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DISCONTINUOUS WAVES

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1. Singular Surfaces

Notation.

E : three-dimensional Euclidean point space.

D : a domain (i.e. open connected point set) of E.

D_0 : the set of position vectors, relative to an arbitrary origin O, of the points of D.

R : the real line.

I : an open interval of R.

Differentiability classes.

A scalar-valued function f is said to be of class C^k ($k \geq 0$ an integer) in $\mathcal{D} = D \times I$ if all the partial derivatives of f of orders $1, \ldots, k$ are continuous in $D_0 \times I$. A function of class C^0 is merely continuous. If, in relation to some basis, each component of a vector- (or tensor-) valued function is of class C^k in $\mathcal{D}$, the function itself is said to have this property.

Let t denote time and regard R as the set of all times. Consider a hypersurface Σ in $E \times R$ (space-time) given by

$$\underline{x} = \underline{\varphi}(\xi_1, \xi_2, t). \qquad (1.1)$$

of t (1·1) defines a surface Σ_t in E on which ξ_1 and ξ_2 are curvilinear coordinates. Thus we can regard Σ as a moving surface in E and Σ_t as its configuration at time t.

Suppose now that for each $t \in I$, Σ_t divides D into two parts, D_t^+ and D_t^-, as shown in Figure 1. Define

$$\mathcal{D}^{\pm} = \{(p, t) : t \in I,\ p \in D_t^{\pm}\}. \qquad (1{\cdot}2)$$

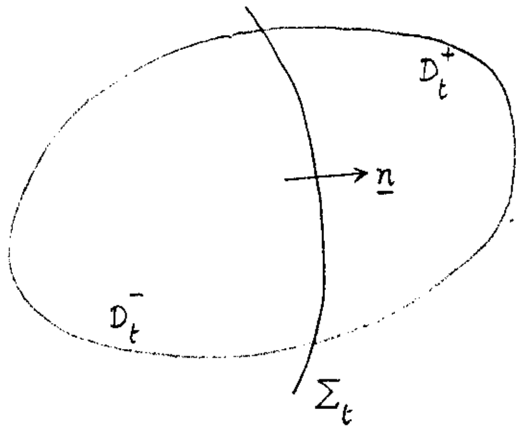

Figure 1

If f is of class C^0 in $\mathcal{D}^+$ and $\mathcal{D}^-$ but not in $\mathcal{D}$, and there are functions f^+ and f^-, continuous in $\mathcal{D}^+ \cup (\Sigma \cap \mathcal{D})$ and $\mathcal{D}^- \cup (\Sigma \cap \mathcal{D})$ respectively and coinciding with f in $\mathcal{D}^+$ and $\mathcal{D}^-$, the difference

$$[f](\xi_1, \xi_2, t) = f^+(\underline{\varphi}(\xi_1, \xi_2, t), t) - f^-(\underline{\varphi}(\xi_1, \xi_2, t), t) \qquad (1{\cdot}3)$$

is called the <u>jump in f on Σ</u>. If $[f] \neq 0$ almost everywhere on $\Sigma \cap \mathcal{D}$, we say that Σ is an <u>absolute singular surface</u> of f.

Further classes of singular surfaces of f can be defined as follows. Suppose that f is of class C^{k-1} $(k \geqslant 1)$ in $\mathcal{D}$ and of class C^k in $\mathcal{D}^+$ and $\mathcal{D}^-$ and that there are functions f^+ and f^- in terms of which the jumps on Σ of the kth partial derivatives of f can be defined. If one of surface of f of order k.

It has been tacitly supposed that f is scalar-valued, but exactly similar considerations apply to vector- and tensor-valued functions.

2. Compatibility Conditions

The basic idea here is that the jumps in the partial derivatives of a function on a singular surface are not independent of one another. The relationships satisfied by the jumps in the partial derivatives of a given order are called compatibility conditions.

