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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS TO MECHANICS

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THEORY OF PLATES

Part V - Some boundary value problems in the theory of elastic plates

Part VI - Variational formulations

Part VII - Functional operators and their properties

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V - Some boundary value problems.

The virtual work principle (or Green's formulas) suggests various boundary problems, associated with plates theory. In this section, we describe some such problems, that we will study in the following.

If only bending problems can be considered in the Love-Kirchhoff theory, in the von Karman's one, it is necessary to consider simultaneously tension and bending problems; in this theory, we meet boundary conditions for horizontal displacements and vertical ones.

A) Horizontal displacements. On a part $\Sigma_1 \subset \partial\Omega$, we assume

$$u = 0 \quad \text{on } \Sigma_1$$

and on the complementary, we have

$$\sigma_{\alpha\beta} n_\beta = \lambda t_\alpha ,$$

where $t = (t_\alpha)$ is a density of horizontal forces at the boundary, defined on $\partial\Omega$, and λ is a real parameter which caracterizes the intensity of these external forces.

For $\lambda = 0$, we have to consider only bending problems. But if $\lambda \neq 0$, we can have buckling phenomena: the plate can bend if it is subjected at only horizontal forces, with λ sufficiently large

B) Vertical displacement.

1- Bilateral problems.

- $y = 0$, $\dfrac{\partial y}{\partial n} = 0$ on Γ_1 : the plate is clamped on Γ_1,
- $y = 0$, $M = 0$ on Γ_2 : the plate is simply supported on Γ_2,
- $F = 0$, $M = 0$ on Γ_3 : the plate is free on Γ_3.

2 - Problems with unilateral conditions

We use some examples ~~for~~ introduce general situations we will consider.

a) Unilateral conditions on Ω ; classical conditions on Γ.

Example 1. on Γ, we suppose, for ~~example~~ instance, that the plate is clamped:

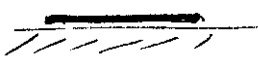
$$y = 0 \ , \quad \frac{\partial y}{\partial n} = 0 \quad \text{on } \Gamma.$$

In Ω, exterior body forces f are not ~~known~~ known ; we assume that

$$(1) \quad \begin{cases} f = p + R & (x \mapsto p(x) \text{ is given}) \\ y \geq 0, \ R \geq 0, \ y.R = 0 \end{cases}$$

This is the situation of the plate placed on a rigid support :

$y \geq 0$: vertical desplacement is restrained



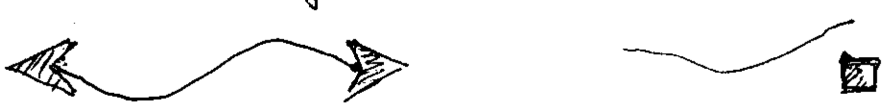
R is the reaction of the support on the plate : $R \geq 0$

$yR = 0$: $R = 0$ where the plate is not at the contact of the support.

b) Unilateral conditions on the boundary Γ (classical ones in Ω).

In this case, f is given as function defined in Ω.

We can find many examples ; for ~~example~~ instance, the vertical desplacement y (or the rotation $\frac{\partial y}{\partial n}$) is restrained by an abutment (as example 1)



Give other examples ;

Example 2

$$(2) \quad \left. \begin{cases} |M| \leq \alpha \\ |M| < \alpha \Rightarrow \dfrac{\partial y}{\partial n} = 0 \\ M = -\alpha \Rightarrow \dfrac{\partial y}{\partial n} \leq 0 \\ M = \alpha \Rightarrow \dfrac{\partial y}{\partial n} \geq 0. \end{cases} \right\} \text{ on } \Gamma$$

and the classical condition $y = 0$.

That is the situation where the plate is clamped at its boundary . On a plastic hinge

Example 3. It is ~~com~~parable with the previous:

$$
\text{ii)}\quad \begin{cases} |F| \leqslant \alpha & \\ |F| < \alpha \Rightarrow \varphi = 0 & \\ F = -\alpha \Rightarrow \varphi \geqslant 0 & \\ F = \alpha \Rightarrow \varphi \leqslant 0. & \end{cases} \qquad \text{on } \Gamma_1 \subset \Gamma
$$

and classical condition for φ on $\Gamma - \Gamma_1$ and for $\frac{\partial \varphi}{\partial n}$ on Γ, for example

$$\varphi = 0 \quad \text{on} \quad \Gamma - \Gamma_1$$

$$M = 0 \quad \text{on} \quad \Gamma$$

Here, we have the conditions of a vertical displacement with friction on Γ_1.

$\boxed{\text{Convexity}}$

To describe general situations, we must use the Convexity theory.

We recall the following (to see lectures notes of M$^{\text{me}}$ D. Cioranescu)

We have to consider $\overset{\text{function}}{j} : \mathbb{R} \to (-\infty, +\infty]$

with j is convex,

proper $(j \not\equiv +\infty)$,

and, in addition, l.s.c. (lower semi-continuous).

Définition. Let be ξ in $\mathbb{R}$.

We define the subdifferential of j at the point ξ, noted $\partial j(\xi)$:

$\partial j(\xi)$ is a part of $\mathbb{R}$ such that

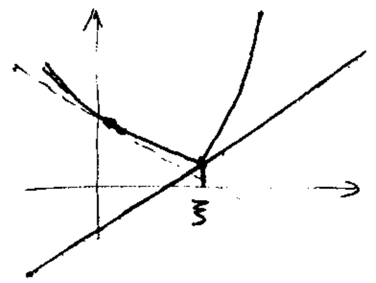
$$y \in \partial j(\xi) \iff j(\eta) \geqslant j(\xi) + y(\eta - \xi) \quad \forall \eta \in \mathbb{R}$$

Now, y is a subgradient

Interpretation.

The function $\eta \mapsto j(\xi) + y(\eta - \xi)$ is affine (ξ, y fixed).

y is a subgradient if this affine function is a minorant function of j, 'exact' at the point ξ

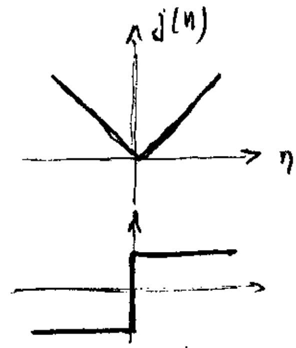


Observe that, if j is differentiable at the point ξ,

$\partial j(\xi) = \{ j'(\xi) \}$.

Remark $\partial j(\xi) \neq \phi \implies j(\xi) < +\infty$.

