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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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SPECTRAL ANALYSIS IN VIBRATION MECHANICS

(Part I)

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Spectral analysis in vibration mechanics.

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III – Numerical analysis of vibration problems.

1 – preliminary
2 – an elementary example
3 – general results in eigenvalue approximation
4 – bibliography.

Introduction

Vibration problems are an important part of applied structural and fluid mechanics mainly because it leads to stability studies. The aim of these lectures is to connect these problems with fundamental theory in functional analysis and to show that the functional formulation is often necessary to introduct computational techniques-

Part I : Compact operators on Hilbert Spaces

I-1- Preliminary

We recall some fundamental results.

1) Compact operators

Let X and Y be two normed spaces and u be a linear mapping from X to Y.
u is called compact if it maps every bounded set E of X into a set $u(E)$ whose closure is compact in Y ($u(E)$ is then called relatively compact)

Definition 1.1 (equivalent)
u is called compact $\Leftrightarrow$ we can select from an arbitrary bounded sequence $\{x_n\}$, $x_n \in X$, a subsequence $\{x_{n_k}\}$ such that $u(x_{n_k})$ is convergent in Y.
u compact is of course, continuous..

2) Fundamental examples

2-1- The first known result is the following one: If X is a reflexive Banach space and u a compact mapping from X to Y, then the image of the unit sphere of X is a compact set.

2-2- Let Ω be a bounded open set of $\mathbb{R}^n$ with boundary Γ. The boundary Γ is supposed smooth in the following sense. Γ is a $(n-1)$ dimensional manifold of class $\mathcal{C}^r$ $r \geq 1$, and Ω is locally located on one side of Γ. Then:

__theorem 1.1__: The mapping from $H^1(\Omega)$ into $L^2(\Omega)$ is compact.

For the proof of compactness theorem the reader is referred to the references [3] (mentionned at the end of chapter I.)

3°/ __compact operators on Hilbert spaces.__

In the following the Hilbert spaces are supposed __separable__. We can recall the characteristic property. [7] t_2 p 63

__Theorem 1.2__: Let V and W be two Hilbert spaces and A an operator such that $A \in \mathcal{L}(V, W)$. The operator A is compact if and only if A is the limit of a denumerable sequence of finite operators

__Remarks 1.1__

* $\{\varphi_i\}$ being a basis for V we write

$$(1) \qquad Ax = \underbrace{\sum_{i=1}^{n} (x, \varphi_i) A\varphi_i}_{} + \underbrace{\sum_{i=n+1}^{\infty} (x, \varphi_i) A\varphi_i}_{}$$

Then, theorem 1.2 means that: for any $\varepsilon > 0$ there exists n_0 such that $\| A_\varepsilon x \|_W < \varepsilon$ for $n \geqslant n_0$.

An operator A is called <u>completely continuous</u> if it verifies the formula (1) where A_n is a finite dimensional operator and $\| A_\varepsilon x \| < \varepsilon$.

In fact, theorem 1.2, includes the trivial property: a compact linear operator on a Hilbert space is completely continuous.

* Theorem 2 can be extended if V is a Banach space and W a Hilbert space.

I-2- The spectrum of self adjoint, linear, compact operators on Hilbert spaces. (separable)

H^{ses}. • Let V be a Hilbert space. (V is complex or real). The inner product is defined by $(\cdot,\cdot)$

• a linear operator A on V is called a <u>self adjoint</u> (or hermitian) operator if:

(2) $$(Ax, y) = (x, Ay) = \overline{(Ay, x)}$$

(In case of a real space V, A is symetric, then $(Ax, y) = (Ay, x)$)

• We denote by λ an eigenvalue of the operator A, and by E_λ the subspace generated by elements associated with the eigenvalue λ.

The spectrum of A is the set of values μ for which $(A-\mu I)$ has no inverse operator. Of course the eigenvalues belong to the spectrum.

- We prove now a fundamental Riesz theorem for spectral decomposition of a self adjoint compact operator. Some lemmas are first necessary.

