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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS TO MECHANICS

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MATHEMATICAL RESULTS FOR BINGHAM FLUIDS
(Part IV)

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§ 6. Continuous dependence on the data and on the parameters.

Theorem 1 (continuous dependence on the data)

The solution u of the Problem (5.1) depends continuously on $u_0 \in H$ and $f \in L^2(0,T;V')$ in the space $L^2(0,T;V) \cap L^\infty(0,T;H)$ i.e.:

$$|u_1 - u_2|^2_{L^\infty(0,T;H)} + \|u_1 - u_2\|^2_{L^2(0,T;V)} \leq$$

$$\leq C_1 |u_0^1 - u_0^2|^2_H + C_2 \|f - g\|^2_{L^2(0,T;V')}$$

if u_1 (resp. u_2) is the solution of (5.1) with data u_0^1, f (resp u_0^2, g).

Proof.

We have u_1 and u_2 solutions of the inequalities:

$$(6.1)\quad \begin{cases} (u_1', v-u_1) + \mu a(u_1, v-u_1) + b(u_1,u_1,v-u_1) + gj(v) - gj(u_1) \geq \\ \qquad \geq (f, v-u_1) \\ u_1(0) = u_0^1 \end{cases}$$

and

$$(6.2)\quad \begin{cases} (u_2', v-u_2) + \mu a(u_2, v-u_2) + b(u_2,u_2,v-u_2) + gj(v) - gj(u_2) \geq \\ \qquad \geq (g, v-u_2) \\ u_2(0) = u_0^2 \end{cases}$$

We shall take $v = u_2$ in (6.1) and $v = u_1$ in (6.2) and if we add $\Rightarrow$

$$((u_1-u_2)', u_1-u_2) + \mu a(u_1-u_2, u_1-u_2) -$$

$$- b(u_1-u_2, u_1, u_1-u_2) \leq (f-g, u_1-u_2) \leq$$

$$\leq |f-g|_* \|u_1-u_2\| \leq$$

It follows that

$$(6.3)\begin{cases} \frac{1}{2}\frac{d}{dt}|U_1-U_2|^2+\frac{3\mu}{4}\|U_1-U_2\|^2 \le C|f-g|_*^2 + b(U_1-U_2,U_1,U_1 \\ \le C|f-g|_*^2 + \tilde{C}\|U_1-U_2\|\,|U_1-U_2|\,\|U_1\| \le \\ \le C|f-g|_*^2 + \bar{C}|U_1-U_2|^2\|U_1\|^2 + \frac{\mu}{4}\|U_1-U_2\|^2 \end{cases}$$

therefore

$$\frac{1}{2}\frac{d}{dt}|U_1-U_2|^2 \le C|f-g|_*^2 + \bar{C}|U_1-U_2|^2\|U_1\|^2$$

and after integration from 0 to t:

$$\frac{1}{2}|U_1(t)-U_2(t)|^2 \le \frac{1}{2}|U_1(0)-U_2(0)|^2 + C\int_0^T |f-g|_*^2\,dt + \bar{C}\int_0^T \|U_1\|^2|U_1-U_2|^2\,dt =$$

$$= \frac{1}{2}|U_0^1-U_0^2|^2 + C\|f-g\|^2_{L^2(0,T;V')} + \bar{C}\int_0^T \|U_1\|^2|U_1-U_2|^2\,dt$$

and we can apply Gronwall's inequality with

$$A = |U_0^1-U_0^2| + 2C\|f-g\|^2_{L^2(0,T;V')}$$

$$\varphi(t) = |U_1(t)-U_2(t)|^2$$

$$B(s) = 2\bar{C}\|U_1(s)\|^2 \in L^1(0,T)$$

$\Longrightarrow$

$$|U_1(t)-U_2(t)|^2 \le \left(|U_0^1-U_0^2|^2 + 2C\|f-g\|^2_{L^2(0,T;V')}\right)\times e^{2\int_0^t \bar{C}\|U_1(s)\|^2 ds}$$

But $U_1 \in L^2(0,T;V)$ so that $e^{\int_0^t \bar{C}\|U_1(s)\|^2 ds} \le$ const.