Example 1. $j(\eta) = |\eta|$



Then $\partial j(\eta) = \begin{cases} -1 & \text{if } \eta < 0 \\ [-1, 1] & \text{if } \eta = 0 \\ 1 & \text{if } \eta > 0. \end{cases}$

Example 2. $J(\eta) = \begin{cases} +\infty & \text{if } \eta < 0 \\ 0 & \text{if } \eta \geq 0 \end{cases}$

J is the indicator of $\mathbb{R}^+$ (*)

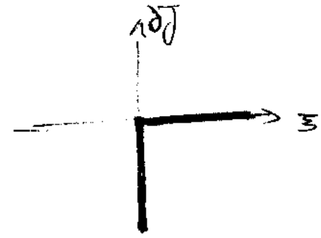
$\partial J(\eta) = \phi$ if $\eta < 0$ (because $J(\eta) = +\infty$; see previous remark)

Let be $\xi \geq 0$; $y \in \partial J(\xi) \iff 0 \geq y(\eta - \xi), \ \forall \eta \geq 0$

Then this relation implies $y = 0$ if $\xi > 0$
and $y \leq 0$ if $\xi = 0$

Therefore, we obtain:

$$\partial J(\xi) = \begin{cases} \phi & \text{if } \xi < 0 \\ \mathbb{R}^- & \text{if } \xi = 0 \\ 0 & \text{if } \xi > 0 \end{cases}$$



Or, it is the same: $\boxed{y \in \partial J(\xi) \iff \left[\begin{matrix} y \leq 0 \\ \xi \geq 0 \end{matrix} \quad y\xi = 0 \right.}$

(*) Let be K a closed convex set in an Hilbert space Z.
The function χ_K defined on Z by $\chi_K(z) = \begin{cases} 0 & z \in K \\ +\infty & z \notin K \end{cases}$

Now, let be Z an Hilbert space and ψ a convex proper l.s.c. function from Z into $(-\infty, +\infty]$.

We define as previous the ~~subgradient~~ subdifferential $\partial\psi(y)$ by:

$\partial\psi(y) \subset Z$ and

$$u \in \partial\psi(y) \iff \psi(z) \geq \psi(y) + (u, z-y) \quad \forall z \in Z$$

In the following, we will use the property:

Monotonicity of the subdifferential. (see. lectures Notes of D. Cioranescu)

Let be $u_1 \in \partial\psi(y_1)$, $u_2 \in \partial\psi(y_2)$; then:

$$(u_1 - u_2, y_1 - y_2) \geq 0$$

(When ψ is differentiable, we have
$$(\psi'(y_1) - \psi'(y_2), y_1 - y_2) \geq 0 \quad).$$

We use convexity notions ~~to~~ formulate unilateral problems; we begin by the examples

<u>Example 1.</u> Let be J the indicator of $\mathbb{R}^+$.

We put $j(\xi) = -\xi p + J(\xi)$; j is a convex function

and $\partial j(\xi) = -p + \begin{cases} \mathbb{R}^- & \text{if } \xi = 0 \\ 0 & \text{if } \xi > 0 \end{cases}$, $\partial j(\xi) = \phi$ if $\xi < 0$

The conditions (1) can be written:

$y \geq 0$, $+p - f \leq 0$, $(p-f)y = 0$

therefore $p - f \in \partial J(y)$, or

$$\boxed{-f \in \partial j(y)} \text{ in } \Omega$$

<u>Example 2.</u> Let be $k(\xi) = \alpha|\xi|$;

The conditions (2) can be written

$$\boxed{M \in \partial k\left(\frac{\partial y}{\partial n}\right) \quad \text{on } \Gamma}$$

<u>Example 3.</u> We put $l(\xi) = \alpha|\xi|$; the conditions (3) can be written

$$\boxed{-F \in \partial l(y) \quad \text{on } \Gamma_1}$$

From these examples, we deduce the general form of the problem which will be studied in the following.

Problem 1   Let be j a proper, l.s.c., convex function from $\mathbb{R}$ into $(-\infty, +\infty]$. We assume that:

$$-f \in \partial j(y) \quad \text{in } \Omega$$

In addition, boundary conditions are classical; for instance:

$$y = 0 \qquad \text{on } \Gamma$$
$$\frac{\partial y}{\partial n} = 0$$

Problem 2.   Let be k a proper, l.s.c., convex function from $\mathbb{R}$ into $(-\infty, +\infty]$. We assume that

$$M \in \partial k\left(\frac{\partial y}{\partial n}\right) \quad \text{on } \Gamma$$

In addition f is given (as function of x); and the boundary condition on y is classical; we assume that

$$y = 0 \quad \text{on } \Gamma$$

Problem 3.   Let be l a proper, l.s.c, - - - - - - - - . We assume that

$$-F \in \partial l(y) \quad \text{on } \Gamma$$

In addition f is given (as function of x), and the boundary condition on the rotation $\frac{\partial y}{\partial n}$ is classical; we assume that

$$M(y) = 0 \quad \text{on } \Gamma.$$

Remark 1. j (k, or l) can depend on x in Ω (or Γ), according to the case. Especially, in the third problem, we will have to suppose that $y = 0$ on a part Γ_1, with Γ_1 non rectilinear and measure $(\Gamma_1) > 0$; that is possible by choosing

$$\{ +\infty \quad \text{if } \xi \neq 0,$$

Remark 2. For instance, consider the condition

$$-f \in \partial j(\sigma)$$

If we suppose that j is differentiable, this relation can be written $-f = -j'(\sigma)$: exterior forces f possess a potential j. In unilateral case, we have a comparable situation : j is called a superpotential : This terminology has been introduced by J. J. MOREAU

Obviously, this remark is valid for problems 2 and 3 -

VI. Variational formulations

Variational formulations are directly arising from Green's formulas.

We write these formulations only for the non linear case; we can obtain the linear one, puting down the non linear term from the von Karman's equation.

Formal aspect. First, we give only the ideas of variational formulation; for this, we consider, for instance, the case of the problem 1, in the linear case:

$$y = 0$$

$$\frac{\partial y}{\partial n} = 0$$

$$-f \in \partial j(y).$$

The Green's formula can be written

$$a(y, z-y) - (f, z-y) = (F, z-y)_\Gamma - \left(M, \frac{\partial z}{\partial n} - \frac{\partial y}{\partial n}\right)_\Gamma \quad \forall z.$$

F and M are unknown: they are the reactions of the support which clamps the plate; the second member of Green's formula is the work of these reactions in the displacement $z-y$. It is usual only to consider displacements for which this work is zero: they are the "kinematically admissible desplacements"

$$Z_{ad} = \left\{ z \;/\; z=0,\ \frac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma \right\}$$

Green's formula gives:

$$y \in Z_{ad},\ a(y, z-y) = (f, z-y) \quad \forall z \in Z_{ad}..$$

Now, we have

$$j(z) \geq j(y) - f\,(z-y) .$$

and, by integrating:

$$\int_\Omega j(z)\, dx \geq \int_\Omega j(y)\, dx \; - \; (f, z-y)$$

We put

$$\psi(z) = \int_\Omega j(z)\, dx .$$

and we have

$$\psi(z) - \psi(y) \geq -\,(f, z-y) .$$

From Green's formula, we obtain:

$$\boxed{a(y, z-y) + \psi(z) - \psi(y) \geq 0 \qquad \forall z \in Z_{ad}}$$

That is the formal variational formulation of our problem; we have a variational inequality.