1°/- Lemma 1.1

i) the eigenvalues of a linear, self adjoint operator A are real numbers

ii) The corresponding subspaces are orthogonal

Proof: If $\lambda \neq 0$ is an eigenvalue then

$$(Ax, x) = \lambda(x, x) = (x, Ax) = \overline{(Ax, x)}$$

(Ax, x) is real for every $x \in V$. Then λ is real because (x, x) is real

To prove the property ii), let $x_\lambda \in E_\lambda$ and $x_\mu \in E_\mu$, $\lambda \neq \mu$. Then

$$(Ax_\lambda, x_\mu) = \lambda(x_\lambda, x_\mu)$$

$$(x_\lambda, Ax_\mu) = \mu(x_\lambda, x_\mu)$$

which implies $(\lambda-\mu)(x_\lambda, x_\mu) = 0$

and therefore $(x_\lambda, x_\mu) = 0$ since $\lambda \neq \mu$.

lemma 1.2.

The spectrum of a self adjoint bounded operator A lies entirely in the interval $[m, M]$ of the real axis where:

$$M = \sup_{\|x\|=1} (Ax, x) \quad ; \quad m = \inf_{\|x\|=1} (Ax, x)$$

Proof : We admit than eigenvalues exist

Let for instance $\lambda > M$, therefore $\lambda = M + d$, $d > 0$

We have

$$(A_\lambda x, x) = (Ax, x) - \lambda(x, x) \leq M(x, x) - \lambda(x, x)$$
$$\leq -d(x, x)$$

implying $|(A_\lambda x, x)| \geq d\|x\|^2$

On the other hand $|(A_\lambda x, x)| \leq \|A_\lambda x\| \|x\|$

Consequently $\|A_\lambda x\| \geq d\|x\|^2$

Then the equation $A_\lambda x = 0$ has a unique solution $x = 0$, λ is not an eigenvalue.

We get the same proof with $m - d$, $d > 0$

2/ <u>Self adjoint linear compact operators</u>

Let A be a self adjoint linear compact (A is of course continuous) operator on V and Λ the set of eigenvalues of A.

From the last lemmas we obtain :

$\lambda \notin \Lambda$; E_λ is $\{0\}$.

$\lambda \in \Lambda$; E_λ is arbitrary if $\lambda = 0$ and

the different subspaces E_λ are orthogonal
Λ lies entirely in the interval $[-\|A\|, \|A\|]$ because from $Ax = \lambda x$, $x \neq 0$, it follows easily that $|\lambda| \leq \|A\|$.
We shall admit the next lemma 3 [7]

lemma 1.3.

If A is a compact linear operator on V and λ an eigenvalue (non equal to zero) then the subspace E_λ is finite dimensional.

lemma 1.4.

Let A be a compact self adjoint operator on the Hilbert space V. Λ is either finite, either denumerable and the set of eigenvalues can have only one limit point $\lambda = 0$

Proof : Suppose that is not true.
Then we could find a set of values $\lambda_0, \lambda_1, \dots \lambda_m \dots \in \Lambda$ with $|\lambda_i| \geq \varepsilon > 0$.
This sequence lies inside the set $\varepsilon \leq |z| \leq \|A\|$, which is compact. Then we could extract a convergent subsequence $\{\lambda_n\}_{n \in N}$, whose limit λ is different from zero.
For each λ_n there exists $x_n \in E_{\lambda_n}$ with
Because of the orthogonality of

eigenelements $\|x_n - x_m\| = \sqrt{2}$ for $n \neq m$.
But $\{Ax_n\}$ lies inside the mapping $A(\mathcal{B})$ of the unit sphere $\mathcal{B}$ belonging to V. $A(\mathcal{B})$ is a compact set. Then we could find a subsequence $\{Ax_{n_k}\}$ with a limit y. The sequence $x_{n_k} = \frac{Ax_{n_k}}{\lambda_{n_k}}$ has a limit point $\frac{y}{\lambda}$. This contradicts $\|x_n - x_m\| = \sqrt{2}$.

<u>Lemma 1.5</u>.
Let A be a linear compact self adjoint operator. A has at least an eigenvalue λ such that $|\lambda| = \|A\|$.

