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BASIC IDEAS IN APPLIED FUNCTIONAL ANALYSIS
(continued)

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Remark 4: It seem now that the space $H^1(\Omega)$ is quite convenient for the Dirichlet problem (2) but we have yet to give a meaning to the boundary condition $\gamma_0 u = 0$ on Γ; now, it is important to remember that, except in dimension 1, the functions v of $H^1(\Omega)$ are not necessarily continuous in Ω, nor naturally on $\bar{\Omega}$. Nevertheless, it is possible to define the values of v on the boundary Γ of Ω (more generally on a regular variety of dimension $n-1$ included in $\bar{\Omega}$). The definition of $\gamma_0 v$ for $v \in H^1(\Omega)$ is the motivation of the following paragraph.

II. 4. Properties of Sobolev Spaces $H^m(\Omega)$ for m integer

Assumptions. Ω is supposed bounded with a boundary $n-1$ times continuously differentiable. Ω being locally on the same side of Γ. (these hypothesis can be enlarged)

(cf remark 3)

It is interesting to notice than $C^1(\bar{\Omega})$ being dense in $H^1(\Omega)$ that implies a kind of regularity of the functions of Ω on the boundary; for instance $C^\infty(\Omega)$ is not included in $H^1(\Omega)$. The idea of the following study will be to study the behavior of the mapping

$$u \longmapsto \gamma_0 u \text{ in } C^1(\bar{\Omega}) \text{ supplied with the norm (16)}$$

and to see what happen in $H^1(\Omega)$ in passing to the limit. We obtain then the first theorem of traces.

Theorem 5: *Under the preceding assumptions. We can define in an unic way, the trace $\gamma_0 v$ of $v \in H^1(\Omega)$ on Γ in order that $\gamma_0 v$ coincide with the usual definition if $v \in C^1(\bar{\Omega})$; $\gamma_0 v \in L^2(\Gamma)$ and the mapping*

$$v \longmapsto \gamma_0 v$$

is linear and continuous from $H^1(\Omega)$ in $L^2(\Gamma)$

Remark 5: the mapping $v \longrightarrow D^\alpha v$ is linear and continuous from $H^{|\alpha|+1}$ in $H^1(\Omega)$ (that is a consequence of the theorem 4) it is therefore possible to define

(23) $\{\gamma_0 (D^\alpha v)$ for $|\alpha| \leq m-1\}$ as soon as $v \in H^m(\Omega)$

but, because the knowledge of v on Γ involves that of its tangential defined by:

$$(23) \quad \gamma v = \left\{ v, \frac{\partial v}{\partial n}, \ldots\ldots \frac{\partial^{m-1}}{\partial n^{m-1}} \right\}_\Gamma \in \left(L^2(\Gamma)\right)^m \qquad \forall \ v \in H^m(\Omega)$$

where $\vec{n}$ is the unitary outward normal to Γ

We shall write still: $\gamma v = \{\gamma_0 v, \gamma_1 v, \ldots, \gamma_{m-1} v\}$

Remark 6: In the Homogeneous Dirichlet problem, the solution u has to be found in the kernel of γ_0 that we denote by.

$$(24) \quad H_0^1(\Omega) = \{ v / \ v \in H^1(\Omega), \ \gamma_0 v = 0 \}.$$

Now this subset of $H^1(\Omega)$ plays an essential part in the following, as well as:

$$(25) \quad H_0^m(\Omega) = \{ v / \ v \in H^m(\Omega), \ \gamma v = 0 \}, \text{ kernel of } \gamma$$

It is then interesting to note that, thanks to the continuity of γ_0 (resp γ), $H_0^m(\Omega)$ is a closed subset of $H^m(\Omega)$ and therefore also a Hilbert space.