Why formal?

It is necessary to specify

- Z_{ad}: integrations (in the bilinear form a) must be possible).
- the sense of $\psi(z)$: for instance j can take the value $+\infty$; the definition of $\psi(z)$ is not clear.

Accurate variational formulation, Functional spaces. In fact, the choose of functional spaces is clear, and will be specified in each case.

Horizontal displacements must be chosen in

$V \subset (H^1(\Omega))^2$,

$(L^1(\Omega))^2$

Vertical displacement must be chosen in

$$Z_1 \subset H^2(\Omega),$$

Z_1 being a closed subspace of $H^2(\Omega)$

Boundary conditions on the horizontal displacement are

$$\sigma_{\alpha\beta} n_\beta = \lambda t_\alpha \text{ on } \Sigma_1$$
$$u_\alpha = 0 \quad \text{on } \Sigma_2$$

With $\Sigma_1 \cap \Sigma_2 = \phi$, $\Sigma_1 \cup \Sigma_2 = \Gamma$; t_α is a density of horizontal exterior forces given on Σ_1 and λ is a real parameter measuring the intensity of these load. Then

$$V = \left\{ v = (v_\alpha) \in \left(H^1(\Omega)\right)^2 \;/\; v = 0 \text{ on } \Sigma_2 \right\}.$$

Boundary conditions on the vertical displacement. According to the situations, we have:

- the plate is clamped on Γ $\Rightarrow$ $Z_1 = H^2_0(\Omega)$
- the plate is simply supported on Γ $\Rightarrow$ $Z_1 = H^2(\Omega) \cap H^1_0(\Omega)$
- a more general situation is: the plate is clamped on Γ_1, simply supported on Γ_2, free on Γ_3; then

$$Z_1 = \left\{ \zeta \in H^2(\Omega) \;/\; \zeta = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \;;\; \frac{\partial \zeta}{\partial n} = 0 \text{ on } \Gamma_1 \right\}.$$

We exclude some situations (discussed in the Book of G. Duvaut & J. L. Lions): we want (except for buckling problems) that the considered problem are coercive. The used bilinear form $a(\zeta, \zeta)$, is continuous on $H^2(\Omega)$. In addition

$$a(\zeta,\zeta) = D \int_\Omega \left\{ \nu\, \zeta_{,\gamma\gamma}\, \delta_{\alpha\beta} + (1-\nu)\, \zeta_{,\alpha\beta} \right\} \zeta_{,\alpha\beta}\, dx$$

$$= D \int \left\{ \nu (\Delta \zeta)^2 + (1-\nu)\, \zeta_{,\alpha\beta}\, \zeta_{,\alpha\beta} \right\} dx$$

because $0 < \nu < \frac{1}{2}$, $a(z,z) = 0$ implies $z_{,\alpha\beta} = 0$ in Ω. therefore $z = a + b_1 x_1 + b_2 x_2$: z is a polynom of degree 1 in Ω. Then $\{x / z(x) = 0\}$ is a straight line. If the plate is simply supported on a part Γ_S of Γ, non rectilinear there is not any polynom of degree 1 in Z_1, and

Hyp. $z \in Z_1$, $z \neq 0 \implies a(z,z) \neq 0$

In the following, we suppose this property. (In other words, the plate is sufficiently fixed and cannot possess rigid motions).

$*$ We admit the coercivity property
$$a(z,z) \geq \alpha \|z\|^2 \quad (1) \quad \forall z \in Z_1 \text{ (with } \alpha > 0\text{)}$$
when the last hypothesis is true. (to see G. Duvaut & J.L. Lions; Mikhlin, for instance).

We do the second, and third

Hyp $t_\alpha = 0$ on part of Γ where $\mathcal{I}$ is free.

Hyp $p_\alpha = 0$ (there is no horizontal body force)

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(1) $\| \ \|$ is the natural norm on $H^2(\Omega)$

Classical problems. (bilateral situations)

We assume that given external forces f are (for instance) in $L^2(\Omega)$.

The variational formulation is :

To find $y \in Z_1$, $u \in V$ such that
$$a(y,z) + h \int_\Omega \sigma_{\alpha\beta}\, y_{,\beta}\, z_{,\alpha}\, dx = (f, z) \qquad \forall z \in Z_1$$
$$\int_\Omega \sigma_{\alpha\beta}\ \varepsilon_{\alpha\beta}(v)\, dx = \int_{\Sigma_1} \lambda\, t_\alpha\, v_\alpha\, d\sigma \qquad \forall v \in V$$

This last relation can be written too

$$\int_\Omega a_{\alpha\beta\gamma\delta}\left\{\varepsilon_{\gamma\delta}(u) + \frac{1}{2}\, y_{,\gamma}\, y_{,\delta}\right\} \varepsilon_{\alpha\beta}(v)\, dx = \int_{\Sigma_1} \lambda t_\alpha v_\alpha\, d\sigma \quad \forall v \in V$$

For the linear problem, (1) is reduced to

$$a(y,z) = (f, z) \qquad \forall z \in Z_1.$$

Remark. For converses (here and in the following) we can follow the same way as in lecture Notes of Misses D. Cioranescu.