The simplest situation, which we now consider, concerns the jumps in the first derivatives of a scalar-valued function f on a singular surface Σ of order 1. In this case f is continuous in $\mathscr{D}$, but at least one of its first partial derivatives fails to be continuous on Σ. On Σ we have

$$f^{+}(\underline{\varphi}(\xi_1,\xi_2,t),t) = f^{-}(\underline{\varphi}(\xi_1,\xi_2,t),t) = F(\xi_1,\xi_2,t), \text{ say,} \qquad (2{\cdot}1)$$

and differentiation with respect to ξ_A $(A=1,2)$ and t gives

$$\frac{\partial F}{\partial \xi_A} = \frac{\partial f^{\pm}}{\partial x_p}\frac{\partial \varphi_p}{\partial \xi_A} = (\operatorname{grad} f^{\pm})\cdot\frac{\partial \underline{\varphi}}{\partial \xi_A} \quad (A=1,2), \qquad (2{\cdot}2)$$

$$\frac{\partial F}{\partial t} = \frac{\partial f^{\pm}}{\partial x_p}\frac{\partial \varphi_p}{\partial t} + \frac{\partial f^{\pm}}{\partial t} = (\operatorname{grad} f^{\pm})\cdot\frac{\partial \underline{\varphi}}{\partial t} + \frac{\partial f^{\pm}}{\partial t}. \qquad (2{\cdot}3)$$

Here x_i $(i=1,2,3)$ are rectangular Cartesian coordinates (i.e. the components of convention applies over the repeated suffix p. On forming the differences between the '+' and '−' members of equations (2·2) and (2·3) and recalling the definition (1·3) of a jump on Σ we obtain

$$[\mathrm{grad}\, f]\cdot \frac{\partial \underline{\varphi}}{\partial \xi_A} = 0 \quad (A = 1,2), \quad [\mathrm{grad}\, f]\cdot \frac{\partial \underline{\varphi}}{\partial t} + \left[\frac{\partial f}{\partial t}\right] = 0. \tag{2·4}$$

The linearly independent vectors $\partial \underline{\varphi}/\partial \xi_A$ $(A = 1,2)$ are directed tangentially to Σ_t (see equation (1·1)). Thus $(2·4)_1$ tells us that the jump $[\mathrm{grad}\, f]$ is orthogonal to the tangent plane of Σ_t. This means that there is a scalar (i.e. real number) α such that

$$[\mathrm{grad}\, f] = \alpha\, \underline{n}, \tag{2·5}$$

where $\underline{n}$ is a unit normal vector field on Σ_t. By convention $\underline{n}$ is taken to point in the direction of travel of Σ. The result of forming the scalar product of each side of (2·5) with $\underline{n}$ is

$$\alpha = [\mathrm{grad}\, f]\cdot \underline{n} = [(\mathrm{grad}\, f)\cdot \underline{n}] = \left[\frac{\partial f}{\partial n}\right], \tag{2·6}$$

$\partial f/\partial n$ being the derivative of f in the direction normal to Σ_t.

Equation $(2·4)_2$, when combined with (2·5), becomes

$$\left[\frac{\partial f}{\partial t}\right] = -\alpha\, \frac{\partial \underline{\varphi}}{\partial t}\cdot \underline{n}. \tag{2·7}$$

From (1·1) $\partial \underline{\varphi}/\partial t$ is seen to be the velocity of propagation of Σ relative to

$$V_n = \frac{\partial \varphi}{\partial t} \cdot \underline{n} \tag{2.8}$$

is the speed of travel of Σ_t normal to itself. We therefore arrive at the relations

$$[\operatorname{grad} f] = \alpha \underline{n}, \quad \left[\frac{\partial f}{\partial t}\right] = -V_n \alpha, \quad \text{with} \quad \alpha = \left[\frac{\partial f}{\partial n}\right]. \tag{2.9}$$

