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STRATIFIED FLUID FLOWS: INTERNAL GRAVITY WAVES

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1. Kelvin-Helmholtz instability

To illustrate some of the mechanisms and concepts of stability we shall now work through a classic problem that demands little mathematics. Consider the basic flow of incompressible inviscid fluids in two horizontal parallel infinite streams of different velocities and densities, one stream above the other. Then the basic flow is given by

$$\underset{\sim}{U} = \begin{cases} U_2 \underset{\sim}{i} \\ U_1 \underset{\sim}{i} \end{cases}, \quad \rho = \begin{cases} \rho_2 \\ \rho_1 \end{cases}, \quad P = \begin{cases} p_o - g\rho_2 z & (z>0) \\ p_o - g\rho_1 z & (z<0) \end{cases} \tag{1}$$

say, where U_1, U_2 are the velocities of the two streams, ρ_1, ρ_2 the densities, p_o a constant pressure, z is the height, and g is the acceleration due to gravity. Helmholtz [1] in 1868 remarked that "every perfect geometrically sharp edge by which a fluid flows must tear it asunder and establish a surface of separation, however slowly the rest of the fluid may move," thereby recognising the basic flow, but the problem of instability was first posed and solved by Kelvin [2] in 1871; it is now called Kelvin-Helmholtz instability.

Kelvin assumed that the disturbed flow was irrotational on each side of the vortex sheet. This follows if the initial disturbance of the flow is irrotational, because irrotational flow of inviscid fluid persists. However, also initial rotational disturbances are possible. To simplify the mathematics we shall adopt Kelvin's restrictive assumption, remembering that it allows a proof of instability but not stability because it gives no information about rotational disturbances. In fact rotational disturbances are no more unstable and so Kelvin did find a necessary as well as a sufficient condition for instability. Thus we assume the existence of a velocity potential ϕ on each side of the interface between the two streams with $\underset{\sim}{u} = \text{grad}\,\phi$, where

$$\phi = \begin{cases} \phi_2 & (z>\zeta) \\ \phi_1 & (z<\zeta), \end{cases} \tag{2}$$

the interface having elevation

$$z = \zeta\,(x,y,t) \tag{3}$$

when the flow is disturbed. Then the equations of continuity and incompressibility give $\text{div}\,\underset{\sim}{u} = 0$ and therefore the Laplacians of the potentials vanish,

$$\Delta\phi_2 = 0 \;(z>\zeta), \quad \Delta\phi_1 = 0 \;(z<\zeta). \tag{4}$$

Note that Euler's equations of motion have been used only implicitly in taking the irrotational flow as persistent.

The boundary conditions are as follows:

(a) The initial disturbance may be supposed to occur in a finite region so that for all time

$$\text{grad}\,\phi \to \underset{\sim}{U} \quad \text{as} \quad z \to \pm\infty\,. \tag{5}$$

(b) The fluid particles at the interface just move with the interface without the two fluids occupying the same point at the same time and without a cavity forming between the fluids. Therefore the vertical velocity at the interface is given by

$$\frac{\partial\phi}{\partial z} = \frac{D\zeta}{Dt} = \frac{\partial\zeta}{\partial t} + \frac{\partial\phi}{\partial x}\cdot\frac{\partial\zeta}{\partial x} + \frac{\partial\phi}{\partial y}\cdot\frac{\partial\zeta}{\partial y} \qquad (z = \zeta),$$

the material derivative of the surface elevation [cf. 3, p.7]. This kinematic condition is the same as that for surface gravity waves, which occur as a special form of this instability. There is a discontinuity of tangential velocity at the interface, the above giving two conditions,

$$\frac{\partial\phi_i}{\partial z} = \frac{\partial\zeta}{\partial t} + \frac{\partial\phi_i}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi_i}{\partial y}\frac{\partial\zeta}{\partial y} \qquad (z = \zeta,\ \ i = 1,2). \tag{6}$$