$\Rightarrow$

$$|U_1(t)-U_2(t)|^2 \le c_1 \, |u_0^1 - u_0^2|^2 + c_2 \, \| f-g \|^2_{L^2(0,T;V')}$$

From (6.3) it follows that

$$\frac{1}{2}\frac{d}{dt}|U_1(t)-U_2(t)|^2 + \frac{\mu}{2}\|U_1(t)-U_2(t)\|^2 \le$$
$$\le c|f-g|_*^2 + \bar{c}|u_1-u_2|^2 \|u_1\|^2.$$

and therefore

$$|U_1(T)-U_2(T)|^2 + \mu \int_0^T \|u_1(s)-u_2(s)\|^2 ds \le$$
$$\le 2c\|f-g\|^2_{L^2(0,T;V')} + |u_0^1-u_0^2| +$$
$$+ 2\int_0^T \bar{c}\, |u_1-u_2|^2 \, \|u_1\|^2 ds \quad \le \text{ by Gronwall's inequality}$$
$$\le \left(2c\|f-g\|^2_{L^2(0,T;V')} + |u_0^1-u_0^2|\right) \exp\left(2\bar{c}\int_0^T \|u_1\|^2 ds\right)$$

and the statement of the theorem follows as $2\bar{c}\int_0^T \|u_1\|^2 ds \le \text{Const}.$ ∎

<u>Theorem 2</u> (continuous dependence on $g \in [0, g_0]$, $g_0 > 0$ finite)

For $g_1, g_2 \in [0, g_0]$, $g_0 > 0$ finite, arbitrary $\Rightarrow$ $\exists\, c(g_0) = c$ such that

$$|u_{g_1} - u_{g_2}\|^2_{L^\infty(0,T;H)} + |u_{g_1} - u_{g_2}\|^2_{L^2(0,T;V)} \le$$
$$\le c\,(g_1-g_2)^2$$
∎

<u>Proof.</u>

We have to do the same calculations as for

We have

$$(6.4)\quad \begin{cases} (u_{g_1}', v-u_{g_1}) + \mu a(u_{g_1}, v-u_{g_1}) + b(u_{g_2}, u_{g_1}, v-u_{g_1}) + \\ \qquad + g_1 j(v) - g_1 j(u_{g_1}) \geq (f, v-u_{g_1}) \end{cases}$$

$$(6.5)\quad \begin{cases} (u_{g_2}', v-u_{g_2}) + \mu a(u_{g_2}, v-u_{g_2}) + b(u_{g_2}, u_{g_2}, v-u_{g_2}) + \\ \qquad + g_2 j(v) - g_2 j(u_{g_2}) \geq (f, v-u_{g_2}) \end{cases}$$

Taking $v = u_{g_2}$ in (6.4) and $v = u_{g_1}$ in (6.5) it follows:

$$(u_{g_1}' - u_{g_2}', u_{g_1} - u_{g_2}) + \mu a(u_{g_1} - u_{g_2}, u_{g_1} - u_{g_2}) -$$
$$- b(u_{g_1} - u_{g_2}, u_{g_1}, u_{g_1} - u_{g_2}) - (g_1 - g_2)(j(u_{g_1}) - j(u_{g_2})) \leq$$

so that

$$(6.6)\quad \frac{1}{2}\frac{d}{dt}|u_{g_1} - u_{g_2}|^2 + \mu \|u_{g_1} - u_{g_2}\|^2 \leq$$
$$\leq |g_1 - g_2||j(u_{g_1}) - j(u_{g_2})| + |b(u_{g_1} - u_{g_2}, u_{g_1}, u_{g_1} - u_{g_2})|$$

We have

$$j(u_{g_1}) - j(u_{g_2}) \leq j(u_{g_1} - u_{g_2}) \leq C\|u_{g_1} - u_{g_2}\|$$

and

$$|b(u_{g_1} - u_{g_2}, u_{g_1}, u_{g_1} - u_{g_2})| \leq C_1 \|u_{g_1}\| \, |u_{g_1} - u_{g_2}| \, \|u_{g_1} - u_{g_2}\|$$

and so from (6.6) we have

$$\frac{1}{2}\frac{d}{dt}|u_{g_1} - u_{g_2}|^2 + \mu\|u_{g_1} - u_{g_2}\|^2 \leq$$
$$\leq C|g_1 - g_2| \, \|u_{g_1} - u_{g_2}\| + C_1 \|u_{g_1}\| \, |u_{g_1} - u_{g_2}| \, \|u_{g_1} - u_{g_2}\| \leq$$
$$\leq C_2 (g_1 - g_2)^2 + \frac{\mu}{4}\|u_{g_1} - u_{g_2}\|^2 + C_3 \|u_{g_1}\|^2 |u_{g_1} - u_{g_2}|^2 +$$

$+ \mu \quad \| \quad \quad \quad {}^2$

and therefore

$$\frac{d}{dt}|u_{g_1}-u_{g_2}|^2+\mu\|u_{g_1}-u_{g_2}\|^2\leq \tag{6.7}$$

$$\leq C_4(g_1-g_2)^2+C_5\|u_{g_1}\|^2|u_{g_1}-u_{g_2}|^2$$

We can now integrate on t ($t\leq T$):

$$|u_{g_1}(t)-u_{g_2}(t)|^2\leq C_4T(g_1-g_2)^2+C_5\int_0^T\|u_{g_1}\|^2|u_{g_1}-u_{g_2}|ds \tag{6.8}$$

$\leq C_6(g_1-g_2)^2$ (by Gronwall's inequality applied with $A=C_4T(g_1-g_2)^2$, $\varphi(t)=|u_{g_1}(t)-u_{g_2}(t)|^2$ and $B(s)=C_5\|u_{g_1}(s)\|^2$) We have also used the fact that $u_{g_1}\in$ bounded set of $L^2(0,T;V)$ if $g_1\in[0,g_0]$, g_0 finite, arbitrary.