Problems with unilateral conditions

Problem 1. $Z_1 = H_0^2(\Omega)$. (because classical boundary conditions)

If $z \in Z_1$, we put

$$\psi(z) = \begin{cases} \int_\Omega j(z)\,dx & \text{if } j(z) \in L^1(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

As $-f \in \partial j(y)$, we have $j(z) \geq j(y) - f(z-y)$, and after integration :

$$\int_\Omega f(z-y)\,dx \geq \psi(y) - \psi(z)$$

Then, the variational formulation of the problem 1 introduces a variational inequality :

Problem 1 : To find $u \in V$ and $y \in Z_1$ such that

$$a(y, z-y) + h\int_\Omega \sigma_{\alpha\beta}\, y_{,\beta}(z_{,\alpha} - y_{,\alpha})\,dx + \psi(z) - \psi(y) \geq 0 \quad \forall z \in Z_1,$$

$$\int_\Omega \sigma_{\alpha\beta}(u)\,\varepsilon_{\alpha\beta}(v)\,dx = \int_{\Sigma_1} \lambda t_\alpha v_\alpha\,d\sigma \quad \forall v \in V$$

with $\sigma_{\alpha\beta}(u) = a_{\alpha\beta\gamma\delta}\, V_{\gamma\delta} = a_{\alpha\beta\gamma\delta}\left(\varepsilon_{\gamma\delta}(u) + \frac{1}{2} y_{,\gamma} y_{,\delta}\right)$

Problem 2. (If the plate is simply supported, we choose) $Z_1 = H^2(\Omega)\cap H^1_0(\Omega)$

For $z \in Z_1$, we put

$$\psi(z) = -\int_\Omega f z\,dx + \begin{cases} \int_\Gamma k\left(\frac{\partial z}{\partial n}\right) d\sigma & \text{if } k\left(\frac{\partial z}{\partial n}\right) \in L^1(\Omega), \\ +\infty & \text{otherwise}. \end{cases}$$

here, f are given exterior forces; we assume $f \in L^2(\Omega)$.

As previously, we have

$$+\int_\Gamma M\left(\frac{\partial z}{\partial n} - \frac{\partial y}{\partial n}\right) d\sigma - \int_\Omega f(z-y)\,dx \leq \psi(z) - \psi(y).$$

We deduce the variational formulation of ~~problem~~

Problem 2. To find $u \in V$ and $y \in Z_1$ such that:

$$a(y, z-y) + h\int_\Omega \sigma_{\alpha\beta}\, y_{,\beta}(z_{,\alpha} - y_{,\alpha})\,dx + \psi(z) - \psi(y) \geq 0 \quad \forall z \in Z_1,$$

$$\int_\Omega \sigma_{\alpha\beta}(u)\,\varepsilon_{\alpha\beta}(v)\,dx = \int_{\Gamma_1} \lambda\, t_\alpha v_\alpha\,d\sigma \qquad \forall v \in V$$

where $\sigma_{\alpha\beta}(u) = a_{\alpha\beta\gamma\delta}\left(\varepsilon_{\gamma\delta}(u) + \frac{1}{2} y_{,\gamma} y_{,\delta}\right)$

Problem 3 It is here that the previous hypothesis is used:

On Γ_1, Γ_1 non rectilinear and measure $(\Gamma_1) > 0$, we suppose $y = 0$ (the plate is simply supported on Γ_1)

Then $Z_1 = \{ z \in H^2(\Omega) \,/\, z = 0 \text{ on } \Gamma_1 \}$

Let be Γ_2 the complementary part; we have $-F \in \partial\ell(y)$ on Γ_2. We put

$$\psi(z) = -\int_\Omega f z\,dx + \begin{cases} \int_{\Gamma_2} \ell(z)\,dx & \text{if } \ell(z) \in L^1(\Gamma_2), \end{cases}$$

where f are given exterior forces ; we assume $f \in L^2(\Omega)$

As above, we obtain the variational formulation of the

Problem 3. To find $u \in V$, $\zeta \in Z_1$, such that

$$a(\zeta, \xi - \zeta) + h \int_\Omega \sigma_{\alpha\beta} \zeta_{,\beta} (\xi_{,\alpha} - \zeta_{,\alpha}) \, dx + \psi(\xi) - \psi(\zeta) \geq 0 \quad \forall \xi \in Z_1,$$

$$\int_\Omega \sigma_{\alpha\beta} \varepsilon_{\alpha\beta}(v) \, dx = \int_{\Gamma_1} \lambda t_\alpha v_\alpha \, d\sigma$$

$$\sigma_{\alpha\beta} := a_{\alpha\beta\gamma\delta} \left(\varepsilon_{\gamma\delta}(u) + \frac{1}{2} \zeta_{,\gamma} \zeta_{,\delta} \right)$$

We are going to study the previous problem, following this scheme.

Recall the variational formulation : $\mathcal{Y} \in Z_1$, $u \in V$ such that

(1) $$a(\mathcal{Y}, \zeta-\mathcal{Y}) + h\int_\Omega \sigma_{\alpha\beta}\mathcal{Y}_{,\beta}(\zeta_{,\alpha}-\mathcal{Y}_{,\alpha})\,dx + \psi(\zeta)-\psi(\mathcal{Y}) \geq 0 \quad \forall \zeta \in Z_1,$$

(2) $$\int_\Omega \sigma_{\alpha\beta}\,\varepsilon_{\alpha\beta}(v)\,dx = \lambda\int_\Gamma t_\alpha v_\alpha\, d\sigma \qquad \forall v \in V,$$

3) $$\sigma_{\alpha\beta} = a_{\alpha\beta\gamma\delta}\left(\varepsilon_{\gamma\delta}(u) + \tfrac{1}{2}\mathcal{Y}_{,\gamma}\mathcal{Y}_{,\delta}\right)$$

1- For each fixed $\mathcal{Y} \in Z_1$, (3) and (4) is a classical problem of linear elasticity in the plane of the plate. The solution of this problem can be written

(4) $$\begin{cases} u = \lambda u^0 + w(\mathcal{Y}), \\ \sigma_{\alpha\beta} = \lambda\sigma^0_{\alpha\beta} + s_{\alpha\beta}(\mathcal{Y}). \end{cases}$$

2- (1) and (4) give

(5) $$\begin{cases} a(\mathcal{Y},\zeta-\mathcal{Y}) + \underline{\lambda h\int_\Omega \sigma^0_{\alpha\beta}\mathcal{Y}_{,\beta}(\zeta_{,\alpha}-\mathcal{Y}_{,\alpha})\,dx} + \underline{h\int_\Omega s_{\alpha\beta}(\mathcal{Y})\mathcal{Y}_{,\beta}(\zeta_{,\alpha}-\mathcal{Y}_{,\alpha})\,dx} + \\ \quad + \psi(\zeta)-\psi(\mathcal{Y}) \geq 0 \quad \forall \zeta \in Z_1. \end{cases}$$

That is a problem only for $\mathcal{Y}$.

3- It is necessary to study underlined terms. By Riesz Theorem, we can write

$$h\int_\Omega \sigma^0_{\alpha\beta}\mathcal{Y}_{,\beta}\zeta_{,\alpha}\,dx = a(L\mathcal{Y},\zeta)$$

$$h\int_\Omega s_{\alpha\beta}(\mathcal{Y})\mathcal{Y}_{,\beta}\zeta_{,\alpha}\,dx = a(C\mathcal{Y},\zeta)$$

where L (resp. C) is a linear (resp. non linear) operator on Z_1. We must study properties of L and C.

