$$\frac{du_\varepsilon^m}{dt} \in \text{bounded set of } L^2(0,T;\, V').$$

We can thus extract from u_ε^m a subsequence $u_\varepsilon^{m_k}$ such that:

$$(5.11)\quad \begin{cases} u_\varepsilon^{m_k} \xrightarrow[m_k \to +\infty]{} u_\varepsilon \text{ in } L^2(0,T;\, V) \text{ weakly} \\ u_\varepsilon^{m_k} \xrightarrow[m_k \to +\infty]{} u_\varepsilon \text{ in } L^2(0,T.\, H) \text{ strongly (by compactness theorem)} \\ u_\varepsilon^{m_k} \xrightarrow[m_k \to +\infty]{*} u_\varepsilon \text{ in } L^\infty(0,T;\, H) \text{ weakly}* \\ \dfrac{du_\varepsilon^{m_k}}{dt} \xrightarrow[m_k \to +\infty]{} \dfrac{du_\varepsilon}{dt} \text{ in } L^2(0,T;\, V') \text{ weakly} \\ i'(u^{m_k}) \longrightarrow \chi \text{ in } L^2(0,T;\, V') \text{ weakly} \end{cases}$$

We have

$$b(u_\varepsilon^{m_k}, u_\varepsilon^{m_k}, w_i) = -b(u_\varepsilon^{m_k}, w_i, u_\varepsilon^{m_k}) =$$

$$= -\sum_{l,j=1}^{2} \int_\Omega (u_\varepsilon^{m_k})_l \frac{\partial (w_i)_j}{\partial x_l} (u_\varepsilon^{m_k})_j \, dx$$

Now we know that

$u_\varepsilon^{m_k} \longrightarrow u_\varepsilon$ in $L^2(0,T;H)$ strong that is in $L^2(0,T;L^2(\Omega))$

Using lemma 3 §4 (i.e. $|u|_{L^4}^2 \leq \|u\| \, |u|$) $\Rightarrow$

$u_\varepsilon^{m_k} \longrightarrow u_\varepsilon$ in $L^4(0,T;L^4(\Omega))$ weakly

so that

$(u_\varepsilon^{m_k})_l (u_\varepsilon^{m_k})_j \longrightarrow (u_\varepsilon)_l (u_\varepsilon)_j$ in $L^1(0,T;L^1(\Omega))$ strongly

$\longrightarrow (u_\varepsilon)_l (u_\varepsilon)_j$ in $L^2(0,T;L^2(\Omega))$ weakly

it follows then that

(5.12) $\quad b(u_\varepsilon^{m_k}, u_\varepsilon^{m_k}, w_i) \longrightarrow b(u_\varepsilon, u_\varepsilon, w_i)$

6. Passage to the limit $m_k \to +\infty$.

Using now all the informations about the subsequence $\{u_\varepsilon^{m_k}\}$ (5.11) & (5.12) we can now pass to the limit with $m_k \longrightarrow +\infty$ in (5.4) and we get :

$$\begin{cases} \left(\dfrac{du_\varepsilon}{dt}, w_i\right) + \mu a(u_\varepsilon, w_i) + b(u_\varepsilon, u_\varepsilon, w_i) + g(\chi, w_i) = \\ \qquad = (f, w_i) \qquad i = 1, \dots, m \\ u_\varepsilon(0) = 0 \end{cases}$$

and as the subspace spanned by $\{w_i\}$ is dense in V, u_ε satisfies

(5.13) $$\left\{ \left(\frac{du_\varepsilon}{dt}, v\right) + \mu a(u_\varepsilon, v) + b(u_\varepsilon, u_\varepsilon, v) + g(\chi, v) = (f, v) \right.$$