These are <u>compatibility conditions for jumps in first partial derivatives on a first-order singular surface.</u> In component form they read

$$\left[\frac{\partial f}{\partial x_i}\right] = \alpha n_i, \quad \left[\frac{\partial f}{\partial t}\right] = -V_n \alpha, \quad \text{with} \quad \alpha = \left[\frac{\partial f}{\partial x_p}\right] n_p. \tag{2.10}$$

<u>Exercise.</u> By replacing f by $\partial f/\partial x_j$ and $\partial f/\partial t$ in turn in (2.10) derive the following <u>compatibility conditions for jumps in second partial derivatives on a second-order singular surface.</u>

$$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right] = \beta n_i n_j, \quad \left[\frac{\partial^2 f}{\partial t\, \partial x_i}\right] = -V_n \beta n_i, \quad \left[\frac{\partial^2 f}{\partial t^2}\right] = V_n^2 \beta, \tag{2.11}$$

where

$$\beta = \left[\frac{\partial^2 f}{\partial x_p \partial x_q}\right] n_p n_q = \left[\frac{\partial^2 f}{\partial n^2}\right]. \tag{2.12}$$

It is apparent from (2.10) and (2.11) that the jumps in the kth partial derivatives of a scalar function on a kth-order singular surface depend on a single jump, namely $[\partial^k f/\partial n^k]$. More elaborate compatibility conditions apply to jumps in partial derivatives of order higher than the order of the singular surface, the curvature of Σ_t then entering into

Truesdell and Toupin in Vol. III/1 of the Handbuch der Physik (Springer, 1960).

3. Singular Surfaces in a Moving Body

A motion of a body (or continuum) can be represented in the form

$$\underline{x} = \underline{\Psi}(\underline{X}, t), \tag{3.1}$$

where $\underline{x}$ is the position vector of the point occupied at time t by a representative particle of the body which has position $\underline{X}$ in an arbitrarily chosen reference configuration.

A classification of singular surfaces in a moving body can conveniently be based upon the vector-valued deformation function $\underline{\Psi}$ in (3.1). A singular surface of order k (≥ 1) is one on which the derivatives of $\underline{\Psi}$ of orders $1, \ldots, k-1$ are continuous and the derivatives of order k almost everywhere discontinuous. We exclude from consideration absolute singular surfaces of $\underline{\Psi}$ since they give rise to dislocations of the body. Of course dislocations are very important in the mechanics of solids, but it is not profitable to treat them by the methods under discussion here.

The velocity

$$\underline{v} = \frac{\partial \underline{\Psi}}{\partial t} \quad \left(\text{with components } v_i = \frac{\partial \Psi_i}{\partial t}\right) \tag{3.2}$$

and the deformation gradient

are discontinuous on a singular surface of order 1. When such a surface carries a jump in the normal component of velocity (i.e. when $[\mathbf{v}]\cdot\mathbf{n} \neq 0$) it is called a <u>shock wave</u>. Otherwise (i.e. when $[\mathbf{v}]\cdot\mathbf{n} = 0$, $[\mathbf{v}]\times\mathbf{n} \neq \mathbf{0}$) the term <u>vortex sheet</u> is used.

The term <u>wave</u> in the present context means a singular surface (of $\boldsymbol{\Psi}$) propagating <u>relative to the material</u>. A non-propagating singular surface is occupied at all times by the same particles of the body: in other words it is a material surface. A singular surface of this kind is called a <u>contact discontinuity</u>.

On a second-order singular surface $\mathbf{v}$ and $\mathbf{F}$ are continuous, but the acceleration

$$\mathbf{f} = \frac{\partial^2 \boldsymbol{\Psi}}{\partial t^2} \quad \left(\text{with components } f_i = \frac{\partial^2 \psi_i}{\partial t^2}\right), \tag{3.4}$$

the velocity gradient

$$\mathbf{L} = \operatorname{grad} \mathbf{v} \quad \left(\text{with components } L_{ij} = \frac{\partial v_i}{\partial x_j}\right) \tag{3.5}$$

and the space and time derivatives of $\mathbf{F}$ are discontinuous. A propagating singular surface of order 2 is consequently termed an <u>acceleration wave</u>.