5- Because the coercivity, we can choose $a(\zeta_1,\zeta_2)$ as the scalar product on Z_1, and the problem is reduced to

6) $$\boxed{((I-\lambda L)\mathcal{Y} + C\mathcal{Y}, \zeta-\mathcal{Y}) + \psi(\zeta)-\psi(\mathcal{Y}) \geq 0 \quad \forall \zeta \in Z_1}$$

VII - Functional operators and their properties.

1. Elimination of u.

In various considered problems, we have two variational (in) equations; the unknown are u and y. The principle of the study is to eliminate u. For this, we consider the second equation

$$(1) \quad \begin{cases} u \in V, & \int_\Omega \sigma_{\alpha\beta}\, \varepsilon_{\alpha\beta}(v)\, dx = \lambda \int_\Gamma t_\alpha v_\alpha \, d\sigma \qquad \forall v \in V \\ \sigma_{\alpha\beta} = \varepsilon_{\alpha\beta}(u) + \frac{1}{2} y_{,\alpha} y_{,\beta}. \end{cases}$$

This is a plane elasticity problem that we separate into two such problems.

<u>First problem</u>

To find $w^0 \in V$ such that

$$(2) \quad \begin{cases} \int_\Omega \sigma^0_{\alpha\beta}\, \varepsilon_{\alpha\beta}(v)\, dx = \int_\Gamma t_\alpha v_\alpha \, d\sigma \qquad \forall v \in V \\ \sigma^0_{\alpha\beta} = a_{\alpha\beta\gamma\delta}\, \varepsilon_{\gamma\delta}(w^0) \end{cases}$$

<u>Second problem.</u> Let fixed z in Z_1 (or in $H^2(\Omega)$). To find $w = w(z) \in V$ such that

$$(3) \quad \begin{cases} \int_\Omega s_{\alpha\beta}(z)\, \varepsilon_{\alpha\beta}(v)\, dx = 0 \qquad \forall v \in V \\ s_{\alpha\beta} = a_{\alpha\beta\gamma\delta}\left(\varepsilon_{\gamma\delta}(w) + \frac{1}{2} z_{,\gamma} z_{,\delta}\right). \end{cases}$$

It is clear that the solution of (1) is:

$$\begin{cases} u = \lambda w^0 + w(y) \\ \sigma_{\alpha\beta} = \lambda \sigma^0_{\alpha\beta} + s_{\alpha\beta}(y) \end{cases}$$

After the resolution of (2) and (3), we reintroduce $u = u(y)$ in the first variational inequality (for y); we obtain a new formulation only for y, which is:

$$(4) \quad \begin{cases} a(y, z - y) + \lambda h \int_\Omega \sigma^0_{\alpha\beta}\, y_{,\beta}(z_{,\alpha} - y_{,\alpha})\, dx + \end{cases}$$

Resolution of (2) and (3).

The problem (2) is a classical problem of linear elasticity, without body forces, but with non homogeneous boundary conditions. The problem (3) have homogeneous boundary conditions, but body forces are

$$- \frac{1}{2} a_{\alpha\beta\gamma\delta} \zeta_{,\gamma} \zeta_{,\delta}$$

We can apply general results for problems of linear elasticity (to see the book of G. Duvaut and J. L. Lions), if $\zeta_{,\gamma}\zeta_{,\delta}$ is in $L^2(\Omega)$ (because $a_{\alpha\beta\gamma\delta}$ are constant).

This result is true via the embedding theorem

•) Thm(1) In the plane (dimension = 2) $H^1(\Omega) \subset L^4(\Omega)$; in addition, the injection from $H^1(\Omega)$ into $L^4(\Omega)$ is compact.

By this theorem, $\zeta_{,\gamma}\zeta_{,\delta}$ is in $L^2(\Omega)$; $a_{\alpha\beta\gamma\delta} \zeta_{,\gamma}\zeta_{,\delta} \varepsilon_{\alpha\beta}(v)$ is in $L^1(\Omega)$ for each v in V, and

$$v \mapsto \int_\Omega a_{\alpha\beta\gamma\delta} \zeta_{,\gamma}\zeta_{,\delta} \varepsilon_{\alpha\beta}(v)\,dx$$

is a continuous linear form on V.

The conclusions are:

$$(6) \begin{cases} w^0, w(\zeta) \text{ exist, and are unique (except for an additive rigid displacement)} \\ \sigma^0_{\alpha\beta}, \sigma_{\alpha\beta}(\zeta) \in L^2(\Omega), \text{ and are unique.} \end{cases}$$

In addition

$$(7) \begin{cases} \sigma_{\alpha\beta}(t\zeta) = t^2 \sigma_{\alpha\beta}(\zeta) \\ w(t\zeta) = t^2 w(\zeta) \end{cases} \qquad \forall t \in \mathbb{R}.$$

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(1) To see, for instance, the book
J. NEČAS - Les méthodes directes dans la théorie des

<u>Lemma 1</u> $\quad z \mapsto s_{\alpha\beta}(z)$ is <u>compact</u> from Z_1 into $L^2(\Omega)$.

<u>Proof</u>. We have , for all $v \in V$:

$$\int_\Omega a_{\alpha\beta\gamma\delta}\, \varepsilon_{\gamma\delta}(u)\, \varepsilon_{\alpha\beta}(v)\,dx = -\int_\Omega \tfrac{1}{2} a_{\alpha\beta\gamma\delta}\, z_{,\gamma} z_{,\delta}\, \varepsilon_{\alpha\beta}(v)\,dx = L(v);$$

L is in V', dual of V; L is associated with z.