By monotonicity arguments we shall now prove that

$$\chi = j_\varepsilon'(u_\varepsilon)$$

Let denote by $X_\varepsilon^{m_k}$ the following integral:

$$X_\varepsilon^{m_k} = g\int_0^T \left(j_\varepsilon'(u_\varepsilon^{m_k}) - j_\varepsilon'(\varphi),\ u_\varepsilon^{m_k} - \varphi\right) dt +$$

$$+\mu \int_0^T a(u_\varepsilon^{m_k} - \varphi,\ u_\varepsilon^{m_k} - \varphi) + \int_0^T \left(\frac{du_\varepsilon^{m_k}}{dt} - \frac{d\varphi}{dt},\ u_\varepsilon^{m_k} - \varphi\right) dt$$

where φ is a function in $L^2(0,T;V)$ such that

$$\frac{d\varphi}{dt} \in L^2(0,T;V') \text{ and } \varphi(0) = 0$$

Now it is easy to see that $X_\varepsilon^{m_k} \geq 0$. In fact the first term is ≥ 0 as $j_\varepsilon'(r)$ is monotone (see lemma 3 § 3), the second one is equal to $\mu \int_0^T \| u_\varepsilon^{m_k} - \varphi \|^2 dt$. The third term is equal to

$$\int_0^T \frac{1}{2}\frac{d}{dt} |u_\varepsilon^{m_k} - \varphi|^2 dt = \frac{1}{2} |u_\varepsilon^{m_k}(T) - \varphi(T)|^2 \geq 0$$

On the other hand using (5.6) $\Rightarrow$

$$X_\varepsilon^{m_k} = \int_0^T (f, u_\varepsilon^{m_k})\, dt - g\int_0^T \left(j_\varepsilon'(\varphi),\ u_\varepsilon^{m_k} - \varphi\right) dt -$$

$$-\mu \int_0^T a(\varphi,\ u_\varepsilon^{m_k} - \varphi)\, dt - \int_0^T (\varphi',\ u_\varepsilon^{m_k} - \varphi)\, dt$$

Now we can to limit in $X_\varepsilon^{m_k}$ if $m_k \to +\infty$

$$X_\varepsilon^{m_k} \longrightarrow X_\varepsilon = \int_0^T (f, u_\varepsilon)\, dt - g\int_0^T \left(j_\varepsilon'(\varphi),\ u_\varepsilon - \varphi\right) dt$$

$$-\mu \int_0^T a(\varphi,\ u_\varepsilon - \varphi)\, dt - \int_0^T (\varphi',\ u_\varepsilon - \varphi)\, dt \geq 0$$

We can now use (5.13) with $v = u_\varepsilon$ (which is a test function):

$$(f, u_\varepsilon) = \left(\frac{du_\varepsilon}{dt}, u_\varepsilon\right) + \mu a(u_\varepsilon, u_\varepsilon) + g(\chi, u_\varepsilon)$$

so that

$$X_\varepsilon = g\int_0^T (\chi - j_\varepsilon'(\varphi), u_\varepsilon - \varphi)\,dt + \mu\int_0^T a(u_\varepsilon - \varphi, u_\varepsilon - \varphi)\,dt +$$

$$+ \int_0^T (u_\varepsilon' - \varphi', u_\varepsilon - \varphi)\,dt \geq 0$$

(where $\varphi \in L^2(0,T;V)$, $\varphi' \in L^2(0,T;V')$, $\varphi(0) = 0$)

We shall take $\varphi = u_\varepsilon - \lambda\psi$, $\psi \in L^2(0,T;V)$, $\psi' \in L^2(0,T;V')$, $\psi(0) = 0$; this operation is allowed as $u_\varepsilon \in L^2(0,T;V)$, $u_\varepsilon' \in L^2(0,T;V')$ and $u_\varepsilon(0) = 0$.

It follows that

$$X_\varepsilon = \lambda g\int_0^T (\chi - j_\varepsilon'(u_\varepsilon - \lambda\psi), \psi)\,dt +$$

$$+ \mu\lambda^2\int_0^T a(\psi,\psi)\,dt + \lambda^2\int_0^T (\psi', \psi)\,dt \geq 0$$

By dividing with λ it follows that

$$g\int_0^T (\chi - j_\varepsilon'(u_\varepsilon - \lambda\psi), \psi) + \mu\lambda\int_0^T a(\psi,\psi)\,dt + \lambda\int_0^T (\psi',\psi)\,dt \geq 0$$