As for the proof of theorem 1 it follows from (6.7) using (6.8) that

$$\mu\int_0^T\|u_{g_1}-u_{g_2}\|^2dt\leq C_7|g_1-g_2|^2$$

■

<u>Theorem 3</u> ($g\to+\infty$)

If $g\to+\infty$ $u_g\to0$ in $L^2(0,T;V)$ weakly

<u>Proof</u>.

From the proof of existence and uniqueness theorem it follows that

$$\|u_g\|_{L^2(0,T;V)}\leq c \quad \text{indep. from } g$$

(see estimates in 4 §5)

so that we can extract a subsequence u_{g_i} from " such that

u_g satisfies the inequality :

$$(6.9)\begin{cases}(u_g', v-u_g) + \mu a(u_g, v-u_g) + b(u_g, u_g, v-u_g) + \\ \qquad + gj(v) - gj(u_g) \geq (f, v-u_g) \\ u_g(0,x) = u_0\end{cases}$$

Let $v=0$ in (6.9) $\Rightarrow$

$$\frac{1}{2}\frac{d}{dt}|u_g|^2 + \mu \|u_g\|^2 + gj(u_g) \leq (f, u_g) \leq$$

$$\leq |f|_* \|u_g\| \leq c|f|_*^2 + \mu \|u_g\|^2$$

so that

$$2gj(u_g) + \frac{d}{dt}|u_g|^2 \leq 2c\,|f|_*^2$$

and after integration from 0 to T:

$$2g\int_0^T j(u_g)dt + |u_g(T)|^2 \leq c_1 \|f\|^2_{L^2(0,T;V')} + |u_0|^2$$

Therefore :

$$(6.10) \qquad \int_0^T j(u_g)dt \leq \frac{\text{const}}{g}$$

But now if we let $g_i \longrightarrow +\infty$ as $\int_0^T j(u_{g_i})$ is l.s.c for the weak-topology of $L^2(0,T;V)$

$\Longrightarrow$

$$\underline{\lim}_{g_i\to\infty} \int_0^T j(u_{g_i})dt \geq \int_0^T j(w)\,dt$$

which together with (6.10) gives:

Theorem 4 ($g \to 0$).

Let u_g be the solution of

$$(6.11)\begin{cases} (u_g', v-u_g) + \mu a(u_g, v-u_g) + b(u_g, u_g, v-u_g) + gj(v) - \\ \qquad - gj(u_g) \geq (f, v-u_g) \quad \forall v \in V \\ u_g(0,x) = u_0 \end{cases}$$

If $g \to 0$ then

$u_g \longrightarrow u$ in $L^2(0,T;V)$ weakly

$u_g' \longrightarrow u'$ in $L^2(0,T;V')$ weakly

where u is the solution of Navier - Stokes equation:

$$(6.12)\begin{cases} (u',v) + \mu a(u,v) + b(u,u,v) = (f,v) \\ u(0,x) = u_0 \end{cases}$$

Proof.

From the proof of existence and uniqueness theorem it follows that

$$\|u_g\|_{L^2(0,T;V)} \leq C$$

and

$$\|u_g'\|_{L^2(0,T;V')} \leq C \quad \text{if } g > 0 \text{ and bounded}$$

(we didn't have the same if $g \to +\infty$ → see estimate on $\frac{du_\varepsilon^m}{dt}$ in 4 § 5)

It follows that if $g \to 0$ we can extract a subsequence u_{g_e} such that

$$u_{g_e} \xrightarrow[g_e \to 0]{} w \text{ in } L^2(0,T;V) \text{ weakly}$$

$$u'_{\varrho} \xrightarrow[\varrho\to 0]{} w' \text{ in } L^2(0,T;V') \text{ weakly}$$
$$u_{\varrho} \xrightarrow[\varrho\to 0]{} w \text{ in } L^2(0,T;H) \text{ strongly (by compactness theorem)}$$

$$(u_{\varrho})_i\,(u_{\varrho})_j \longrightarrow (w)_i\,(w)_j \text{ in } L^2(0,T;L^2(\Omega)) \text{ weakly}$$

(so that $b(u_{\varrho},u_{\varrho},v) = -\sum_{ij}\int (u_{\varrho})_i \dfrac{\partial v_j}{\partial x_i}(u_{\varrho})_j\,dx \xrightarrow[\varrho\to 0]{}$
$-b(w,v,w) = b(w,w,v)$ in $L^1(0,T)$)