emark 7: We saw in the paragraph II.1, that it is sufficient o suppose f in $L^2(\Omega)$ to give a meaning to (f, v) but because . and v have to be in $H^1(\Omega)$ it is important to notice that e injection of any $H^m(\Omega)$ in $L^2(\Omega)$ is continuous because the bvious inequality

$$\|u\|_{L^2(\Omega)} \leq \|u\|_{H^m(\Omega)}$$

Otherwise, it can be proved that $\mathcal{D}(\Omega)$ is dense in $H_0^m(\Omega)$, then if e denote by $H^{-m}(\Omega)$ the dual space of $H_0^m(\Omega)$, $H^{-m}(\Omega)$ is also a ubspace of $\mathcal{D}'(\Omega)$. We can write

$$(26) \quad \mathcal{D}(\Omega) \subset H_0^m(\Omega) \subset L^2(\Omega) = H^0(\Omega) \subset H^{-m}(\Omega)$$

nd to remark then, that we are in the frame described in the aragraph I.2

mark 8: The theorem 5 is not the best possible, in this sense that the pping $v \longmapsto \gamma_0 v$ is not surjective of $H^1(\Omega)$ in $L^2(\Gamma)$. In order characterize the space $\gamma_0[H^1(\Omega)]$, it is necessary to introduce some ew Sobolev spaces

II.5. Introduction of the Sobolev Spaces $H^s(R^n)$ for s non necessarily integer

In the special case of $\Omega = R^n$, we can define $H^m(R^n)$ by Fourier transformation. If v is a continuous function with compact support its Fourier Transform $\hat{v}$ is defined by.

$$(27) \qquad \hat{v}(\xi) = \int_{R^n} e^{-2\pi i x.\xi} v(x)\,dx$$

where $\xi = \{\xi_1, \dots, \xi_n\}$ and $x.\xi = x_i \xi_i \cdots$

It can be proved (that is the Plancherel theorem) that $\hat{v} \in L^2(R^n_\xi)$ and:

$$(28) \qquad \|\hat{v}\|_{L^2(R^n_\xi)} = \|v\|_{L^2(R^n_x)}$$

We can therefore by continuity define $\hat{v}$ for all $v \in L^2(R^n)$.
The mapping $v \longmapsto \hat{v}$ is invertible. So if $\hat{v} = \mathcal{F}v$

$$(29) \qquad v(x) = \lim_{j\to\infty} \text{ in } L^2(R^n_x) \text{ of } v_j(x) = \int_{|\xi|<j} \exp(2\pi i x.\xi)\,\hat{v}(\xi)\,d\xi$$

and we shall write symbolically.

$$v(x) = \int_{R^n_\xi} \exp(2\pi i x.\xi)\,\hat{v}(\xi)\,d\xi\,.$$

Then we can check that.

$$(30) \qquad v \in H^m(R^n) \iff (1+|\xi|^2)^{m/2}\,\hat{v} \in L^2(R^n_\xi)$$

and therefore, introduce in a natural way the following definition:

$$(31) \qquad H^s(R^n) = \{v \,/\, (1+|\xi|^2)^{s/2}\,\hat{v} \in L^2(R^n_\xi)\}$$

$H^s(R^n)$ being a Hilbert space for the norm

$$(32) \qquad \|v\|_{H^s(R^n)} = \|(1+|\xi|^2)^{s/2}\,\hat{v}\|_{L^2(R^n_\xi)}$$

which is equivalent to the norm introduced in (21) for $p = 2$, $s = m$ and $\Omega = R^n$

The definitions (31) and (32) are also available for s negative number and, when $H^0(R^n) \equiv L^2(R^n)$ is identified with its dual we have

(33) $$[H^s(\mathbb{R}^n)]' = H^{-s}(\mathbb{R}^n);$$

That justifies the notation $H^{-m}(\Omega)$ introduced in the remark 7

Thanks to the following property of localisation:

(34) $$\begin{cases} \underline{\forall \varphi \in \mathcal{D}(\mathbb{R}^n) \text{ and } \forall v \in H^s(\mathbb{R}^n), \; \varphi v \in H^s(\mathbb{R}^n) \text{ and}} \\ \underline{\text{the mapping } v \longrightarrow \varphi v \text{ is continuous linear from } H^s(\mathbb{R}^n)} \\ \underline{\text{in } H^s(\mathbb{R}^n)} \end{cases}$$

it is possible to define $H^s(\Gamma)$ for $s \geq 0$ and

(35) $$H^\sigma(\Gamma) = [H^{-\sigma}(\Gamma)]' \quad \text{if } \sigma < 0$$

and also to introduce a norm on $H^s(\Gamma)$. We can then complete the theorem 5 by the following one.