(c) The normal stress of the fluid is continuous at the interface. For an inviscid fluid, this gives the dynamical condition that the pressure is continuous. Therefore

$$\rho_1\{C_1 - \tfrac{1}{2}(\text{grad}\ \phi_1)^2 - \partial\phi_1/\partial t - gz\} = \rho_2\{C_2 - \tfrac{1}{2}(\text{grad}\ \phi_2)^2 - \partial\phi_2/\partial t - gz\} \qquad (z = \zeta), \tag{7}$$

by Bernoulli's theorem for irrotational flow, which is valid on each side of the vortex sheet $z = \zeta$. In order that the basic flow satisfies this condition, the constants C_1, C_2 must be related so that

$$\rho_1(C_1 - \tfrac{1}{2}U_1^2) = \rho_2(C_2 - \tfrac{1}{2}U_2^2). \tag{8}$$

Equations (2) - (8) pose the non-linear problem for instability of the basic flow (1). For linear stability we first put

$$\phi_2 = U_2 x + \phi_2'(z>\zeta), \qquad \phi_1 = U_1 x + \phi_1'\ (z<\zeta) \tag{9}$$

and neglect products of the small increments ϕ_1', ϕ_2', ζ. There is no length scale in the basic velocity so it is far from clear how small ζ must be in order that the linearization is valid. But we can plausibly justify the linearization if the surface and its slopes are small, i.e. $\partial\zeta/\partial x$, $\partial\zeta/\partial y \ll 1$ and $g\zeta \ll U_1^2$, U_2^2. If this is granted, linearization of equations (4) - (7) is straightforward, giving

$$\Delta\phi_2' = 0(z>0), \qquad \Delta\phi_1' = 0(z<0); \tag{10}$$

$$\text{grad}\phi' \to 0 \quad \text{as} \quad z\to\pm\infty; \tag{11}$$

$$\partial\phi_i'/\partial x = \partial\zeta/\partial t + U_i\partial\zeta/\partial x\ (z = 0;\ \ i = 1,2); \tag{12}$$

$$\rho_1(U_1\partial\phi_1'/\partial x + \partial\phi_1'/\partial t + g\zeta) = \rho_2(U_2\partial\phi_2'/\partial x + \partial\phi_2'/\partial t + g\zeta)\ (z = 0). \tag{13}$$

It can be seen that all coefficients of this linear partial differential system are constants and that the boundaries are horizontal. So we use the method of normal modes, assuming that an arbitrary disturbance may be resolved into independent modes of the form

$$(\zeta, \phi_1, \phi_2) = (\hat{\zeta}, \hat{\phi}_1, \hat{\phi}_2) \exp\{i(kx + \ell y) + st\}. \tag{14}$$

Equations (10) now give

$$\hat{\phi}_2 = A_2 \exp(-\tilde{k} z) + B_2 \exp(\tilde{k} z)$$

for some constants A_2, B_2, where the total wave-number is defined by

$$\tilde{k} = +(k^2 + \ell^2)^{\frac{1}{2}}. \tag{15}$$

The boundary condition (11) at infinity gives $B_2 = 0$, and therefore

$$\hat{\phi}_2 = A_2 \exp(\exp(-\tilde{k} z). \tag{16}$$

Similarly, we find

$$\hat{\phi}_1 = A_1 \exp(\tilde{k} z) . \tag{17}$$

Now equations (12), (13) give three homogeneous linear algebraic equations for the three unknown constants $\hat{\zeta}$, A_1, A_2. Equation (12) gives

$$A_2 = -(s + i_k U_2)\hat{\zeta}/\tilde{k},\ A_1 = (s + ikU_1)\hat{\zeta}/\tilde{k} \tag{18}$$

and thence gives the eigenfunctions (14) except for an arbitrary multiplicative constant. Then equation (13) gives the eigenvalue relation,

$$\rho_1\{\tilde{k}g + (s + ikU_1)^2\} = \rho_2\{\tilde{k}g - (s + ikU_2)^2\}. \tag{19}$$