Let $v = v(t) \in L^2(0,T;V)$ in (6.11) $\Rightarrow$ as we are allowed to integrate on t

$$\int_0^T \{(u'_{\varrho},v) + \mu a(u_{\varrho},v) + b(u_{\varrho},u_{\varrho},v) + \varrho j(v) - \varrho j(u_{\varrho}) - (f, v-u_{\varrho})\}dt \geq \int_0^T \{(u'_{\varrho},u_{\varrho}) + \mu a(u_{\varrho},u_{\varrho})\}dt$$

We shall now take limit as $\varrho \to 0$ and using the lower semicontinuity of the terms in the right hand (as in Existence & uniqueness theorem) $\Longrightarrow$

$$\int_0^T \{(w',v) + \mu a(w,v) + b(w,w,v) - (f,v-w)\}dt \geq$$
$$\geq \int_0^T ((w',w) + \mu a(w,w))\,dt$$

We have used also the fact that

$$\varrho\int_0^T j(u_{\varrho})\,dt \xrightarrow[\varrho\to 0]{} 0$$

Therefore w is solution of the inequality

$$(6.13)\quad \int_0^T (w', v-w) + \mu a(w,v-w) + b(w,w,v-w)\,dt \geq$$
$$\geq \int_0^T (f, v-w)\,dt \qquad \forall v \in L^2(0,T;V$$

Moreover $w(0,x) = u_0$

By the same argument as in the last paragraph of the proof of the existence (see § 5), (6.13) is equivalent to the following inequality

$$(6.14)\begin{cases}(w', v-w) + \mu a(w, v-w) + b(w, w, v-w) \geq (f, v-w) \\ \qquad\qquad\qquad\qquad \forall v \in V \\ w(0,x) = u_0\end{cases}$$

Now, taking $v = w \pm \varphi$, φ in V from (6.14) it follows that w is solution of the equation

$$\begin{cases}(w', \varphi) + \mu a(w, \varphi) + b(w, w, \varphi) = (f, \varphi) \quad \forall \varphi \in V \\ w(0,x) = u_0\end{cases}$$

and by the uniqueness of the solution of (6.12) $\Rightarrow$ $w = u$. ∎

§7. The plasticity yield.

We have seen that for g sufficiently small the behavior of a Bingham fluid is not very different then the behavior of a classical viscous incompressible fluid (newtonian fluid) - see theorem 4 § 6.

The particularity of a Bingham fluid is that it is moving as a rigid body if a certain stress-function ($\sigma_{II}^{1/2}$) is less then the yield g. If g is positive on can observe rigid zones in the fluid and if g is increasing these rigid zones increase equally and they can block completely the flow. We have seen (theorem 3 § 6) that if $g \to +\infty$, $u_g \to 0$ in $L^2(0,T; V)$ - this result is clear of mechanical point of view -

More precisely, if $f \in L^\infty(0,T; L^2(\Omega))$ and $u^0 = 0$ we can show that $u_g = 0$ if $g \geq g_c$ (a critical yield). We have thus to prouve that $u = 0$ is solution of the problem for g sufficiently large (as we have uniqueness) It is therefore sufficient to prouve that in this case

(7.1) $\quad g j(v) \geq (f, v) \quad \forall v \in V$

But $\quad |(f(t), v)| \leq |f|\,|v|$

On the other hand, by an inequality due to Nirenberg & Strauss (see Bibliography at the end of this chapter)

(7.2) $\quad |v| \leq c\, j(v)$

and therefore from (7.1) we have

$$g j(v) \leq c\|f\|_\infty j(v) \qquad \forall v.$$

so that if

$$g > c\|f\|_\infty$$

$\|\cdot\|_\infty$ the norm of f in $L^\infty(0,T; L^2(\Omega))$

$\Rightarrow u = 0.$

If $g > c\|f\|_\infty$ and $u_0 \neq 0$ we can show that there exists t_0 such that for $t > t_0$: $u(t) = 0$. ($f \in L^\infty(0,T; L^2(\Omega))$)

Let consider the inequality verified by u:

(7.3) $$\begin{cases} (u', v-u) + \mu a(u, v-u) + b(u, u, v-u) + g j(v) - g j(u) \\ \qquad \geq (f, v-u) \end{cases}$$

Let be $v = 0$ and then $v = 2u$ in (7.3) It follows that u satisfies

$\frac{1}{2}\frac{d}{dt}|u|^2 + \mu\|u\|^2 + g j(u) = (f, u)$

hence

$$(7.4) \quad \frac{1}{2}\frac{d}{dt}|u|^2 + \nu\lambda_1 |u|^2 + \frac{g}{c}|u| \leq |f|\,|u|$$