Proof:
$\|A\| = \sup_{\|x\| \leq 1} |(Ax, x)|$; moreover assume that $\sup |(Ax, x)| = \sup (Ax, x)$ (or $-\inf(Ax, x)$)
Let $\mathcal{B}$ be the unit sphere of V, there exists a sequence $\{x_n\}$, $x_n \in \mathcal{B}$ such that
$\lim_{n \to \infty} (Ax_n, x_n) = \|A\|$.
The sequence $\{Ax_n\}$ lies inside $A(\mathcal{B})$, so there exists a subsequence such that $Ax_{n'} \to y$ as $n \to \infty$. $(Ax_{n'}, x_{n'}) \leq \|Ax_{n'}\| \leq \|A\|$. It converges to $\|A\|$ only if $\lim_{n' \to \infty} \|Ax_{n'}\| = \|A\|$. Then $\|y\| = \|A\|$
... that $x_{n'} \to \frac{y}{\|\ \|}$

$\| y - \|A\| x_{n'} \|^2 = \|y\|^2 + \|A\|^2 - 2 \,\mathrm{Re}\,(y, \|A\| x_{n'})$

$(y - A x_{n'}, \|A\| x_{n'}) \leq \| y - A x_{n'} \| \, \|A\| \to 0$

Then $(y, \|A\| x_{n'})$ and $(A x_{n'}, \|A\| x_{n'})$ have the same limit $\|A\|^2$. So $\| y - \|A\| x_{n'} \| \xrightarrow[n' \to \infty]{} 0$, and

$\|A\| x_{n'}$ and $A x_{n'}$ have the same limit y.

By the continuity of A, we obtain $Ay = \|A\| y$.

y is an eigenelement associated with the eigenvalue $\|A\|$. q.e.d.

lemma 1.6.

Let A be a compact, self adjoint operator on V. the subspaces E_λ associated with $\lambda \in \Lambda$ are dense in V.

Proof: If $x \in E_\lambda$, $Ax \in E_\lambda$ because we have $A(Ax) = A(\lambda x) = \lambda A x$. E_λ is stable by A.

Let $E_\Lambda = \bigoplus_{\lambda \in \Lambda} E_\lambda$, and let G be the subspace orthogonal to E_Λ. For $x \in G$, $Ax \in G$ since

$(Ax, x_\lambda) = (x, A x_\lambda) = (\overline{A x_\lambda, x}) = \lambda (x, x_\lambda) = 0$

As A, A_G $(G \to G)$ is compact and self adjoint. Consequently there exists in G an eigenelement of the operator A_G and therefore of A. This contradicts the definition of G. $G = \{0\}$ and

$$V = E_\Lambda = \bigoplus_{\lambda \in \Lambda} E_\lambda$$

From lemma 1.6 it follows

<u>Corollary</u>.

a self adjoint compact operator has a pure point spectrum

3°/. <u>Spectral decomposition : the Riesz theorem</u>.

<u>Theoreme 1.3</u> : Let V be an Hilbert space (real or complex) and let A be a linear self adjoint compact operator on V

(1). The eigenvalues of A are real. Their set Λ is finite dimensional or denumerable.
For $\lambda \in \Lambda$ we have $0 \leq |\lambda| \leq \|A\|$

(2) E_λ associated with every $\lambda \in \Lambda\ (\lambda \neq 0)$ is finite dimensional. The sets E_λ are orthogonal and

(3) $V = \bigoplus_{\lambda \in \Lambda} E_\lambda$.

<u>Remarks 2</u>

* (3) is called the spectral decomposition of the space V. Let $\varphi_i \in E_{\lambda_i}\ \forall i$, then (3) implies:

$$\forall x \in V \quad Ax = \sum_{i=1}^{\infty} (Ax, \varphi_i)\varphi_i = \sum_{i=1}^{\infty} \lambda_i (x, \varphi_i)\varphi_i .$$

The set of eigenelements is an orthonormal basis of the Hilbert space V : $(\varphi_i, \varphi_j) = \delta_{ij}$

* If A is positive, then the eigenvalues are

* A more general form (usual in mechanics) of the theorem is given in [5]. We consider the spectral problem

(4) $$A x = \lambda B x$$

A is already defined and B is a symetric and V elliptic operator on V. So the form (Bx, y) defines an inner product in V and the results (1) (2) of theorem (3) can be extended.

