INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

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SMR/23 /41

AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

22 September - 26 November 1976

BODIES WITH MICROSTRUCTURE

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1. Introduction

The goal of these lectures is an exposition of some aspects of the theory of bodies affected by dislocations. We begin by quoting the approach of Noll [1, pp 211-244], [2, Ch.], but we proceed in a less ambitious vein. Nevertheless the student with a specific interest can refer to the works quoted above and strengthen by himself a connexion, which is left here of necessity rather loose.

Let us introduce the class $\mathcal{D}$ of complete deformations as the set of couples $(\underset{\sim}{a}, \underset{\sim}{G})$, where $\underset{\sim}{a}$ is chosen within a class $\mathcal{D}_1$ and $\underset{\sim}{G}$ within a class $\mathcal{D}_2$, with the following properties:

(i) the members of $\mathcal{D}_1$ are invertible mappings (of class C^2 at least), whose domains and codomains are open subsets of $\mathcal{E}$; we denote by $\nabla \underset{\sim}{a}$ the gradient of $\underset{\sim}{a}$ and assume that the value of ∇a at any $\underset{\sim}{x}$ (element of the domain of $\underset{\sim}{a}$) is a member of Lin^+.

(ii) the members of $\mathcal{D}_2$ are mappings (of class C^1 at least) from open subsets of $\mathcal{E}$ into Lin^+; in a couple $(\underset{\sim}{a}, \underset{\sim}{G})$ the domains of $\underset{\sim}{a}$ and $\underset{\sim}{G}$ are the same.

(iii) If $(\underset{\sim}{a}^{(1)}, \underset{\sim}{G}^{(1)})$ and $(\underset{\sim}{a}^{(2)}, \underset{\sim}{G}^{(2)})$ belong to $\mathcal{D}$ and if the codomain of $\underset{\sim}{a}^{(1)}$ coincides with the domain of $\underset{\sim}{a}^{(2)}$, then $\mathcal{D}$ contains also the composition

$$(\underset{\sim}{a}, \underset{\sim}{G}) = (\underset{\sim}{a}^{(2)}, \underset{\sim}{G}^{(2)}) \circ (\underset{\sim}{a}^{(1)}, \underset{\sim}{G}^{(1)}),$$

which is defined as follows: $\underset{\sim}{a}$ is simply $\underset{\sim}{a}^{(2)} \circ \underset{\sim}{a}^{(1)}$, whereas $\underset{\sim}{G}$ is specified pointwise through the formula

$$\underset{\sim}{G} = (\underset{\sim}{G}^{(2)} \circ \underset{\sim}{a}^{(1)}) \cdot \underset{\sim}{G}^{(1)}$$

(iv) All rigid displacements, i.e. couples $(\underset{\sim}{a}^{R}, \underset{\sim}{Q})$, where

$$\underset{\sim}{a}^{R}\big|_{\underset{\sim}{x}} = \underset{\sim}{a}^{R}\big|_{\underset{\sim}{y}} + \underset{\sim}{Q}\,(\underset{\sim}{x} - \underset{\sim}{y}) \quad ; \quad \underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{E},$$

$$\underset{\sim}{Q} \in \mathrm{Orth}^{+},$$

belong to $\mathcal{D}$.

We can now specify the notion of a continuous body $\mathcal{B}$ with affine structure. $\mathcal{B}$ is a set of particles X with a number of properties; first of all $\mathcal{B}$ is a smooth continuous body in the usual sense, i.e. a class $\mathcal{P}_1$ of mappings π exist, $\pi : \mathcal{B} \to \mathcal{E}$, such that:

(i) Every $\pi \in \mathcal{P}_1$ is one-to-one and its range $\pi(\mathcal{B})$ is an open subset of $\mathcal{E}$, which is called the region occupied by $\mathcal{B}$ in the placement π. The spatial point $\underset{\sim}{x} = \pi\big|_{X}$ is called the place of the particle X in the placement π.

(ii) If $\pi^{(1)}, \pi^{(2)} \in \mathcal{P}_1$, then there exists a member $\underset{\sim}{a}^{(1,2)}$ of $\mathcal{D}_1$ such that

$$\underset{\sim}{a}^{(1,2)} = \pi^{(2)} \circ \left[\pi^{(1)} \right]^{-1};$$

$\underset{\sim}{a}^{(1,2)}$ is called the <u>apparent deformation</u> from $\pi^{(1)}$ into $\pi^{(2)}$.