<u>Theorem 6</u>: <u>with the assumptions of the theorem 5, the mapping $v \longmapsto \gamma_0 v$ is continuous linear and surjective from $H^1(\Omega)$ in $H^{1/2}(\Gamma)$. More generally, if $v \in H^m(\Omega)$, the mapping $v \longmapsto \gamma v$ is continuous linear and surjective from $H^m(\Omega)$ in $H^{m-1/2}(\Gamma) \times H^{m-\frac{3}{2}}(\Gamma) \times \dots \times H^{1/2}(\Gamma)$ and moreover the retreatment $\mathcal{R} = \gamma^{-1}$ is also linear and continuous</u>

<u>Remark 9</u> If we have now a look on the Dirichlet problem choosen as example, every things seem to be clear except that we have not yet justified the application of the green integral transformation. that will be our last point before being able to apply the Lax Milgram. Theorem

II. 6 green Formula in Sobolev Spaces.

If Γ is sufficiently regular on the one hand and if u and v are twice continuously differentiable $\overset{\text{in } \bar{\Omega}}{\vee}$, on the other hand, the classic green formula gives.

If now, u and v belong to $H^2(\Omega)$ and thanks to the density of $C^2(\bar{\Omega})$ in $H^2(\Omega)$ we can generalize this formula by continuity in writing:

$$\sum_{i=1}^{n} (u_{,i}, v_{,i})_{L^2(\Omega)} = -(\Delta u, v)_{L^2(\Omega)} + (\gamma_1 u, \gamma_0 v)_{L^2(\Gamma)}$$

every things are a meaning from the preceding considerations.

But in our example, we have the pretention to apply this formula for u and v in $H^1(\Omega)$; what about $\gamma_1 u$ in this case? We need therefore of the following result.

<u>Théorème 7: Let W be a vector function defined in Ω such that each component $w_i \in L^2(\Omega)$ and also $\operatorname{div} w \in L^2(\Omega)$. Then $w_i n_i$ can be defined on each regular variety of dimension $n-1$ included in Ω as an element of $H^{-1/2}(\Gamma)$ dual of $H^{1/2}(\Gamma)$</u>

Let first show how this statement can be useful for us, after what its proof will be a good exercise.

Here, $W = \operatorname{grad} u$; if u is solution of (2) in $H^1(\Omega)$, then $u_{,i} \in L^2(\Omega)$ and also $\operatorname{div}(\operatorname{grad} u) = \Delta u = \lambda u - f$, then from this theorem $\gamma_1 u \in H^{-1/2}(\Gamma)$ and the term $\int_\Gamma \frac{\partial u}{\partial n} v \, d\sigma$ has to be interpreted as the scalar product in the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$

<u>Proof of the theoreme</u>: Let g be an element of $H^{1/2}(\Gamma)$, because the surjectivity of the mapping $v \longmapsto \gamma_0 v$ from $H^1(\Omega)$ in $H^{1/2}(\Gamma)$, the exist at least on element $v \in H^1(\Omega)$ such that $\gamma_0 v = g$; Let us write then by definition

$$\int_\Gamma w_i n_i g \, d\Gamma = \int_\Omega w_{i,i} v \, dx + \int_\Omega w_i v_{,i} \, dx$$

the second member being clearly defined. We have just now to prove that L define by

$$L(g) = \int_\Gamma w_i n_i g \, d\Gamma$$ is a continuous linear form on $H^{1/2}(\Gamma)$

$$|L(g)| \le \|w_{i,i}\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|w_i\|_{L^2(\Omega)} \|v_{,i}\|_{L^2(\Omega)}$$

if then we write

$$M = \left(\|w_{i,i}\|^2_{L^2(\Omega)} + \|w_1\|^2_{L^2(\Omega)} + \dots + \|w_n\|^2_{L^2(\Omega)} \right)^{1/2}.$$

$$|L(g)| \leq M \left(\|v\|^2_{L^2(\Omega)} + \sum_{i=1}^{M} \|v_{,i}\|^2_{L^2(\Omega)} \right)^{1/2} = M \|v\|_{H^1(\Omega)}$$

and because the continuity of the relevement $\gamma_0^{-1} : H^{1/2}(\Gamma) \longrightarrow H^1(\Omega)$,

$$\|v\|_{H^1(\Omega)} \leq C \|g\|_{H^{1/2}(\Gamma)}$$

hence.