We have used (7.2) and the inequality

$$\lambda_1 |u|^2 \leq \|u\|^2$$ (see definition of the special basis and norms in V, H & V' in § 5)

Suppose now that $|u(t)| \neq 0 \;\; \forall t$. It follows from (7.4) that

$$\frac{d}{dt}|u| + \nu\lambda_1 |u| + \frac{g}{c} \leq |f|$$

and so

$$\frac{d}{dt}\left(e^{\nu\lambda_1 t}|u(t)|\right) \leq \left(\|f\|_\infty - \frac{g}{c}\right)e^{\nu\lambda_1 t}$$

Therefore

$$e^{\nu\lambda_1 t}|u(t)| \leq |u_0| + \left(\|f\|_\infty - \frac{g}{c}\right)\int_0^t e^{\nu\lambda_1 s}ds =$$

$$= |u_0| + \frac{\|f\|_\infty - \frac{g}{c}}{\nu\lambda_1}\left(e^{\nu\lambda_1 t} - 1\right) =$$

$$= \left[|u_0| - \frac{\|f\|_\infty - \frac{g}{c}}{\nu\lambda_1}\right] + \frac{\|f\|_\infty - \frac{g}{c}}{\nu\lambda_1}e^{\nu\lambda_1 t}$$

By assumption $\|f\|_\infty - \frac{g}{c} \leq 0$ so that

$|u_0| - \frac{\|f\|_\infty - \frac{g}{c}}{\nu\lambda_1} \geq 0$ and we have

$$|u(t)| \leq \left[|u_0| - \frac{\|f\|_\infty - \frac{g}{c}}{\nu\lambda_1}\right]e^{-\nu\lambda_1 t} + \frac{\|f\|_\infty - \frac{g}{c}}{\nu\lambda_1} = F(t)$$

The first term in the right side $\to 0$ as $t \to +\infty$ and the second one is negative so that there $\exists$ t_0 such that $F(t_0) = 0$. For $t \geq t_0$ $F(t_0) < 0$ so that $u(t) = 0$ if $t \geq t_0$.

<u>Bibliography</u>

W. Strauss - Variations of Korn's and Sobolev's

§8. Steady case

Theorem. Let $f \in V'$. Then there exists $u \in V$ such that

$$\mu a(u, v-u) + b(u,u,v-u) + gj(v) - gj(u) \geq (f, v-u) \quad \forall v \in V$$

If $\mu^2 > c_1 |f|_*$ the solution is unique.

Proof.

The ideas of the proof are the same than in the evolution case - we shall give therefore here only a sketch of the proof.

We begin with the regularizated problem:

$$(8.1) \quad \mu a(u_\varepsilon, v) + b(u_\varepsilon, u_\varepsilon, v) + g(j'_\varepsilon(u_\varepsilon), v) = (f, v) \quad \forall v \in V$$

By monotonicity arguments this problem has a solution (see also H. Lanchon's lectures, Lax-Milgram theorem). Moreover from

$$\mu a(u_\varepsilon, u_\varepsilon) + g(j'_\varepsilon(u_\varepsilon), u_\varepsilon) = (f, u_\varepsilon)$$

it follows

$$\|u_\varepsilon\| \leq C |f|_* \qquad \text{independently of } \varepsilon$$

We can therefore extract a subsequence u_{ε_k} weakly convergent in V to u. As for the proof of existence and uniqueness theorem §5, let

$$(8.2) \quad Y_{\varepsilon_k} = \mu a(u_{\varepsilon_k}, v - u_{\varepsilon_k}) + b(u_{\varepsilon_k}, u_{\varepsilon_k}, v - u_{\varepsilon_k}) + $$
$$+ gj_{\varepsilon_k}(v) - gj_{\varepsilon_k}(u_{\varepsilon_k}) - (f, v - u_{\varepsilon_k})$$

or by (8.1) $\Rightarrow$

and we pass at the limit with $\varepsilon_k \to 0$ in (8.2) by the same way as for the existence theorem (§5) for the evolution case. As $y_{\varepsilon_k} \geq 0$ from (8.2) we have

$$\mu a(u_{\varepsilon_k}, v) + b(u_{\varepsilon_k}, u_{\varepsilon_k}, v) + g j_{\varepsilon_k}(v) - (f, v - u_{\varepsilon_k}) \geq$$

$$\geq \mu a(u_{\varepsilon_k}, u_{\varepsilon_k}) + g j_{\varepsilon_k}(u_{\varepsilon_k})$$

and as

$$\underline{\lim}_{\varepsilon_k \to 0} a(u_{\varepsilon_k}, u_{\varepsilon_k}) \geq 0$$

$$\underline{\lim}_{\varepsilon_k \to 0} j_{\varepsilon_k}(u_{\varepsilon_k}) \geq j(u)$$

the existence of a solution of the steady case follows.