Let be z^n in Z_1, $z^n \to z$ in Z_1 weak.

u^n, the corresponding solution ; L^n the corresponding linear form. If we can proove that $L^n \to L$ in V', then $u^n \to u$ in V (via continuity with respect the datas) and $\varepsilon_{\alpha\beta}(u^n) \to \varepsilon_{\alpha\beta}(u)$ in $L^2(\Omega)$ (strong)

Then, with (5),

$$s_{\alpha\beta}(z^n) \to s_{\alpha\beta} \quad \text{in } L^2(\Omega) \quad \text{strong} -$$

Therefore, we study $-2(L^n - L)$. We have

$$z^n_{,\gamma} z^n_{,\delta} - z_{,\gamma} z_{,\delta} = z^n_{,\gamma}\,(z^n_{,\delta} - z_{,\delta}) + z_{,\delta}\,(z^n_{,\gamma} - z_{,\gamma});$$

then :

$$2\left|(L^n - L)(v)\right| \leq c\left\{ \|z^n_{,\gamma}\|_{L^4} \|z^n_{,\delta} - z_{,\delta}\|_{L^4} + \|z_{,\delta}\|_{L^4} \|z^n_{,\gamma} - z_{,\gamma}\|_{L^4}\right\} \|v\|_V,$$

and

$$2\left|(L^n - L)v\right| \leq c\ \underbrace{\left\{\|z^n\|_{W^{1,4}(\Omega)} + \|z\|_{W^{1,4}(\Omega)}\right\}}_{\text{bounded}}\ \underbrace{\|z^n - z\|_{W^{1,4}(\Omega)}}_{\downarrow\, 0}\ \underbrace{\|v\|_V}_{\text{fixed}}.$$

Therefore

$$\|L^n - L\|_{V'} = \sup_{\|v\|=1} \left|(L^n - L)v\right| \to 0.$$

▨

Elimination of u gives the variational formulation of the problem for y:

$$\begin{cases} a(y, z-y) + \lambda h \int_\Omega \sigma^0_{\alpha\beta}\, y_{,\beta} (z_{,\alpha} - y_{,\alpha})\, dx + \\ + h \int_\Omega s_{\alpha\beta}(y)\, y_{,\beta} (z_{,\alpha} - y_{,\alpha})\, dx + \psi(z) - \psi(y) \geq 0 \quad \forall z \in Z_1 \end{cases}$$

2- Functional operators.

Previously, we have observed that the bilinear form $a(z_1, z_2)$ is coercive (it is an hypothesis; to see p. 13), and continuous on Z_1. Therefore, $a(z_1, z_2)$ is a scalar product on Z_1 in each situation we consider; in addition, $a(z,z)^{1/2}$ is a norm equivalent to the natural norm on Z_1. In the following, we take $a(z_1, z_2)$ as scalar product on Z_1, and we put

$$a(z_1, z_2) = (z_1, z_2)$$

The related norm is noted $\| \ \|$.

Now, consider the applications from Z_1, with fixed y in Z_1:

$$z \mapsto \int_\Omega \sigma^0_{\alpha\beta}\, y_{,\beta}\, z_{,\alpha}\, dx$$

and

$$z \mapsto \int_\Omega s_{\alpha\beta}(y)\, y_{,\beta}\, z_{,\alpha}\, dx .$$

We have

$$\left| \int_\Omega \sigma^0_{\alpha\beta}\, y_{,\beta}\, z_{,\alpha}\, dx \right| \leq |\sigma^0|_{(L^2(\Omega))^6} \, |y|_{W^{1,4}} \, |z|_{W^{1,4}} \leq c_1 \|z\|_Z$$

$$\left| \int_\Omega s_{\alpha\beta}(y)\, y_{,\beta}\, z_{,\alpha}\, dx \right| \leq |s|_{(L^2(\Omega))^6} \, |y|_{W^{1,4}} \, |z|_{W^{1,4}} \leq c_2 \|z\|$$

Via Riesz Theorem, there exists $Ly \in Z_1$ and $Cy \in Z_1$ such that

(8) $$\begin{cases} h \int_\Omega \sigma^0_{\alpha\beta}\, y_{,\beta}\, z_{,\alpha}\, dx = -(Ly, z) & \forall z \in Z_1. \end{cases}$$

With these notations, the formulation (4) can be written in the form :

$$\boxed{y \in Z,\ \big((I-\lambda L)y + Cy,\ z-y\big) + \psi(z) - \psi(y) \geq 0,\ \forall z \in Z} \tag{9}$$

In the following, the problems are studied from this form.

<u>Remark</u>. In the bilateral case, we have only

$$(I-\lambda L)y + Cy = p$$

When ψ is differentiable, we have

$$(I-\lambda L)y + Cy + \psi'(y) = 0$$

Obviously, it is necessary to study the properties of functional operators L and C.

Theorem 1 (properties of L).
L is a bounded, self adjoint, and compact linear operator on Z.

Theorem 2 (properties of C).

(i) $C(\tau z) = \tau^3\, C(z) \quad \forall z \in Z,\ \tau \in \mathbb{R}$;

(ii) C is continuous ;

(iii) C is compact ;

(iv) $z \mapsto (Cz, z)$ is differentiable on Z; its differential is the application $h \mapsto 4(Cz, h)$;

(v) $(Cz, z) \geq 0$;

(vi) $(Cz, z) > 0$, if $z \neq 0$ (in fact, this is an assumption which will be analysed in the proof)

Proofs.

Proof of the theorem 1. We have, from (8), $(Ly, z) = (Lz, y)$, and L is selfadjoint. The non trivial result is the compactness of L: we have

$$\left| \int_\Omega \sigma^0_{\alpha\beta}\, y_{,\beta}\, z_{,\alpha}\, dx \right| = |(Ly, z)| \leq c\, \|y\|_{W^{1,4}(\Omega)} \|z\|_{W^{1,4}(\Omega)} ;$$

then

$$\|Ly\| = \sup_{\|z\|=1} |(Ly, z)| \leq c\, \|y\|_{W^{1,4}(\Omega)} .$$

The compactness follows, via (5) (if $y^n \to 0$ in Z weak, then $y^n \to 0$ in $W^{1,4}$ strong and $Ly \to 0$ in Z strong).

Proof of theorem 2.

(i) from $s_{\alpha\beta}(\tau z) = \tau^2 s_{\alpha\beta}(z)$.

(ii) and (iii) are prooved simultaneously (then, C is completely continuous = continuous + compact).

We must proove that if $z^n \to z$ in Z weak, then $Cz^n \to Cz$ in Z strong.