The solution of this quadratic equation gives two modes with

$$s = -ik\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left\{\frac{k^2\rho_1\rho_2(U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} - \frac{\tilde{k}g(\rho_1 - \rho_2)}{\rho_1 + \rho_2}\right\}^{\frac{1}{2}} . \tag{20}$$

Both are neutrally stable if

$$\tilde{k}g(\rho_1^2 - \rho_2^2) \geq k^2\rho_1\rho_2(U_1 - U_2)^2 , \tag{21}$$

equality giving marginal stability. One mode is asymptotically stable but the other is unstable if

$$\tilde{k}g(\rho_1^2 - \rho_2^2) < k^2\rho_1\rho_2(U_1 - U_2)^2. \tag{22}$$

This is accordingly a necessary and sufficient condition for instability of the mode with wave-numbers k,ℓ. Thus the flow is always unstable (to short waves) if $U_1 \neq U_2$.

To interpret this result it is simplest to consider special cases separately. When $\rho_2 = 0$ and $U_1 = U_2 = 0$ we have the model of <u>surface gravity waves</u> on deep water. They are stable with phase-velocity,

$$c = is/\tilde{k} = \pm(g/\tilde{k})^{\frac{1}{2}}, \tag{23}$$

as is well known. This illustrates the identity of waves and oscillatory stable normal modes. It is often helpful to regard waves as a special case of hydrodynamic stability.

When the basic flow is at rest $(U_1 = U_2 = 0)$, we find

$$s = \pm\{\tilde{k}g(\rho_2 - \rho_1)/(\rho_1 + \rho_2)\}^{\frac{1}{2}}. \tag{24}$$

There is instability if and only if $\rho_1 < \rho_2$, that is heavy fluid rests above light fluid. The stable waves have phase-velocity, given by

$$c = \pm\{g(\rho_1 - \rho_2)/\tilde{k}(\rho_1 + \rho_2)\}^{\frac{1}{2}}. \tag{25}$$

The eigenfunctions (14) for the velocity potential die away exponentially with distance from the interface, as in all cases of Kelvin-Helmholtz instability, so the motion is confined to the vicinity of the interface between the two fluids. These waves are accordingly called <u>internal gravity waves</u>. They can be observed between layers of fresh and salt water that occur in estuaries; the upper surface of the fresh water may be very smooth while strong internal gravity waves occur at the interface of the salt water a metre or two below, because for fresh and salt water $(\rho_1 - \rho_2)/(\rho_1 + \rho_2) \simeq 10^{-2} \ll 1$ and so equations (18) give relatively small fluid velocities for given amplitude of interfacial elevation.

The eigenvalues (24) can be interpreted differently if the whole fluid system has an upward vertical acceleration f. Then by the principle of equivalence in dynamics, or by solution of the problem of normal modes,

$$s = \pm\{\tilde{k}g'(\rho_2 - \rho_1)/(\rho_1 + \rho_2)\}^{\frac{1}{2}}, \tag{26}$$

where $g' = f + g$ is the apparent gravitational or net vertical acceleration of the system. It follows that there is instability if and only if $g' < 0$, i.e. the net acceleration is directed from the lighter towards the heavier fluid. This called Rayleigh-Taylor instability after Rayleigh's [4] theory of the stability of a stratified fluid at rest under the influence of gravity and Taylor's [5] recognition of the significance of accelerations other than gravity. Rayleigh-Taylor instability can be simply observed by rapidly accelerating a beaker of water downwards (and standing clear).