We can let now $\lambda \to 0$ so that

$$g\int_0^T (\chi - j_\varepsilon'(u_\varepsilon), \psi) \geq 0 \qquad \forall \psi \in L^2(0,T;V),\ \psi' \in L^2(0,T;V'),\ \psi(0) = 0$$

$\Rightarrow$ $\chi = j_\varepsilon'(u_\varepsilon)$ and u_ε is solution of:

(5.14) $\left\{ (u_\varepsilon', v) + \mu a(u_\varepsilon, v) + b(u_\varepsilon, u_\varepsilon, v) + g(j_\varepsilon'(u_\varepsilon), v) = (f, v) \right.$

7. Limit for $\varepsilon \to 0$

Let us consider the estimates obtained in 4. It is easy to observe that all are independent of ε (not only of m) so that all are valable for u_ε too. By the same technique as in 4 & 5 we can extract from u_ε a subsequence u_{ε_k} such that (see (5.11)):

$$
(5.15)\quad \left\{
\begin{array}{ll}
u_{\varepsilon_k} \xrightarrow[\varepsilon_k \to 0]{} u & \text{in } L^2(0,T;V) \text{ weakly} \\
u_{\varepsilon_k} \xrightarrow[\varepsilon_k \to 0]{} u & \text{in } L^2(0,T;H) \text{ strongly (use of compactness theorem)} \\
u_{\varepsilon_k} \xrightarrow[\varepsilon_k \to 0]{*} u & \text{in } L^\infty(0,T;H) \text{ weak } * \\
\dfrac{du_\varepsilon^{m_k}}{dt} \xrightarrow[\varepsilon_k \to 0]{*} \dfrac{du_\varepsilon}{dt} & \text{in } L^2(0,T;V') \text{ weakly} \\
(u_{\varepsilon_k})_e (u_{\varepsilon_k})_j \longrightarrow u_e u_j & \text{in } L^2(0,T;L^2(\Omega)) \text{ weakly}
\end{array}
\right.
$$

Let be $v \in L^2(0,T;V)$ and Y defined by

$$
Y = g\int_0^T [j_\varepsilon(v) - j_\varepsilon(u_{\varepsilon_k}) - (j'_\varepsilon(u_{\varepsilon_k}), v - u_{\varepsilon_k})]\,dt
$$

By the properties of the G-differential of a convex functional (see lemma 1 §3)

$$
Y \geq 0
$$

But the last term $(j'_\varepsilon(u_{\varepsilon_k}), v - u_{\varepsilon_k})$ is equal from (5.14) with v replaced by $v - u_{\varepsilon_k}$, to:

$$
-(u'_{\varepsilon_k}, v - u_{\varepsilon_k}) - \mu a(u_{\varepsilon_k}, v - u_{\varepsilon_k}) - b(u_{\varepsilon_k}, u_{\varepsilon_k}, v - u_{\varepsilon_k}) + (f, v - u_{\varepsilon_k})
$$

and thus

$$
Y = \int_0^T \{ g j_\varepsilon(v) - g j_\varepsilon(u_{\varepsilon_k}) + (u'_{\varepsilon_k}, v - u_{\varepsilon_k}) + \mu a(u_{\varepsilon_k}, v - u_{\varepsilon_k}) +
$$

It follows that

$$\int_0^T \{(u'_{\varepsilon_k}, v) + \mu a(u_{\varepsilon_k}, v) + b(u_{\varepsilon_k}, u_{\varepsilon_k}, v) - (f, v - u_{\varepsilon_k}) + g j_\varepsilon(v)\}\, dt \geq \int_0^T \underset{\frac{1}{2}\frac{d}{dt}|u_{\varepsilon_k}(t)|^2}{\underset{\|}{(u'_{\varepsilon_k}, u_{\varepsilon_k})}}\, dt + \mu \int_0^T a(u_{\varepsilon_k}, u_{\varepsilon_k}) + \int_0^T g j_\varepsilon(u_{\varepsilon_k})\, dt = \frac{1}{2}|u_{\varepsilon_k}(T)|^2 + \mu \int_0^T \|u_{\varepsilon_k}\|^2 dt + \int_0^T g j_\varepsilon(u_{\varepsilon_k})\, dt$$