We put $s_{\alpha\beta}(z^n) = s^n_{\alpha\beta}$, and we write

$$s^n_{\alpha\beta}\, y^n_{,\beta} - s_{\alpha\beta}\, y_{,\beta} = (s^n_{\alpha\beta} - s_{\alpha\beta})\, y^n_{,\beta} + s_{\alpha\beta}(y^n_{,\beta} - y_{,\beta}).$$

We have

$$|(Cz^n - Cz, w)| = \left| \int_\Omega (s^n_{\alpha\beta}\, y^n_{,\beta} - s_{\alpha\beta}\, y_{,\beta})\, w_{,\alpha}\, dx \right|$$

$$\leq |s^n_{\alpha\beta} - s_{\alpha\beta}|_{L^2(\Omega)} \, |y^n|_{W^{1,4}(\Omega)} \|w\| + |s_{\alpha\beta}|_{L^2(\Omega)} |y^n - y|_{W^{1,4}(\Omega)} \|w\|$$

Now

$$\|Cz^n - Cz\| = \sup_{\|w\|=1} |(Cz^n - Cz, w)|$$

$$\leq \underbrace{|s^n_{\alpha\beta} - s_{\alpha\beta}|_{L^2(\Omega)}}_{\downarrow \atop 0\ (\text{by lemm[illegible]})} \underbrace{\|y^n|_{W^{1,4}}}_{\downarrow \atop \text{bounded in } Z} + |s_{\alpha\beta}|_{L^2(\Omega)} \underbrace{|y^n - y|_{W^{1,4}}}_{\downarrow \atop 0\ \text{by (5) (p. 20).}}$$

(iv) We must compute

$$(C(\zeta+h),\zeta+h)-(C\zeta,\zeta),$$

or, it is the same

$$\int_\Omega \left\{ s_{\alpha\beta}(\zeta+h)(\zeta_{,\beta}+h_{,\beta})(\zeta_{,\alpha}+h_{,\alpha}) - s_{\alpha\beta}(\zeta)\zeta_{,\beta}\zeta_{,\alpha} \right\} dx$$

Previously, it has been observed that $s(\zeta)$ is quadratic. Now, it is useful to bilinearize this quadratic function. For this, we consider the intermediate problem:

$$\left|\begin{array}{l} f, g \text{ are given in } Z_1; \quad \text{to find } w = w(f,g) \in Z_1 \text{ such that:} \\ s_{\alpha\beta} = s_{\alpha\beta}(f,g) = a_{\alpha\beta\gamma\delta}\left(\varepsilon_{\gamma\delta}(w) + \frac{1}{2} f_{,\gamma}\, g_{,\delta}\right) \\ s_{\alpha\beta,\beta} = 0 \\ s_{\alpha\beta}\, n_\beta = 0 \text{ on the part } \Sigma_1 \end{array}\right.$$

(Compare with (3), p. 19). Obviously,

$$w(\zeta) = w(\zeta,\zeta) \quad \text{and} \quad s_{\alpha\beta}(\zeta) = s_{\alpha\beta}(\zeta,\zeta).$$

By the symmetry relations $a_{\alpha\beta\gamma\delta} = a_{\alpha\beta\delta\gamma} = a_{\gamma\delta\alpha\beta}$, we have

$$s_{\alpha\beta}(f,g) = s_{\alpha\beta}(g,f)$$

Using the classical reciprocity theorem of linear elasticity, we obtain the

<u>Lemma</u>. $\displaystyle\int_\Omega s_{\alpha\beta}(f,g)\, \bar f_{,\alpha} \bar g_{,\beta}\, dx = \int_\Omega s_{\alpha\beta}(\bar f,\bar g)\, f_{,\alpha} g_{,\beta}\, dx \quad \forall f, g, \bar f, \bar g \in Z_1$

<u>Proof of Lemma</u>. The reciprocity theorem says:

$$\int_\Omega \bar F_i U_i\, dx + \int_{\partial\Omega} \bar f_i U_i\, d\sigma = \int_\Omega F_i \bar U_i\, dx + \int_{\partial\Omega} f_i \bar U_i\, d\sigma \; ;$$

we consider the same elastic material in two states:

	first state :	second state
body forces	F_i	$\bar F_i$
boundary "	f_i	$\bar f_i$
displacement	U_i	$\bar U_i$

and the reciprocity theorem says that
the work of forces (F_i, f_i) in the displacements field $\bar{v}_i$
is the same as the work of $(\bar{F}_i, \bar{f}_i)$ in the displacement field v_i.

The previous problem is:

$$\left(a_{\alpha\beta\gamma\delta}\, \varepsilon_{\gamma\delta}(w)\right)_{,\beta} = -\frac{1}{2}\left(a_{\alpha\beta\gamma\delta}\, f_{,\gamma} g_{,\delta}\right) = F_\alpha$$

and we have

$$\int_\Omega F_\alpha \bar{w}_\alpha\, dx = \int_\Omega w_\alpha \bar{F}_\alpha\, dx$$

let

$$\int_\Omega \left(a_{\alpha\beta\gamma\delta} \bar{f}_{,\gamma} \bar{g}_{,\delta}\right)_{,\beta} w_\alpha\, dx = \int_\Omega \left(a_{\alpha\beta\gamma\delta} f_{,\gamma} g_{,\delta}\right)_{,\beta} \bar{w}_\alpha\, dx$$

or

$$\int_\Omega a_{\alpha\beta\gamma\delta} \bar{f}_{,\gamma} \bar{g}_{,\delta} w_{\alpha,\beta}\, dx = \int_\Omega a_{\alpha\beta\gamma\delta} f_{,\gamma} g_{,\delta} \bar{w}_{\alpha,\beta}\, dx$$

but, by symmetry properties of elasticity coefficients a, we have

$$\int_\Omega a_{\alpha\beta\gamma\delta}\, \varepsilon_{\gamma\delta}(\bar{w})\, \varepsilon_{\alpha\beta}(w)\, dx = \int_\Omega a_{\alpha\beta\gamma\delta}\, \varepsilon_{\gamma\delta}(w)\, \varepsilon_{\alpha\beta}(\bar{w})\, dx$$

by addition of these two last relations, we obtain the result and the lemma is proved. ▨

Now, we return to the proof of the property (iv).
We compute $(C(y+h), y+h) - (Cy, y)$; for this, we write the integrand, ordering the terms according to the "powers" of h, using

$$s(y+h) = s(y) + 2\, s(y,h) + s(h).$$

We have

$$s_{\alpha\beta}(y+h)(y_{,\beta} + h_{,\beta})(y_{,\alpha} + h_{,\alpha}) - s_{\alpha\beta}\, y_{,\beta}\, y_{,\alpha} =$$

$$s_{\alpha\beta}(y)\left(y_{,\alpha} h_{,\beta} + y_{,\beta} h_{,\alpha}\right) + \quad \left(\longrightarrow \text{ this term is } 2\, s_{\alpha\beta}(y)\, y_{,\alpha} h_{,\beta} \text{ by symmetry}\right)$$

$$+\ 2\, s_{\alpha\beta}(y;h)\, y_{,\alpha} y_{,\beta} \ +$$

$$+\ s_{\alpha\beta}(y)\, h_{,\alpha} h_{,\beta} + 2\, s_{\alpha\beta}(y,h)\left(y_{,\beta} h_{,\alpha} + y_{,\alpha} h_{,\beta}\right) + s_{\alpha\beta}(h)\, y_{,\alpha} y_{,\beta} +$$

We conserve only the terms of first order in h (because the differentiability is not a problem); this term is:

$$2\int_\Omega \{ s_{\alpha\beta}(y)\, y_{,\alpha} h_\beta + s_{\alpha\beta}(y,h)\, y_{,\alpha} y_{,\beta} \}\, dx$$

$$= 4\int_\Omega s_{\alpha\beta}(y)\, y_{,\alpha} h_{,\beta}\, dx \qquad \text{(by previous lemma)}$$

$$= 4\ (Cy, h),$$

and the property (iv) is prooved. ☐

$\boxed{(v)}$ First $(Cz, z) \geq 0$, or, it is the same: $\int_\Omega s_{\alpha\beta}(z) z_{,\alpha} z_{,\beta}\, dx \geq 0$.