For the uniqueness let be u_1 and u_2 two solutions and $U = u_1 - u_2$. It follows that

$$\mu a(U, U) \leq b(U, u_1, U) \leq \leq c_1 \|u_1\| \, \|U\|^2 \tag{8.3}$$

Now as

$$\mu \|u_1\|^2 = \mu a(u_1, u_1) \leq (f, u_1) \leq |f|_* \|u_1\|$$

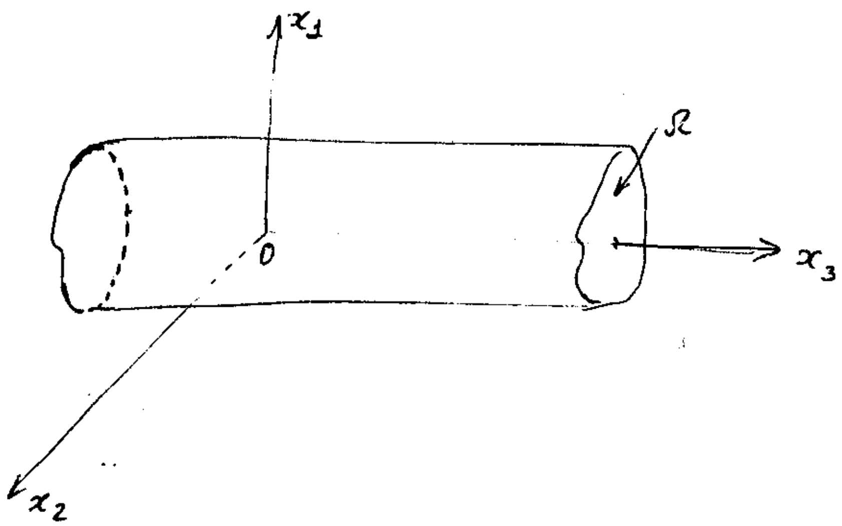
$$\Rightarrow \quad \|u_1\| \leq \frac{|f|_*}{\mu}$$

and from (8.3) we have

$$\mu \|U\|^2 \leq c_1 \frac{|f|_*}{\mu} \|U\|^2$$

so that if $\mu^2 > c_1 |f|_* \Rightarrow U = 0$ hence $u_1 = u_2$

§ 9. Flow in a pipe



We have to study the flow of a Bingham fluid in a pipe by drop pressure. $\Omega \subset \mathbb{R}^2$ is a of the cylinder. The flow is studied between $x_3 = 0$ and $x_3 = L$, with the pressure satisfying

$$(9.1) \qquad p\Big|_{x_3=0} = 0 \qquad p\Big|_{x_3=L} = -cL$$

$c > 0$ is the drop pressure.

If we are looking for u such that

$$(9.2) \qquad \sigma_{ij,j} = -(f-\gamma)_i \quad \text{in } \Omega \qquad f = 0 \text{ here}$$

$$\operatorname{div} u = 0$$

$$u\big|_{\partial\Omega} = 0$$

$$\begin{cases} \sigma_{II}^{1/2} < g \iff D_{ij} = 0 \\ \sigma_{II}^{1/2} \geq g \iff \sigma_{ij} = -p\,\delta_{ij} + g\dfrac{D_{ij}}{D_{II}^{1/2}} + 2\mu D_{ij} \end{cases}$$

and (9.1) the problem is in general not well-posed.

We shall consider ~~a~~ well-posed problem by imposing to the flow to be laminary i.e: the velocity are parallel to the axis satisfied if c is not too large (the Reynolds number of the flow is not too large, consequently)

If laminarity is imposed, the velocity field to find is $(0, 0, u)$ and as $\operatorname{div} u = 0 \Rightarrow$ $\frac{\partial u}{\partial x_3} = 0$ so that $u = u(x_1, x_2)$. We have therefore

$$D = (D_{ij})_{i,j=1,3} = \begin{pmatrix} 0 & 0 & \frac{1}{2}\frac{\partial u}{\partial x_1} \\ 0 & 0 & \frac{1}{2}\frac{\partial u}{\partial x_2} \\ \frac{1}{2}\frac{\partial u}{\partial x_1} & \frac{1}{2}\frac{\partial u}{\partial x_2} & 0 \end{pmatrix}$$

and

$$\sigma^D = (\sigma^D_{ij})_{i,j=1,3} = \begin{pmatrix} 0 & 0 & \sigma^D_{13} \\ 0 & 0 & \sigma^D_{23} \\ \sigma^D_{13} & \sigma^D_{23} & 0 \end{pmatrix}$$