We shall now take $\underline{\lim}_{\varepsilon_k \to 0}$ in this inequality. In fact the first term has a limit, using the convergences of (5.15), therefore

$$\int_0^T \{(u', v) + \mu a(u, v) + b(u, u, v) - (f, v - u) + g j(v)\}\, dt \geq \frac{1}{2} \underline{\lim}_{\varepsilon_k \to 0} |u_{\varepsilon_k}(T)|^2 + \underline{\lim} \int_0^T \|u_{\varepsilon_k}\|^2 dt + \underline{\lim}_{\varepsilon_k \to 0}\, g \int_0^T j_\varepsilon(u_{\varepsilon_k})\, dt \geq \frac{1}{2}|u(T)|^2 + \int_0^T \|u(t)\|^2 dt + g \int_0^T j(u)\, dt = \int_0^T (u', u)\, dt + \int_0^T \big(\mu a(u, u) + g j(u) + \underset{0}{\underset{\|}{b(u,u,u)}}\big)\, dt$$

We have used the lower semicontinuity of the norms (see definition and examples in § 3) and the fact that $j_\varepsilon(v)$ is also lower semicontinuous (it is convex and continuous for the weak topology of $L^2(0, T; V)$)

It follows then that

$$(5.16)\quad \left\{ \int_0^T \{(u', v - u) + \mu a(u, v - u) + b(u, u, v - u) + g j(v) - g j(u) - (f, v - u)\}\, dt \geq 0 \right.$$

for all $v \in L^2(0, T; V)$.

We have to solve the problem (5.1). It is clear that if $v \in L^2(0,T;V)$ in (5.1) we are allowed to integrate in (5.1) from 0 to T (all the terms $\in L^1(0,T)$) so that

$$(5.1) \Longrightarrow (5.16)$$

We shall now prouve that (5.16) $\Rightarrow$ (5.1) so that the proof of the existence is completely done. Let w be an arbitrary element in V and t_0 a point in $]0,T[$. Let be

$$O_j = \left]t_0 - \frac{1}{j}, t_0 + \frac{1}{j}\right[\qquad j \text{ sufficiently great}$$

Let v be defined by

$$v(t) = \begin{cases} w & \text{if } t \in O_j \\ u(t) & \text{if } t \notin O_j \end{cases}$$

$v(t)$ is a test function and we can put it in (5.16) which becomes:

$$\int_{O_j} \{(u', w-u) + \mu a(u, w-u) + b(u,u,w-u) + gj(w) - gj(u) - (f, w-u)\} dt \geq 0$$

$b(u,u,u) = 0$ and $b(u,u,w) = (\tilde{u}, w)$ with $\tilde{u} \in L^2(0,T;V')$ (as it was proved in lemma 5 §4)

We have, by dividing with $\frac{1}{|O_j|}$ ($|O_j|$ = mesure of O_j)

$$\text{*)} \quad \frac{1}{|O_j|}\int_{O_j} (u' + \mu A u + \tilde{u} - f, w)\, dt - \frac{1}{|O_j|}\int_{O_j}\{(u',u) + \mu a(u,u) + gj(u) - (f,u)\} dt + gj(w) \geq 0$$

We shall now let $j \to +\infty$, and apply the Lebesgue theorem on the differentiation of set-function if g is a scalar or vector-valued function, measurable $\Rightarrow \frac{1}{|O_j|}\int_{O_j} g(t)\,dt \xrightarrow[j\to\infty]{} g(t_0)$ except maybe for t_0 belonging to an exceptional set of zero measure)