When there is a vortex sheet in a homogeneous fluid $(\rho_1 = \rho_2, U_1 \neq U_2)$, equation (20) gives

$$s = -\tfrac{1}{2}ik(U_1 + U_2) \pm \tfrac{1}{2}k(U_1 - U_2). \tag{27}$$

This flow is always unstable, the waves moving with phase-velocity $c = \frac{1}{2}k(U_1 + U_2)/\tilde{k}$, the average velocity of the basic flow resolved in the direction $(k/\tilde{k}, \ell/\tilde{k}, 0)$ of propagation. Waves of all lengths are unstable, there being no length scale of the basic flow. For a real shear layer of finite thickness, we shall show that short waves are stable. However, the above result shows the instability of a shear layer to waves whose lengths are much greater than the thickness of the layer. It can also be seen that the wave of given length $\lambda = 2\pi/\tilde{k}$ that grows most rapidly is the one that propagates in the direction of the basic flow $(k = \tilde{k})$. So, after some time, waves in the direction of the basic flow will be dominant.

The full condition (22) of Kelvin-Helmholtz instability represents an imbalance of the destabilizing effect of inertia over the stabilizing effect of buoyancy when the heavy fluid is below. Kelvin [2] used this theory as a model of the generation of waves by wind. Helmholtz [6] applied the theory to billow clouds, whose presence in regular lines marks instability of winds with strong shear. Reynolds' [7] famous paper on pipe flow described also some experiments on Kelvin-Helmholtz instability, though he did not recognize them as such. Reynolds filled a tube with water above carbon disulphide and tilted it. The ensuing relative motion of the two fluids led to instability at their interface. Thorpe [8] has recently refined this experiment and clearly identified the Kelvin-Helmholtz instability.

2. The Taylor-Goldstein problem for normal modes

(a) Introduction

The interplay of two major themes now appears. We introduced these themes in our discussion of Kelvin-Helmholtz instability, where we described how two layered fluids at rest are stable to internal gravity waves and how a vortex sheet of homogeneous fluid is unstable. We found that for a wave of given length on a vortex sheet between two layered fluids, either the buoyancy is strong enough to overcome the tendency of shear instability and render the wave stable, or it is not and the wave is unstable. Here we shall consider this interplay of the stabilizing influence of gravity on a continuously stratified fluid and of the destabilizing influence of basic shear in a generalized form of the Kelvin-Helmholtz instability, although this generalization is sometimes simply called Kelvin-Helmholtz instability. We shall see that the intuitions that heavy below light fluid is a stabilizing influence, that strong shear is a destabilizing influence, and that light below heavy fluid renders any basic flow unstable are usually correct.

We model the problem by taking a basic state in dynamic equilibrium, with velocity, density and pressure given by

$$\underset{\sim}{u}_* = U_*(z_*)\underset{\sim}{i}, \quad \rho_* = \bar{\rho}_*(z_*), \quad p_* = p_{0*} - g\int^{z_*} \bar{\rho}_*(z_*')dz_*' \qquad \text{for} \quad z_{1*} \leq z_* \leq z_{2*} \tag{1}$$

respectively, where z_* is the height and g the acceleration due to gravity and where each of the horizontal planes at $z_* = z_{1*}$ and z_{2*} is taken to be rigid. We take scales L of length and V of velocity characteristic of the basic velocity distribution $U_*(z_*)$ and ρ_0 characteristic of the basic density $\rho_*(z_*)$. We assume that the fluid is inviscid and incompressible, density being convected but not diffused. Then the equations of motion, incompressibility and continuity in dimensionless form give

$$\left.\begin{aligned} &\rho\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u}\right) = -\nabla p - F^{-1}\rho\underset{\sim}{k}, \\ &\nabla\cdot\underset{\sim}{u} = 0, \\ \text{and} \quad &\frac{\partial\rho}{\partial t} + \underset{\sim}{u}\cdot\nabla\rho = 0\,. \end{aligned}\right\} \tag{2}$$

where the <u>Froude number</u> is defined by

$$F = V^2/gL, \tag{3}$$

and

$$\rho = \rho_*/\rho_0, \quad \underset{\sim}{u} = \underset{\sim}{u}_*/V \quad \text{etc.}$$