4. <u>The Balance Laws of Continuum Mechanics</u>

Notation:

R : the current configuration of an arbitrary material region of the

∂R_t : the boundary of R_t.

$\underline{n}$: outward unit normal on ∂R_t.

ρ : mass density in the current configuration B_t.

$\underline{b}$: extrinsic body force (per unit mass).

r : extrinsic heat supply (per unit mass).

$\underline{t}_{(\underline{n})}$: contact force (or stress vector) on ∂R_t.

$h_{(\underline{n})}$: heat influx on ∂R_t.

ε : internal energy (per unit mass).

The balance laws of continuum mechanics, as they apply to non-polar materials in the absence of extrinsic torques, take the following form.

(a) Mass balance.

$$\frac{d}{dt}\int_{R_t} \rho \, dv = 0. \tag{4.1}$$

(b) Linear momentum balance:

$$\frac{d}{dt}\int_{R_t} \rho \underline{v} \, dv = \int_{R_t} \rho \underline{b} \, dv + \int_{\partial R_t} \underline{t}_{(\underline{n})} \, da. \tag{4.2}$$

(c) Angular momentum balance:

$$\frac{d}{dt}\int_{R_t} \rho \underline{x} \times \underline{v} \, dv = \int_{R_t} \rho \underline{x} \times \underline{b} \, dv + \int_{\partial R_t} \underline{x} \times \underline{t}_{(\underline{n})} \, da. \tag{4.3}$$

(d) Energy balance:

$$\frac{d}{dt}\int_{R_t} \rho\left(\tfrac{1}{2}\underline{v}\cdot\underline{v} + \varepsilon\right) dv = \int_{R_t} \rho(\underline{b}\cdot\underline{v} + r) \, dv + \int_{\partial R_t} (\underline{t}_{(\underline{n})}\cdot\underline{v} + h_{(\underline{n})}) \, da. \tag{4.4}$$

Here v denotes volume and a surface measure. Each of (4.1) to (4.4) holds for all t and for all $R_t \subseteq B_t$.

The four balance equations (4.1) to (4.4) can be expressed in the generic form

$$\frac{d}{dt}\int_{R_t} \rho\pi\, dv = \int_{R_t} \rho s\, dv + \int_{\partial R_t} f_{(\underline{n})}\, da, \tag{4.5}$$

where the quantities π, s (the rate of supply of π per unit mass) and $f_{(\underline{n})}$ (the influx of π per unit area in B_t) are specified in the accompanying table.

Balance law	π	s	$f_{(\underline{n})}$
Mass	1	0	0
Linear momentum	$\underline{v}$	$\underline{b}$	$\underline{t}_{(\underline{n})}$
Angular momentum	$\underline{x}\times\underline{v}$	$\underline{x}\times\underline{b}$	$\underline{x}\times\underline{t}_{(\underline{n})}$
Energy	$\frac{1}{2}\underline{v}\cdot\underline{v}+\varepsilon$	$\underline{b}\cdot\underline{v}+r$	$\underline{t}_{(\underline{n})}\cdot\underline{v}+r$

In order to derive field equations (like the equation of motion) from the balance laws (4.1) to (4.4) it is necessary to attribute to the fields ρ, $\underline{t}_{(\underline{n})}$, $h_{(\underline{n})}$ and ε, which represent properties of the material composing the body certain smoothness properties. Problems of considerable physical interest arise, however, in which these conditions are not met, and of especial importance is the situation in which the failure of ρ, $\underline{t}_{(\underline{n})}$, $h_{(\underline{n})}$ and ε to attain the consider the discontinuity relations induced on such a singular surface by the balance laws.