(iii) If $\pi \in \mathcal{P}_1$, $\underset{\sim}{a} \in \mathcal{D}_1$ and the range of π coincides with the domain of $\underset{\sim}{a}$, then $\underset{\sim}{a} \circ \pi \in \mathcal{P}_1$.

With this partial structure of $\mathcal{B}$ we can define <u>local placements</u> λ_X at each particle as equivalence classes of a partition $\mathcal{E}_X$ within placements $\pi \in \mathcal{P}_1$, induced by the following relation of equivalence at X:

$$\left[\nabla \underset{\sim}{a}^{(1,2)} \right]_X = \left[\nabla \left(\pi^{(2)} \circ [\pi^{(1)}]^{-1} \right) \right]_X = \underset{\sim}{1}\ ; \quad \pi^{(1)}, \pi^{(2)} \in \mathcal{P}_1 .$$

We can also define a <u>reference</u> for $\mathcal{B}$ as a function Λ which associates with each particle a local placement; a very special case is the <u>homogeneous reference</u> $\nabla \pi$, i.e. the reference which associates with each $X \in \mathcal{B}$ the equivalence class deduced from a unique placement π.[1] The set of all references on $\mathcal{B}$ will be indicated by $\mathcal{R}$.

We are now in the position to specify the complete structure of $\mathcal{B}$; it is characterized by a class $\mathcal{P}$ of mappings (π, Λ), called <u>complete placements</u> of $\mathcal{B}$, where $\pi \in \mathcal{P}_1$ and $\Lambda : \mathcal{B} \to \mathcal{R}$

belongs to a class $\mathcal{P}_2$ with the following properties:

(i) Any Λ is smooth in the sense that any placement π of $\mathcal{P}_1$ admits of a continuous relative gradient $\nabla_\Lambda \pi$ with respect to any Λ in $\mathcal{P}_2$; $\nabla_\Lambda \pi$ is defined pointwise as follows: let the placement γ belong to the equivalence class λ_X, then

$$\nabla_\Lambda \pi\Big|_X = \Big[(\nabla \pi)\,(\nabla \gamma)^{-1}\Big]_X . \qquad (1.1)$$

Λ is called relaxed reference of $\mathcal{B}$ in the complete placement (π, Λ); its value λ_X at X is the local relaxed placement.

(ii) If $(\pi^{(1)}, \Lambda^{(1)})$ and $(\pi^{(2)}, \Lambda^{(2)})$ belong to $\mathcal{P}$, then there exists a member $(\underset{\sim}{a}^{(1,2)}, \underset{\sim}{G}^{(1,2)})$ of $\mathcal{D}$ such that: (a) domain $\underset{\sim}{a}^{(1,2)} \equiv$ codomain $\pi^{(1)}$; (b) $\underset{\sim}{a}^{(1,2)} = \pi^{(2)} \circ [\pi^{(1)}]^{-1}$ and (c) the local relaxed placement $\lambda_X^{(2)}$ can be obtained from $\lambda_X^{(1)}$ by

$$\lambda_X^{(2)} = \left(\underset{\sim}{G}^{(1,2)} \circ \pi^{(1)}\right)_X \lambda_X^{(1)} , \qquad (1.2)$$

for all $X \in \mathcal{B}$. $(\underset{\sim}{a}^{(1,2)}, \underset{\sim}{G}^{(1,2)})$ is called the complete deformation from $(\pi^{(1)}, \Lambda^{(1)})$ to $(\pi^{(2)}, \Lambda^{(2)})$; if at X $\nabla \underset{\sim}{a}^{(1,2)} = \underset{\sim}{1}$ (or $\underset{\sim}{G}^{(1,2)} = \underset{\sim}{1}$), then the two complete placements are said to have the same apparent (or relaxed) local placement at X.