It can be verified that the basic state satisfies the equations for arbitrary distributions $U(z)$, $\bar{\rho}(z)$. To study its stability we put

$$\underset{\sim}{u}(\underset{\sim}{x},t) = U(z)\underset{\sim}{i} + \underset{\sim}{u}'(\underset{\sim}{x},t),\ p(\underset{\sim}{x},t) = p_0 - g\int^{z}\bar{\rho}(z')dz' + p'(\underset{\sim}{x},t),$$

$$\rho(\underset{\sim}{x},t) = \bar{\rho}(z) + \rho'(\underset{\sim}{x},t), \tag{4}$$

substitute these expressions into equations (2), and neglect quadratic terms in the small primed quantities to derive the linearized equations for the disturbance. We also take normal modes of the form

$$(\underset{\sim}{u}',\ p',\ \rho') = (\hat{\underset{\sim}{u}}(z),\ \hat{p}(z),\ \hat{\rho}(z))\exp\{i(\alpha x + \beta y - \alpha c t)\}\ . \tag{5}$$

Thus equations (4) give

$$\begin{aligned}
i\alpha\bar{\rho}(U-c)\hat{u}+\bar{\rho}U'\hat{w} &= -i\alpha\hat{p},\\
i\alpha\bar{\rho}(U-c)\hat{v} &= -i\beta\hat{p},\\
i\alpha\bar{\rho}(U-c)\hat{w} &= -D\hat{p} - F^{-1}\hat{\rho},\\
i\alpha\hat{u} + i\beta\hat{v} + D\hat{w} &= 0,\\
i\alpha(U-c)\hat{\rho} + \bar{\rho}'\hat{w} &= 0,
\end{aligned} \tag{6}$$

where differentiation with respect to z of a basic quantity is denoted by a prime and of a perturbed quantity by D. One may eliminate $\hat{u}$ and $\hat{v}$ from the first two of equations (6) and from the fourth; then one may eliminate $\hat{p}$ and ρ in turn with the aid of the third and fifth of equations (6) to find

$$(U-c)\{D^2\hat{w} - (\alpha^2 + \beta^2)\hat{w}\} - U''\hat{w} - \frac{(\alpha^2 + \beta^2)\bar{\rho}'}{\alpha^2 F(U-c)\bar{\rho}}\hat{w}$$

$$+ \frac{\bar{\rho}'}{\bar{\rho}}\{(U-c)D\hat{w} - U'\hat{w}\} = 0\ . \tag{7}$$

The conditions at the rigid boundaries give

$$\hat{w} = 0 \quad \text{at} \quad z = z_1,\ z_2\ . \tag{8}$$

Yih [9] applied Squire's transformation to this system. It can be seen that the characteristics of two-dimensional waves are simply related to those of three-dimensional ones, for each three-dimensional wave with numbers (α,β) there being a two-dimensional one with the same value of the complex velocity c but with wave-numbers $((\alpha^2 + \beta^2)^{\frac{1}{2}}, 0)$ and Froude number $\alpha^2 F/(\alpha^2 + \beta^2)$. Thus two-dimensional waves effectively have reduced gravity and magnified relative growth rate $(\alpha^2 + \beta^2)^{\frac{1}{2}}c_i$, and they are usually found to be the most unstable waves. For these reasons we shall henceforth consider only two-dimensional waves.

We used F^{-1} as a dimensionless measure of gravity because it was the first one at hand. However, it can now be seen from equation (7) that F^{-1}

arises only in a product with $-\bar{\rho}'/\bar{\rho}$. Further, the physical effects of gravity on the modes will be seen to create internal rather than surface gravity waves (if the upper boundary is rigid). So we shall use the overall Richardson number J, defined as a characteristic value of

$$-\bar{\rho}'/F\bar{\rho} = -gL^2 \frac{d\bar{\rho}_*}{dz_*}\bigg/\bar{\rho}_* V^2,$$

rather than the Froude number. It is also convenient to define the Brunt-Väisälä frequency (or the buoyancy frequency) N_* by

$$N_*^2(z_*) = -g\frac{d\rho_*}{dz_*}\bigg/\rho_* = JN^2(z) \;.\; V^2/L^2 \;. \tag{9}$$