5. Basic Jump Conditions

We need a modified form of the convection theorem proved in Professor Benjamin's lectures. The basic ingredients of this result are (a) a material region with current configuration R_t, t belonging to some time interval I, and (b) a scalar field f defined in $\{(p,t): t \in I, p \in R_t\}$. Normally f is taken to be of class C^1 in this domain, but here we suppose that, for all $t \in I$, R_t is intersected by an absolute singular surface of f. The amended form of the convection theorem is then

$$\frac{d}{dt}\int_{R_t} f\,dv = \int_{R_t} \frac{\partial f}{\partial t}\,dv + \int_{\partial R_t} f\,\underline{v}\cdot\underline{n}\,da - \int_{\Lambda_t} [f]\,V_n\,da, \tag{5.1}$$

where $\Lambda_t = \Sigma_t \cap R_t$ and V_n is defined by equation (2.8). The presence of the singular surface is seen to result in an extra term on the right-hand side. Proofs of (5.1) are given in the article by Truesdell and Toupin cited in Section 2 and in the lecturer's book "Continuum Mechanics" (pp. 115-116).

On replacing f by $\rho\pi$ in (5.1) and using the generic balance law (4.5) we obtain the relation

$$\int \left\{\frac{\partial}{\partial t}(\rho\pi) - \rho s\right\} dv + \int \left\{\rho\pi\,\underline{v}\cdot\underline{n} - f_{(n)}\right\} da - \int [\rho\pi]\,V_n\,da = 0. \tag{5.2}$$

The volume of R_t is now made to approach zero in such a way that, in the limit, ∂R_t collapses onto the two sides of Λ_t as illustrated in Figure 2. Assuming that ρ and π are of class C^1 and s of class C^0 away from the singular surface Σ and that $f_{(-\underline{n})} = -f_{(\underline{n})}$, the result of this procedure is

$$\int_{\Lambda_t} \left\{ [\rho\pi\underline{v}\cdot\underline{n} - f_{(\underline{n})}] - [\rho\pi] V_n \right\} da = 0. \qquad (5.3)$$

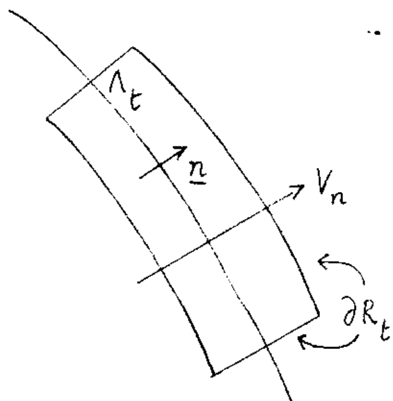

Figure 2

Equation (5.3) holds for all $t \in I$ and for all segments Λ_t of Σ_t formed by intersection with (regular) subregions R_t of the current configuration B_t. Thus, provided that the integrand is continuous on Σ, there follows the general jump condition

$$[\rho V \pi + f_{(\underline{n})}] = 0, \qquad (5.4)$$

where

$$V = V_n - \underline{v}\cdot\underline{n}. \qquad (5.5)$$

Since $\underline{v}\cdot\underline{n}$ is the component of the particle velocity normal to Σ_t and V_n is the speed of travel of Σ_t normal to itself through space, V is the speed with which Σ_t <u>traverses the material</u>. We refer to V as the <u>local speed of propagation</u> of the singular surface.

Substitution into (5.4) from the entries in the second and fourth columns of the table on p. 9 now leads at once to the <u>basic jump conditions</u>

$$[\rho V] = 0,$$

$$[\rho V \underline{v} + \underline{t}_{(\underline{n})}] = \underline{0}, \tag{5.7}$$

$$[\rho V(\tfrac{1}{2}\underline{v}\cdot\underline{v} + \varepsilon) + \underline{t}_{(\underline{n})}\cdot\underline{v} + h_{(\underline{n})}] = 0, \tag{5.8}$$

the condition associated with the balance of angular momentum being implied by (5.7).