I-3- Some properties of eigenvalues

According to the preceding statements the highest eigenvalue is

$$\lambda_1 = \|A\| = \sup_{x \in V, \|x\| = 1} |(Ax, x)|$$

considering now the subspace $\bigoplus_{\substack{\lambda \in \Omega \\ \lambda \neq \lambda_1}} E_\lambda$, we can find λ_2 ------ so we have:

Property 1.1. Let $\lambda_1, \lambda_2, \ldots \lambda_n, \ldots$ the eigenvalues of the operator A and $\varphi_1, \varphi_2, \ldots \varphi_n \ldots$ the relative eigenelements. Then we have

(5) $$|\lambda_p| = \sup_{\substack{\|x\| = 1 \\ x \in \{y \in V;\ (y, \varphi_k) = 0 \quad k = 1 \ldots p-1\}}} |(Ax, x)|$$

In fact, theorem 3 can also be proved by using the property 1.1

We just recall now the property, characteristic of eigenvalues

<u>Property 1.2</u> (Riesz Fisher theorem)

Let λ_k^+ (λ_k^-) be a non negative (resp^{ly} non positive) eigenvalue of the operator A. Then:

(6) $$\lambda_k^+ = \min_{x_i \in V} \max_{\substack{x \in V \\ x \neq 0 \\ (x, x_i) = 0 \quad i = 1, 2 \dots, k-1}} \frac{(Ax, x)}{\|x\|^2}$$

(7) $$\lambda_k^- = \max_{x_i \in V} \min_{\substack{x \in V \\ x \neq 0 \\ (x, x_i) = 0 \quad i = 1, 2, \dots k-1}} \frac{(Ax, x)}{\|x\|^2}$$

One set is an increasing one. The other is decreasing and zero is the only limit point.

<u>Theorem 1.4</u>

If A is a linear continuous, symmetric, positive bounded below operator on a Hilbert space V, with a discrete spectrum, then its inverse operator $G = A^{-1}$ is compact

Proof : G exists thanks to the hypothesis.

Let ($\lambda_1, \lambda_2 \dots \lambda_n \to \infty$) be the spectrum of A and ($\varphi_1, \varphi_2 \dots \varphi_n \dots$) the corresponding orthonormal eigenvalues, we have

$$Gx = \sum_1^\infty \frac{(x, \varphi_k)}{\lambda_k} \varphi_k = \sum_1^n \frac{(x, \varphi_k)}{\lambda_k} \varphi_k + \sum_{n+1}^\infty \frac{(x, \varphi_k)}{\lambda_k} \varphi_k$$

$$= G_n x + G_\varepsilon x$$

$\|G_\varepsilon x\|^2 \leq \frac{1}{\lambda_{n+1}^2} \sum_1^\infty |(x, \varphi_k)|^2$ and by Bessel inequality $\|G_\varepsilon x\|^2 \leq \frac{\|x\|^2}{\lambda^2_{n+1}}$. So $\|G_\varepsilon\| \to 0$ for $n \to \infty$, and we get the

I - h - Variational spectral theory:

H^{sis} (i) - Let H and V be, real (separable) Hilbert spaces with norms $|\cdot|$, $\|\cdot\|$ - We suppose that V is included in H and that the map from V into H is continuous and dense $\iff$ $V \subset_0 H \subset_0 V'$ (dual of V)

(ii) Let $a(\cdot,\cdot) : V \times V \to R$ be a continuous bilinear, V-elliptic form on V (in the sense $\exists \alpha$, $\forall v \in V$; $a(v,v) \geq \alpha \|v\|^2$)

(iii) moreover we suppose that the mapping from V into H is compact.

We immediately remark that hypothesis (i), (ii) are the same as for Lax Milgram theorem (see H. Lanchon lectures) -

Then we can define an operator A and the set $D(A)$ such that

(8) $A \in \mathcal{L}(V; V')$; $a(u,v) = \langle Au, v \rangle$ $\forall u, v \in V$
($\langle \cdot, \cdot \rangle$ is the duality product between V and V')

(9) $D(A) = \{ u \in V ;\ Au \in H \}$
$D(A)$ is dense in V

Property 1.3 In addition to the above assumptions we assume that a is symmetric form on V. Then the operator A, unbounded in H, is symmetric.

Proof:
let A^* be the adjoint operator of A.

$D(A^*) = \{ u \in V \ ; \ v \to (A^* u, v) = (u, Av)$ is continuous on the set V equipped with the norm $|\cdot|$ $\}$

We have

$(u, Av) = a(v, u) = a(u, v) = (Au, v)$

Then

$D(A^*) = \{ u \in V ; Au \in H \} = D(A)$

A is symmetric on $D(A)$. It follows from $D(A) = D(A^*)$ that $A^* = A$; which proves the property.

<u>Consequences</u>

From Lax milgram theorem we deduce that the equation $Au = 0$ has one and only one solution $u = \{0\}$. So A has an inverse operator A^{-1}.