(iii) If $(\pi^*, \Lambda^*) \in \mathcal{P}$, $(\underset{\sim}{a}, \underset{\sim}{G}) \in \mathcal{D}$ and codomain $\pi^* \equiv$ domain $\underset{\sim}{a}$, then the composition

$(\pi, \Lambda) = (\underset{\sim}{a}, \underset{\sim}{G}) \circ (\pi^*, \Lambda^*)$, defined as follows

$$\pi = \underset{\sim}{a} \circ \pi^* \quad , \quad \lambda_{\underline{X}} = (\underset{\sim}{G} \circ \pi)\big|_{\underline{X}} \lambda^*_{\underline{X}} \,, \; \forall \underline{X} \in \mathcal{B},$$

belongs to $\mathcal{P}$. The composition is called the complete placement obtained from (π^*, Λ^*) by the deformation $(\underset{\sim}{a}, \underset{\sim}{G})$.

Footnote (1). To visualize a reference Λ, one has to cut the body into infinitesimal pieces and to view the piece corresponding to the material point $\underline{X}$ in a configuration that belongs to the equivalence class $\lambda_{\underline{X}}$. The pieces will not fit together to form a coherent region in space if Λ is an inhomogeneous reference. If $\Lambda \equiv \nabla\pi$, i.e. if Λ is homogeneous, however, then the pieces can be fit together to form the region $\pi(\mathcal{B})$. (NOLL).

2. Geometric preliminaries

A particular complete placement (π^*, Λ^*), called here primary placement, is often preferred either for greater ease in the description of the material properties of a body or for other reasons. Then any other complete placement (π, Λ) is conveniently specified by [(π^*, Λ^*) and] the deformation $(\underset{\sim}{a}, \underset{\sim}{G})$ from (π^*, Λ^*).

For simplicity, we presume from now on that a cartesian coordinate frame is available so that we can identify the particles $\mathfrak{X}$ with their coordinates X^H $(H=1,2,3)$ in (π^*, Λ^*); correspondingly we can interpret π^* as the mapping

$$\underset{\sim}{X} = \pi^*(\mathfrak{X}) \quad , \quad \underset{\sim}{X} \equiv (X^1, X^2, X^3),$$

and introduce the component form of the deformation

$$\begin{aligned} x^i &= a^i(X^H) \quad ; \\ G^i_H &= G^i_H(X^K) \quad ; \quad i=1,2,3 \; ; \; H, K = 1,2,3 . \end{aligned} \tag{2.1}$$

Furthermore we can also introduce the directors to describe the affine structure of our body [3, pp 88-92]; to this end, choose a placement γ which belongs to the equivalence class $\lambda_{\mathfrak{X}}$, consider the composition $\gamma \circ (\pi^*)^{-1}$ and define the vectors

$$\underset{\sim}{d}_H = \frac{\partial [\gamma \circ (\pi^*)^{-1}]}{\partial X^H} \quad , \quad H = 1,2,3. \tag{2.2}$$

These vectors are the directors at the particle $\mathfrak{X}$; their definition involves a choice of γ but in fact they depend only on λ_X. They form a vector basis at X; the set of all triads $\underset{\sim}{d}_H$ in a certain placement characterizes the relaxed reference for $\mathcal{B}$ in that placement. The reference is, in general, non-holonomic, unless the reference is homogeneous.

We will use the notation $\overset{*}{\underset{\sim}{d}}_H$ for the directors in the primary placement; then, if we interpret $(\underset{\sim}{\mathbb{Q}}, \underset{\sim}{G})$ as the deformation from the primary placement and accept the specifications (2.1), we can rewrite formula (1.2) in the form

$$\underset{\sim}{d}_H = \underset{\sim}{G}\,\overset{*}{\underset{\sim}{d}}_H . \qquad (2.3)$$

Remark. There is, of course, a certain arbitrariness in the choice of directors at least in one of the placements of $\mathcal{B}$. We could choose a different cartesian frame in writing (2.2) or even refer the primary placement to general coordinates $\tilde{X}^H$. If at a place X^K

$$X^H = A^H{}_K\, \tilde{X}^K , \qquad (2.4)$$

then a new set of directors $\tilde{\underset{\sim}{d}}_K$ can be evidenced

$$\tilde{\underset{\sim}{d}}_K = \underset{\sim}{d}_H\, A^H{}_K . \qquad (2.5)$$

On the other hand the introduction of directors allows us to imagine a connection between the kinematics of bodies with local reference and that of bodies made up of complex particles (say, dumbbell or tetrahedral), where directors have an immediate physical appeal.