We have (from $s_{\alpha\beta,\beta} = 0$, $s_{\alpha\beta} n_\beta = 0$ on Γ_1)

$$\int_\Omega s_{\alpha\beta}\, \varepsilon_{\alpha\beta}(w)\, dx = 0 \qquad \forall w \in V.$$

By addition, we have

$$\frac{1}{2}\int_\Omega s_{\alpha\beta}\, z_{,\alpha} z_{,\beta}\, dx = \int_\Omega s_{\alpha\beta}(z) \left(\varepsilon_{\alpha\beta}(w) + \frac{1}{2} z_{,\alpha} z_{,\beta}\right) dx$$

$$= \int_\Omega s_{\alpha\beta}\, s_{\alpha\beta}\, dx \geq 0.$$

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Second. $Cz \neq 0$ implies $(Cz, z) \neq 0$: because if $(Cz, z) = 0$, then z is solution of the variational problem

$$0 = (Cz, z) = \inf_Z (Cw, w)$$

, by the first, and Cz (which is the differential de (Cz, z)) $= 0$.

Third. By the last, it is sufficiently to proove

$$z \neq 0 \implies Cz \neq 0$$

But $Cz = 0 \iff (Cz, h) = 0 \quad \forall h \in Z$

$$\iff \int_\Omega s_{\alpha\beta}(z)\, z_{,\beta} h_{,\alpha}\, dx = 0 \quad \forall h \in Z$$

and, since $\int_\Omega s_{\alpha\beta}(z)\, \varepsilon_{\alpha\beta}(w)\, dx = 0$, we obtain by addition:

Therefore, we have

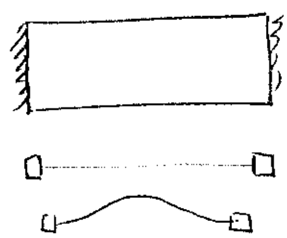
$$\boxed{C_z = 0 \iff \exists\, u \in V \text{ such that } \varepsilon_{\alpha\beta}(u) + \tfrac{1}{2} z_{,\alpha} z_{,\beta} = 0}$$

But $v_{\alpha\beta} = \varepsilon_{\alpha\beta}(u) + \frac{1}{2} z_{,\alpha} z_{,\beta}$ is the metric tensor of the plate, after deformation, and

(*) To say $\exists\,(u, z) \neq 0$, with $v_{\alpha\beta} = 0$ is equivalent to say: "the plate can bend without variation of length"

It is clear that, if the plate is <u>sufficiently fixed</u> at its boundary, the event (*) is not possible. For example, that is not possible if the plate is clamped

On the contrary, there exists some situations where (*) is possible. For example, a rectangular plate, clamped on two opposite bases, and free on other ones



For more accurate study, to see
M. Potier-Ferry, C.R. Ac. Sc., Paris, A, 281, p 1317 - 1319 and p. 1385 - 1387

For us, we assume in the following that

<u>Hyp</u> $$(C z, z) > 0 \qquad \forall\, z \in Z_1,\ z \neq 0$$

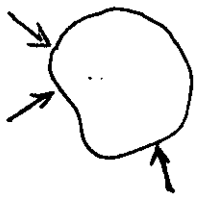
(or, it is the same, (*) is not possible).

In addition, we assume that

<u>Hyp</u> $$\begin{cases} (Lz,z) \geqslant 0 & \text{on } \mathbf{Z} ; \\ (Lz,z) > 0 & \text{if } z \neq 0 \end{cases}$$

this hypothesis (not indispensable) simplify the study.

For example, consider a plate with uniform pressure p at the edge.



We have $$\begin{cases} \sigma^{\circ}_{\alpha\beta,\beta} = 0 \\ \sigma^{\circ}_{\alpha\beta} = a_{\alpha\beta\gamma\delta}\,\varepsilon_{\gamma\delta}(w^{\circ}) \\ \sigma^{\circ}_{\alpha\beta}\, n_\beta = -p\, n_\alpha . \end{cases}$$

$\sigma^{\circ}_{\alpha\beta} = -p\,\delta_{\alpha\beta}$ is the solution. For this, we have

$$-(Ly,z) = -p\int_\Omega y_{,\alpha}\, z_{,\alpha}\, dx = -p\int_\Omega \operatorname{grad} y \operatorname{grad} z\, dx .$$

Then $(Lz,z) = p\int_\Omega |\operatorname{grad} z|^2 dx \geqslant 0$,

and, for example, $(Lz,z) > 0$ if $z \in H^1_0(\Omega)$, $z \neq 0$.

$=$

Now, we summarize previous results ; it is the same to put an abstract problem, with assumptions which are satisfied for the previous concrete situations.

Z is an Hilbert space, with scalar product $(\ ,\ )$ and related norm $\|\ \|$.

I is the identity on Z.

L is a compact self adjoint linear operator on Z.

C is a non linear operator on Z, with following properties:

$$C(\tau z) = \tau^3 Cz \qquad \forall z \in Z,\ \tau \in \mathbb{R};$$

C is continuous and compact;

$$\begin{cases} (Cz, z) = \Phi(z) \text{ is Frechet-differentiable on } Z; \\ \text{its differential is } \quad h \mapsto 4(Cz, h) \end{cases}$$

$$(Cz, z) \geq 0 \quad ; \quad (Cz, z) > 0 \quad \forall\, z \neq 0$$

ψ is a proper convex function on Z, lower semi continuous.

We put the

Problem To find $y \in Z$ such that

$$\big((I - \lambda L) y + Cy\,,\ z - y\big) + \psi(z) - \psi(y) \geq 0 \quad \forall z \in Z$$

Remark For $\lambda = 0$ (or λ sufficiently small) this problem is "coercive"; it is a bending problem. We can hope to use classical methods in coercive situations (Galerkin process, for example), although there is a non linear term Cy.

But for λ sufficiently large, the coercivity is missed; we have a buckling problem.