If Σ is a first-order singular surface and $[\underline{v}]\cdot\underline{n} \neq 0$, it follows from (5.5) that V is discontinuous and hence, from (5.6) (rembering that $\rho > 0$) that Σ_t is in motion relative to the material both immediately ahead of it and directly to its rear. Thus Σ is a wave, in the sense of the definition adopted in Section 3, and the terminology shock <u>wave</u> introduced earlier is justified.

<u>Exercise</u>. Deduce from (5.6) to (5.8) the following results.

(i) $V^+V^-[\rho] = -[\underline{t}_{(\underline{n})}\cdot\underline{n}]$ (the Rankine-Hugoniot relation).

(ii) On a contact discontinuity $[\underline{t}_{(\underline{n})}] = \underline{0}$ and $[\underline{v}\cdot\underline{n}] = 0$.

(iii) On an acceleration wave

$$[\rho] = 0, \quad [\underline{t}_{(\underline{n})}] = \underline{0}, \quad \rho V[\varepsilon] + [h_{(\underline{n})}] = 0. \tag{5.9}$$

(iv) In an isochoric motion (for which ρ is constant at each particle) no shock wave can exist.

6. Referential and Spatial Descriptions of a Singular Surface

at time t, but it is convenient now to give Σ the representation

$$\phi(\underline{x}, t) = 0, \tag{6.1}$$

alternative to (1.1). Invoking equation (3.1), which specifies the motion of the body in which Σ is situated, we can rewrite (6.1) as

$$\phi(\underline{\psi}(\underline{X}, t), t) \equiv \Phi(\underline{X}, t) = 0. \tag{6.2}$$

Equations (6.1) and (6.2) are respectively the spatial (or Eulerian) and referential (or Lagrangian) descriptions of Σ. The surface Σ_r in E given by (6.2) passes through the points occupied in the reference configuration B_r of the body by the particles currently situated on Σ_t. Σ_r may therefore be described as the <u>referential trace</u> of Σ_t. Unlike Σ_t, Σ_r has no direct physical significance.

Let

$$G = |\mathrm{Grad}\,\Phi| = \left(\frac{\partial\Phi}{\partial X_p}\frac{\partial\Phi}{\partial X_p}\right)^{\frac{1}{2}}, \quad g = |\mathrm{grad}\,\phi| = \left(\frac{\partial\phi}{\partial x_p}\frac{\partial\phi}{\partial x_p}\right)^{\frac{1}{2}}. \tag{6.3}$$

Then, from (6.1) and (6.2), the unit normals $\underline{N}$ on Σ_r and $\underline{n}$ on Σ_t in the direction of propagation are given by

$$\underline{N} = G^{-1}\,\mathrm{Grad}\,\Phi, \quad \underline{n} = g^{-1}\,\mathrm{grad}\,\phi. \tag{6.4}$$

Also the speeds of propagation of Σ_r and Σ_t normal to themselves are

$$U_N = -G^{-1}\frac{\partial\Phi}{\partial t}, \quad V_n = -g^{-1}\frac{\partial\phi}{\partial t}, \tag{6.5}$$

To relate the normals and propagation speeds we differentiate with respect to position and time the connection (6.2) between ϕ and Φ:

$$\frac{\partial \Phi}{\partial X_i} = \frac{\partial \phi}{\partial x_p}\frac{\partial \psi_p}{\partial X_i} = F_{pi}\frac{\partial \phi}{\partial x_p}, \quad \text{i.e.} \quad \operatorname{Grad}\Phi = \underset{\sim}{F}^T \operatorname{grad}\phi, \tag{6.6}$$

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \phi}{\partial x_p}\frac{\partial \psi_p}{\partial t} + \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x_p} v_p + \frac{\partial \phi}{\partial t}, \quad \text{i.e.} \quad \frac{\partial \Phi}{\partial t} = (\operatorname{grad}\phi)\cdot \underline{v} + \frac{\partial \phi}{\partial t}. \tag{6.7}$$