Thus

$$\frac{1}{|O_j|}\int_{O_j}(u'+\mu Au+\tilde{u}-f)\,dt \longrightarrow u'(t_0)+\mu Au(t_0)+\tilde{u}(t_0)-f(t_0) \text{ in } V' \text{ for } t_0\notin E_1,\ \text{mes}\, E_1=0$$

and so

$$\frac{1}{|O_j|}\int_{O_j}(u'+\mu Au+\tilde{u}-f, w)\,dt = \left(\frac{1}{|O_j|}\int_{O_j}(u'+\mu Au+\tilde{u}-f)\,dt, w\right)$$

$$\longrightarrow (u'(t_0)+\mu Au(t_0)+\tilde{u}(t_0)-f(t_0), w)$$

We have also

$$\frac{1}{|O_j|}\int_{O_j}\{(u',u)+\mu a(u,u)+gj(u)-(f,u)\}\,dt \longrightarrow$$

$$\rightarrow (u'(t_0),u(t_0))+\mu a(u(t_0),u(t_0))+gj(u(t_0))-(f(t_0),u(t_0))$$

for $t_0\notin E_2$, $\text{mes}\,E_2=0$

Therefore for $t_0\notin E_1\cup E_2$ we can pass to limit in (5.17). It follows that:

$$(u'(t_0), w-u(t_0))+\mu(A(u(t_0)), w-u(t_0))+$$
$$+(\tilde{u}(t_0), w)+gj(w)-g(u(t_0))-(f(t_0), w-u(t_0))\geq 0$$

i.e (5.1).

8. Uniqueness

We shall use for prouving the uniqueness the

If $\varphi(t) \leq A + \int_0^t B(s)\varphi(s)\,ds$ with $\varphi(t) \geq 0$

$\varphi(t) \in L^\infty(0,T)$; $B(t) \geq 0$, $B \in L^1(0,T) \Rightarrow$

$$\varphi(t) \leq A + \int_0^t B(s)\varphi(s) \leq A e^{\int_0^t B(s)ds}$$

Let u_1 and u_2 be two solutions of the problem (5.1):

$$(5.17)\quad (u_1', v-u_1) + \mu a(u_1, v-u_1) + b(u_1, u_1, v-u_1) + g j(v) - g j(u_1) \geq (f, v-u_1) \qquad \forall v \in V, \quad u_1(0,x) = u_0$$

$$(5.18)\quad (u_2', v-u_2) + \mu a(u_2, v-u_2) + b(u_2, u_2, v-u_2) + g j(v) - g j(u_2) \geq (f, v-u_2) \qquad \forall v \in V, \quad u_2(0,x) = u_0$$

We shall take $v = u_2$ in (5.17) and $v = u_1$ in (5.18) and we shall add afterwards. Let denote by $U = u_1 - u_2 \Rightarrow$

$$(5.19)\quad (U', U) + \mu a(U,U) - b(u_1, u_1, U) + b(u_2, u_2, U) \leq 0$$

But $b(u_1, u_1, U) - b(u_2, u_2, U) =$

$= b(U, u_1, U)$

It follows from (5.19) that

$$\frac{1}{2}\frac{d}{dt}|U|^2 + \mu \|U\|^2 \leq |b(U, u_1, U)| \leq \text{by lemmas 3\&5(4)}$$

$$\leq c\,|U|\,\|U\|\,\|u_1\| \leq$$

$$\leq C_1 |U|^2 \|u_1\|^2 + \frac{\mu}{2}\|U\|^2$$

$\left(ab \leq \frac{1}{2K}a^2 + \frac{K}{2}b^2\right)$

$\Rightarrow$

$$|U|^2 \leq \frac{1}{2}\frac{d}{dt}|U|^2 + \overbrace{\frac{\mu}{2}\|U\|^2}^{\text{this term is positive}} \leq C_1 |U|^2 \|u_1\|^2$$

and by integration on t:

But $U(0) = 0$. We can apply Gronwall's inequality with $A = 0$ and $\varphi(t) = |U(t)|^2$, $B(s) = C_2 \|u_2\|^2$ — $B(s) \in L^1(0,T)$ as $u_1 \in L^2(0,T; V)$) $\Rightarrow$

$$|U(t)|^2 \leq 0 \quad \longrightarrow \quad U = 0 \text{ so that } u_1 = u_2.$$

Bibliography

· Dunford - Schwartz : Linear operators, Part. 1, Interscience Publ. (1958)
(III. 12.9 → for Lebesgue thm)