From (9.2) it follows that

$$\frac{\partial p}{\partial x_1} = 0$$

$$\frac{\partial p}{\partial x_2} = 0$$

$$\frac{\partial p}{\partial x_3} = \sigma^D_{31,1} + \sigma^D_{32,2}$$

From the first equations $\Rightarrow$ $p = p(x_3)$ so that the left term in the last equation depens only on x_3; the right by the preceding comments depends only on x_1, x_2 $\Rightarrow$

$$\frac{\partial p}{\partial x_3} = \text{const} \quad \text{and using (9.1)} \Rightarrow$$

$$p = -c\, x_3.$$

It follows then that we have the following problem: find $u(x_1, x_2)$ in Ω such that

$$(9.3) \qquad \sigma^D_{31,1}(u) + \sigma^D_{32,2}(u) + c = 0 \qquad \text{in } \Omega$$

$$u\big|_{\partial\Omega} = 0$$

$$(9.4) \qquad \sigma^D_{3i} = 2\mu D_{3i} + \frac{g\, D_{3i}}{D_{II}^{1/2}} \qquad i = 1, 2$$

$$D_{II} = D_{31}^2 + D_{32}^2 = \frac{1}{4}\left(\frac{\partial u}{\partial x_1}\right)^2 + \frac{1}{4}\left(\frac{\partial u}{\partial x_2}\right)^2$$

We suppose that Ω has a boundary sufficiently smooth.

By definition

$$D_{3i} = \frac{1}{2}\frac{\partial u}{\partial x_i} \qquad i = 1, 2.$$

Let multiply (9.3) by $v - u$ and integrate on Ω $\Longrightarrow$

$$\int_\Omega \left\{\sigma^D_{31,1}(v-u) + \sigma^D_{32,2}(v-u)\right\} dx_1 dx_2 = -\int_\Omega c(v-u)\, dx_1 dx_2$$

By integration by parts it follows that

$$\int_\Omega \left\{\sigma^D_{31,1}(v-u) + \sigma^D_{32,2}(v-u)\right\} dx_1 dx_2 =$$

$$= -\int_\Omega \sigma^D_{31}\frac{\partial(v-u)}{\partial x_1} dx_1 dx_2 - \int_\Omega \sigma^D_{32}\frac{\partial(v-u)}{\partial x_2} dx_1 dx_2 \quad \underline{\underline{\text{using}}} \ (9.4)$$

$$= -\int_\Omega 2\mu \frac{1}{2}\frac{\partial u}{\partial x_1}\frac{\partial(v-u)}{\partial x_1} dx_1 dx_2 - \int_\Omega 2\mu \cdot \frac{1}{2}\frac{\partial u}{\partial x_2}\frac{\partial(v-u)}{\partial x_2} dx_1 dx_2 -$$

$$- g\int_\Omega \frac{1}{2}\frac{\partial u}{\partial x_1}\frac{\partial(v-u)}{\partial x_1} \cdot \frac{1}{\frac{1}{2}\left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right]^{1/2}} dx_1 dx_2 - g\int \frac{1}{2}\frac{\partial u}{\partial x}\frac{\partial(v-u)}{\partial} \cdot \frac{dx_1 dx_2}{}$$

$$\Rightarrow \mu\int_\Omega \overbrace{\left(\frac{\partial u}{\partial x_1}\frac{\partial(v-u)}{\partial x_1}+\frac{\partial u}{\partial x_2}\frac{\partial(v-u)}{\partial x_2}\right)}^{\text{grad}\,u\cdot\text{grad}\,(v-u)}dx_1\,dx_2 +$$

$$(9.5)\quad + g\int_\Omega\left[\frac{\partial u}{\partial x_1}\frac{\partial(v-u)}{\partial x_1}+\frac{\partial u}{\partial x_2}\frac{\partial(v-u)}{\partial x_2}\right]\frac{1}{\left[\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2\right]^{1/2}}dx_1\,dx_2 = \int_\Omega c(v-u)\,dx_1\,dx_2$$

Now

$$\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1}+\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2} \leq \left[\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2\right]^{1/2}\left[\left(\frac{\partial v}{\partial x_1}\right)^2+\left(\frac{\partial v}{\partial x_2}\right)^2\right]^{1/2}$$

$\Rightarrow$ from (9.5) that:

$$\mu\int_\Omega \text{grad}\,u\cdot\text{grad}\,(v-u)\,dx_1\,dx_2 + g\int_\Omega \underbrace{\left[\left(\frac{\partial v}{\partial x_1}\right)^2+\left(\frac{\partial v}{\partial x_2}\right)^2\right]^{1/2}}_{|\text{grad}\,v|}dx_1\,dx_2 +$$

$$+ g\int_\Omega \underbrace{\left[\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2\right]^{1/2}}_{|\text{grad}\,u|}dx_1\,dx_2 \geq \int c(v-u)\,dx_1\,dx_2$$

so that if u satisfies (9.3) & (9.4) it satisfies also

$$\mu\, a(u, v-u) + g j(v) - g j(u) \geq (c, v-u)$$

where

$$a(u,v) = \int_\Omega \text{grad}\,u\cdot\text{grad}\,v\,dx_1\,dx_2$$

$$j(v) = \int_\Omega |\text{grad}\,v|\,dx_1\,dx_2$$

It follows that the precise formulation of the laminary flow in a pipe is

Find $u \in H_0^1(\Omega)$ so that

(9.6) $\mu a(u, v-u) + gj(v) - gj(u) \geq (c, v-u) \quad \forall\, v \in H_0^1(\Omega)$