Thus $JN^2/U'^2 = -g\dfrac{d\bar{\rho}_*}{dz_*}\bigg/\bar{\rho}_*\left(\dfrac{dU_*}{dz_*}\right)^2$ is the local Richardson number of the basic flow at each height z_*, and we shall identify $J^{\frac{1}{2}}N$ as the dimensionless frequency of long internal gravity waves in the case for which $U \equiv 0$ and N is constant (see equation (17)).

In many applications of this theory it happens that $\bar{\rho}_*(z_*)$ varies much more slowly with height than $U_*(z_*)$, so that $-\bar{\rho}'/\bar{\rho} \ll 1$, yet J is nonetheless of order of magnitude unity because $F \ll 1$; in this approximation, which resembles the Bousinesq approximation, we neglect the last two terms of equation (7). Thus the effects of variation of density are neglected in the inertia but retained in the buoyancy.

Considering only two-dimensional disturbances, using the Richardson number instead of the Froude number, and neglecting the inertial effects of the variation of density, we can reduce the system (7), (8) to the form

$$(U-c)(D^2-\alpha^2)\phi - U''\phi + JN^2\phi/(U-c) = 0, \tag{10}$$

$$\alpha\phi = 0 \quad \text{at} \quad z = z_1, z_2, \tag{11}$$

where

$$u' = \partial\psi'/\partial z, \qquad w' = -\partial\psi'/\partial x \tag{12}$$

and

$$\psi' = \phi(z)\exp\{i\alpha(x-ct)\} \;. \tag{13}$$

Equation (10) is called the Taylor-Goldstein equation in honour of its derivation and exploitation by Taylor [10] and Goldstein [11], although the equation was independently published by Haurwitz [12] in the same year, 1931.

(b) <u>Internal gravity waves and Rayleigh-Taylor instability</u>

The important special case of internal gravity waves or Rayleigh-Taylor instability arises when

$$U_* \equiv 0 . \tag{14}$$

Of course this is equivalent to the case when U_* has any constant value, by Galilean transformation. Here there is no scale of the basic velocity, so we use dimensional variables, for which the Taylor-Goldstein problem reduces to

$$c_*^2(D_*^2 - \alpha_*^2)\phi + N_*^2(z_*)\phi = 0, \tag{15}$$

$$\alpha_*\phi = 0 \quad \text{at} \quad z_* = z_{1*},\ z_{2*} , \tag{16}$$

a problem originally due to Rayleigh [4].

The problem has no solution in finite terms for a general function $N_*^2(z_*)$, but there are a few simple solutions known for particular functions $N_*^2(z_*)$. For the simplest, we follow Rayleigh [4] and suppose that $\bar{\rho}_* = \rho_o\exp(-\beta z_*)$ so that $N_*^2 = g\beta$ is constant, and deduce at once that

$$c_*^2 = \{\alpha_*^2 + n^2\pi^2/(z_{2*} - z_{1*})^2\}^{-1} N_*^2,\ \phi = \sin\{n\pi(z_*-z_{1*})/z_{2*}-z_{1*})\} \quad \text{for } n = 1, 2, \ldots . \tag{17}$$

This gives a discrete spectrum of internal gravity waves, stable or unstable according as N_*^2 is positive or negative, with a complete set of eigenfunctions.

Detailed properties of internal gravity waves defined by the standard Sturm-Liouville problem (15), (16), for both general and particular density distributions, are described in the books by Yih [13, Chap. 2] and Krauss [14]. They also treat cases when the upper boundary is a free surface, when the inertial terms due to the variation of density are not negligible, and when the fluid is compressible.