$A^{-1} \in \mathcal{L}(H; H)$ as $A \in \mathcal{L}(D(A); H)$

A^{-1} is a positive bounded operator (it results of V- ellipticity) on H

A^{-1} is symmetric

A^{-1} is a compact operator (because the mapping $V \to H$ is compact)

If Λ is the set of eigenvalues of A^{-1}, it follows from Riesz theorem, that:

$$\Lambda \subset]0\ \alpha]$$

and we have theorem 1.5 which is crucial because it has many applications.

Theorem 1.5.

With the above assumptions (i)..(iv) the operator A (defined by (8)) has a pure point spectrum. The eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$ are real and they define a set of increasing values

$$\lambda \leq \lambda_1 \leq \lambda_2 ---- \leq \lambda_n \to \infty \; .$$

The set of associated eigenelements $(\varphi_i)_{i\in\mathbb{N}}$ is an orthonormal basis on H and an orthogonal basis on V in the following sense:

(10) $$\langle A\varphi_i, \varphi_j \rangle = \lambda_i (\varphi_i, \varphi_j)_H = \lambda_i \delta_{ij}$$

variational spectral problem:

Theorem 1.5 solves the following variational spectral problem: Find $u \in V$ and λ such that:

(11) $$a(u, v) = \lambda (u, v) \qquad \forall v \in V$$

Example 1.1 (see Artola's lectures)

- Let Ω be a bounded open set of $\mathbb{R}^n$ with smooth boundary Γ. We solve the following equations:

(12) $$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \quad \left(\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}\right) \\ u = 0 & \text{on } \Gamma \end{cases}$$

- The equivalent variational problem is: Find u in $H_0^1(\Omega)$ and the scalar λ solution of the equation

$$a(u, v) = \lambda \int_\Omega u v \, d\Omega \qquad \forall v \in H_0^1(\Omega)$$

with $a(u, v) = \int_\Omega \operatorname{grad} u . \operatorname{grad} v \, d\Omega$.

- The operator $A = -\Delta$ is one to one from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. $G = A^{-1} \in \mathcal{L}(H^{-1}(\Omega); H_0^1(\Omega))$

$G \in \mathcal{L}(L^2(\Omega); L^2(\Omega))$

moreover

G is symmetric : $(Gu, v) = (u, Gv) = a(u, v)$

G is positive $(Gu, u) \geq 0$ and $(Gu, u) = 0$ implies $u = 0$

G is compact and bounded (it results easily of V-ellipticity and of the Poincaré inequality on $H_0^1(\Omega)$)

The problem:

$$(13) \quad \begin{cases} Gu = \frac{1}{\lambda} u & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

is equivalent to (11) and (12) and it admits a denumerable set of eigenvalues λ_j:

$0 < \lambda_1 \leq \lambda_2 \dots \leq \lambda_n \leq \dots \to \infty$

The associated eigenelements w_j satisfy:

$$((w_i, w_j))_{H_0^1(\Omega)} = \int_\Omega \operatorname{grad} w_i \cdot \operatorname{grad} w_j \, d\Omega = \lambda_i (w_i, w_j)_{L^2(\Omega)}$$

$$= \lambda_i \delta_{ij}$$

Remarks 1.3.

* For $n=2$, problem 11 can be associated to the vibrations of a clambed membrane. Then u defines the displacement about the rest state. The spectral results are well known in vibration mechanics.

* If $\Omega =]0,1[\times]0,1[$ we get easily the eigenelements by calculation.

$\sin l\pi x \ \sin k\pi y \quad l, k = 1, 2 \dots$

The corresponding eigenvalues are:
$\lambda_{lk} = \pi^2 (l^2 + k^2)$.

* Another physical case is the following one:
(in acoustics u defines the pressure about the rest state)

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma_0 \quad (\text{mes } \Gamma_0 \neq 0) \end{cases}$$

$\partial\Omega = \Gamma_0 \cup \Gamma_1$, n represents the external normal to Ω.
Define the Hilbert space $V = \{ u ;\ u \in H^1(\Omega),\ u/_{\Gamma_1} = 0 \}$
we have the same type of results about spectral properties

* we shall see in chapters II and III how functional analysis allow us to solve some mechanical problems
in a theorical point of view (II) and
in a numerical point of view (III).

I-5. Bibliography (non exhaustive)

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