This problem is equivalent to the minimisation of the functional

$$J(v) = \frac{1}{2}\mu a(v,v) + gj(v) - (c, v) \qquad v \in H_0^1(\Omega)$$

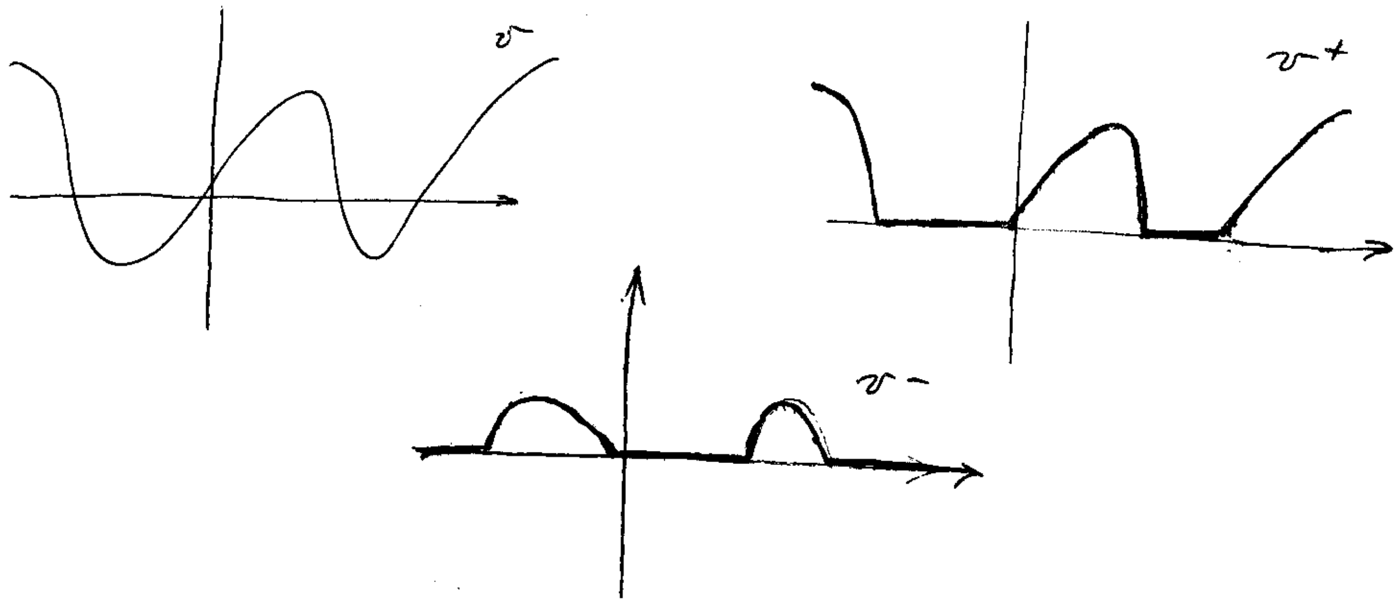
and by the classical results (see H. Lanchon's lecture) this problem has an unique solution.

<u>Property of u</u> : $u \geq 0$ a.e in Ω

<u>Proof</u>. Let define by

$v^+ = \sup(v, 0) \quad \rightarrow$ the positive part of v

$v^- = \sup(-v, 0) \quad \rightarrow$ the negative part of v



$v = v^+ - v^-$

Let $v = u^+$ in (9.6) $\Rightarrow v - u = u^-$ and if we want to prouve that $u \geq 0$ we have to show that $u^- = 0$

From (9.6) $\Rightarrow$

$$\mu a(u, u^-) + g(j(u^+) - g j(u)) \geqslant (c, u^-)$$

but by the definition of u^- and u^+

$$\Rightarrow \mu a(u, u^-) = -\mu a(u^-, u^-) \text{ and}$$

$$j(u) = j(u^+) + j(u^-)$$

$\Rightarrow$

$$(9.7) \quad -\mu a(u^-, u^-) - g j(u^-) \geqslant (c, u^-)$$

but $(c, u^-) = \int_\Omega c\, u^- dx_1 dx_2 \geqslant 0$ and the left term in (9.7) is negative $\Longrightarrow$ $u^- = 0$. $\blacksquare$