With the use of equations (6.4), (6.5) and (5.5) there follow

$$G\underline{N} = g\underset{\sim}{F}^T\underline{n}, \quad G U_N = g(V_n - \underline{v}\cdot\underline{n}) = gV. \tag{6.8}$$

On eliminating G/g we obtain the required relation in the form

$$U_N^{-1}\underline{N} = V^{-1}\underset{\sim}{F}^T\underline{n}. \tag{6.9}$$

7. Further Results for First- and Second-Order Singular Surfaces

(a) First-order surfaces. In this case Σ is a first-order singular surface of the deformation function $\underline{\Psi}$. Since $\underline{\Psi}$ is a function of the referential position $\underline{X}$ and t (see equation (3.1)) we must apply the first-order compatibility conditions (2.10) on the referential trace Σ_r. Remembering to replace $\underline{n}$ by $\underline{N}$ and V_n by U_N we obtain

$$\left[\frac{\partial \psi_i}{\partial X_j}\right] = [F_{ij}] = \alpha_i N_j, \quad \left[\frac{\partial \psi_i}{\partial t}\right] = [v_i] = -U_N\alpha_i, \tag{7.1}$$

$$\text{with} \quad \alpha_i = \left[\frac{\partial \psi_i}{\partial X_p}\right]N_p = [F_{ip}]N_p. \tag{7.2}$$

In direct notation,

The vector $\underline{\alpha}$ is a measure of the <u>amplitude</u> (or <u>strength</u>) of Σ, but this concept has a more immediate physical significance when associated with the <u>velocity jump</u>

$$\underline{u} = [\underline{v}]. \tag{7.4}$$

Then $\underline{\alpha} = -U_N^{-1}\underline{u}$, from $(7.3)_2$, and, with the use of (6.9), equation $(7.3)_1$ assumes the spatial form

$$[\underset{\sim}{F}] = -V^{-1}\underline{u} \otimes (\underset{\sim}{F}^{T}\underline{n}) = -V^{-1}(\underline{u} \otimes \underline{n})\underset{\sim}{F} \tag{7.5}$$

$$\left(\text{with components } [F_{ij}] = -V^{-1}u_i F_{pj} n_p\right). \tag{7.6}$$

We develop a little further the theory of first-order singular surfaces for the case in which Σ is a shock wave advancing into a <u>region of uniform state</u>, that is a region which is everywhere at rest and in which the deformation gradient has the same value $\underset{\sim}{F}^{+}$ at each point and the heat flux is uniformly zero. Thus

$$\underline{u} = \underline{v}^{+} - \underline{v}^{-} = -\underline{v}^{-}, \tag{7.7}$$

since $\underline{v}^{+} = \underline{0}$, and, from (5.5),

$$V^{+} = V_n - \underline{v}^{+}\cdot\underline{n} = V_n, \quad V^{-} = V_n - \underline{v}^{-}\cdot\underline{n} = V_n + \underline{u}\cdot\underline{n}. \tag{7.8}$$

Evidently V_n is simply the speed of Σ_t relative to the stationary material ahead of it.

Equations (7.5) and $(7.8)_1$ give

and from the basic jump conditions (5·6) to (5·8) we obtain, with the aid of (7·7) and (7·8),

$$\rho^+ V_n = \rho^- (V_n + \underline{u}\cdot\underline{n}), \tag{7·10}$$

$$\underline{t}^+_{(\underline{n})} = \underline{t}^-_{(\underline{n})} - \rho^- (V_n + \underline{u}\cdot\underline{n})\,\underline{u} = \underline{t}^-_{(\underline{n})} - \rho^+ V_n\,\underline{u}, \tag{7·11}$$

$$\begin{aligned} \rho^+ V_n \varepsilon^+ &= \rho^+ V_n \left(\tfrac{1}{2}\underline{u}\cdot\underline{u} + \varepsilon^-\right) - \underline{t}^-_{(\underline{n})}\cdot\underline{u} + h^-_{(\underline{n})} \\ &= \rho^+ V_n \left(\varepsilon^- - \tfrac{1}{2}\underline{u}\cdot\underline{u}\right) - \underline{t}^+_{(\underline{n})}\cdot\underline{u} + h^-_{(\underline{n})}. \end{aligned} \tag{7·12}$$