$$|L(g)| \leq M C \|g\|_{H^{1/2}(\Gamma)}$$

and the continuity of L from $H^{1/2}(\Gamma)$ in R - The linearity of f being obvious, we can actually identify $w_i n_i$ with an element of the dual $H^{-1/2}(\Gamma)$ of $H^{1/2}(\Gamma)$. q. e. d.

II. 7 Application of the Lax milgram theorem to the Dirichlet problem

Let us write $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$, these spaces being supplied by their usual Hilbert structure.

$$(f, g) = \int_\Omega f(x)\, g(x)\, dx.$$

$$|f| = (f, f)^{1/2} \quad \forall\ f \text{ and } g \in H.$$

$$((u, v)) = (u, v) + (u_{,i}, v_{,i})$$

$$\|u\| = ((u, u))^{1/2} \quad \forall\, u, v \in V$$

We already noticed in remark. 7 that we are exactly in the situation discribed in the paragraphe I. 2 about the inclusion of two Hilbert spaces.

The bilinear form introduced is

$$a(u, v) = \lambda(u, v) + (u_{,i}, v_{,i}) \qquad \lambda > 0.$$

It is clear that this form is continuous thanks to the inequality of Schwarz:

$$|a(u,v)| \leq \max(1, \lambda) \left\{ (u_{,i}, v_{,i}) + (u, v) \right\} \leq \max(1, \lambda) \|u\| \|v\|$$

The coercivity is also satisfied:

$$a(u, u) \geq \min(1, \lambda) \|u\|^2.$$

ed be resolved thanks to a new result). The Lax Milgram Theorem involves then

$$(36)\quad \underline{\forall f \text{ given in } \Omega,\ \exists\, u, \text{ unic solution of}}$$
$$\underline{u \in H_0^1(\Omega),\quad a(u,v) = (f,v) \quad \forall\, v \in H_0^1(\Omega)}$$

If we interprete now this problem (36), we can apply the relation particularly for $v = \varphi \in \mathcal{D}(\Omega) \subset H_0^1(\Omega)$ that gives

$$\int_\Omega (\operatorname{grad} u \operatorname{grad} \varphi + \lambda u \varphi - f\varphi)\, dx = 0 \quad \forall\ \varphi \in \mathcal{D}(\Omega).$$

that is to say in applying green and the fact that $\varphi = 0$ on Γ.

$$\int_\Omega (-\Delta u + \lambda u - f)\,\varphi\, dx = 0 \quad \forall\ \varphi \in \mathcal{D}(\Omega).$$

or still $\quad -\Delta u + \lambda u = f \quad$ in $\mathcal{D}'(\Omega)$

Otherwise $\quad u \in H_0^1(\Omega)$ implies $\gamma_0 u = 0$ on Γ.

<u>Remark 10</u> : from the Proposition 2, we know also that the solution of (36) minimizes the fonctional

$$E(v) = a(v,v) - 2(f,v)$$
$$= \sum_{i=1}^{m} |v_{,i}|^2 + \lambda |v|^2 - 2(f,v)$$

<u>emark 11</u> : we know now that u, unic solution of (36) is the nly possible solution of (2) but for the moment we have just btained that u is solution in the sens of distribution.
.

$(37)\quad u \in H_0^1(\Omega)\ ,\quad -\Delta u + \lambda u = f$

An interesting and necessary step would be to prove the regularity f. u and in consequence the realization of (37) in a strong sens. he problem of regularity will be introduced later.

<u>emark 12</u> : we have ~~now~~ the existence and the unicity of the solution of (36) at is already very interesting because it shows that probably the