Equations (7·9) to (7·12) may be regarded as determining conditions immediately behind Σ_t when $\underline{u}$ and the propagation speed V_n are known. However, when the nature of the material which transmits the singular surface is specified, $\underline{t}_{(\underline{n})}$, $h_{(\underline{n})}$ and ε can be related to functions describing the thermo-mechanical state of the body and the above equations then provide, in principle, a means of calculating V_n and the direction of the amplitude $\underline{u}$. In practice this calculation can be performed, in a reasonably complete way, only for a (compressible) inviscid fluid.

(b) Second-order surfaces. Σ is now a second-order singular surface of $\underline{\Psi}$ and the appropriate compatibility conditions are (2·11) and (2·12). Again we must apply them on the referential trace Σ_r and the resulting relations are

$$\left.\begin{aligned} &\left[\frac{\partial^2 \psi_i}{\partial X_J \partial X_K}\right] = \left[\frac{\partial F_{iJ}}{\partial X_K}\right] = \beta_i N_J N_K, \quad \left[\frac{\partial^2 \psi_i}{\partial t\, \partial X_J}\right] = [\dot{F}_{iJ}] = -U_N \beta_i N_J, \\ &\qquad\qquad \left[\partial^2 \psi_i \cdots\right] = [f_i \cdots] = U^2\ \beta \cdots \end{aligned}\right\} \tag{7·13}$$

Here $\beta_i = [\partial^2\psi_i/\partial X_p \partial X_q] N_p N_q$ and we have made use of the definitions (3.3) and (3.4). We identify the amplitude of the singular surface with the acceleration jump

$$\underline{a} = [\ddot{\underline{x}}]. \tag{7.14}$$

Equation $(7.13)_2$ then gives $\underline{\beta} = U_N^{-2}\underline{a}$ and equations $(7.13)_{1,2}$ become

$$\left[\frac{\partial F_{iJ}}{\partial X_k}\right] = U_N^{-2} a_i N_J N_k , \quad [\dot{F}_{iJ}] = -U_N^{-1} a_i N_J . \tag{7.15}$$

We conclude this section by calculating the jump $[\partial F_{iJ}/\partial x_k]$ which is needed later. First, using equation (3.1) to pass from the referential to the spatial description, we have

$$\left[\frac{\partial F_{iJ}}{\partial X_k}\right] = \left[\frac{\partial F_{iJ}}{\partial x_p}\frac{\partial \psi_p}{\partial X_k}\right] = \left[\frac{\partial F_{iJ}}{\partial x_p}\right] F_{pk} . \tag{7.16}$$

Then, postmultiplying by $\underset{\sim}{F}^{-1}$ and appealing once more to (6.9), we obtain

$$\left[\frac{\partial F_{iJ}}{\partial x_k}\right] = \left[\frac{\partial F_{iJ}}{\partial X_r}\right] F^{-1}_{rk} = U_N^{-2} a_i N_J N_r F^{-1}_{rk} = V^{-2} a_i F_{pJ} n_p F_{qr} n_q F^{-1}_{rk}$$

$$= V^{-2} a_i F_{pJ} n_p n_k . \tag{7.17}$$

Exercise. Derive the formula $\underset{\sim}{L} = \dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1}$ connecting the velocity and deformation gradients. Show that, on a second-order singular surface,

$$[\dot{\underset{\sim}{F}}] = -V^{-1}(\underline{a}\otimes\underline{n})\underset{\sim}{F} , \quad [\underset{\sim}{L}] = -V^{-1}\underline{a}\otimes\underline{n} , \quad [\dot{\rho}] = \rho V^{-1}\underline{a}\cdot\underline{n} . \tag{7.18}$$