Rayleigh [4] himself proved the outstanding general property, namely that there is instability if and only if light fluid is locally below heavier fluid, i.e. there is instability if and only if N_*^2 is negative somewhere. His proof runs as follows. Multiply equation (15) by the complex conjugate ϕ^* and integrate from z_{1*} to z_{2*} to deduce that

$$c_*^2 \int_{z_{1*}}^{z_{2*}} |D_*\phi|^2 + \alpha_*^2|\phi|^2 dz_* = \int_{z_{1*}}^{z_{2*}} N_*^2|\phi|^2 dz_*, \tag{18}$$

on integration by parts and use of boundary conditions (16). It follows that c_*^2 and therefore ϕ is real, and that c_* is real if $N_*^2>0$ everywhere. Thus there is stability if $N_*^2>0$ everywhere. To prove the converse, Rayleigh noted that the variational principle associated with the Sturm-Liouville problem (15), (16) gives c_*^2 as the minimum of $\int_{z_{1*}}^{z_{2*}} N_*^2 f^2 dz_*$ over the class of functions $f(z_*)$ with square integrable derivatives such that $\int_{z_{1*}}^{z_{2*}} (D_*f)^2 + \alpha_*^2 f^2 dz_* = 1$. It follows at once by the calculus of variations that $c_*^2<0$ if $N_*^2(z_*)<0$ anywhere.

(c) Instability

The interplay of the effects of basic shear and buoyancy is seen in the eigensolutions of the Taylor-Goldstein problem (10), (11). We shall show how, for a given flow and wave-number, the modes may be divided into five classes, some which may be empty:

(1) There is a finite class of non-singular unstable modes. These, giving instability, are the most important and have been given most attention in the literature. The class is empty certainly if the local Richardson number is everywhere greater than or equal to a quarter, the flow then being stable to all waves. These modes are in general the modifications by buoyancy of the unstable modes of shear instability but exceptionally buoyancy with $N^2>0$ everywhere may render unstable a wave of given number which is stable when $N^2 \equiv 0$.

(2) The conjugate damped modes form a finite class of non-singular stable modes.

(3) The marginally stable modes form a finite class of singular neutral modes, each having a branch point at its critical layer.

(4) There is a continuous spectrum of singular neutral modes, each having a singularity no worse than a discontinuity at its critical layer.

(5) The internal gravity waves modified by the basic shear form a discrete class of stable modes when $N^2>0$ everywhere. There are similar unstable modes when $N^2<0$ somewhere.

To discuss the properties of these classes of modes, first note that we may take $\alpha \geq 0$ without loss of generality and that to each unstable mode there corresponds a conjugate stable mode.

The essential mechanism of the instability converts the available kinetic energy of relative motion of layers of the basic flow into kinetic energy of the disturbance, overcoming the potential energy needed to raise or lower fluid when $\frac{d\bar{\rho}_*}{dz_*} < 0$ everywhere. Thus shear tends to destabilize and buoyancy to stabilize the flow. To quantify these tendencies, suppose that two neighbouring fluid particles of equal volumes, at heights z_* and $z_* + \delta z_*$, are interchanged. Then the work δW per unit volume needed to overcome gravity and effect this interchange is given by

$$\delta W = -g\delta\bar{\rho}_* \delta z_*;$$

where $\delta\bar{\rho}_* = \frac{d\bar{\rho}_*}{dz_*}\delta z_*$. In order that horizontal momentum is conserved in the interchange, the particle originally at height z_* will plausibly have final velocity $(U_*+k\delta U_*)\underset{\sim}{i}$ and the other particle $(U_*+(1-k)\delta U_*)\underset{\sim}{i}$, where $\delta U_* = \frac{dU_*}{dz_*}\delta z_*$ and k is some number between zero and one. Then the kinematic energy δT per unit volume released by the basic flow in this way is given by

$$\delta T = \tfrac{1}{2}\,\bar{\rho}_* U_*^2 + \tfrac{1}{2}(\bar{\rho}_* + \delta\bar{\rho}_*)(U_* + \delta U_*)^2 - \tfrac{1}{2}\,\bar{\rho}_*(U_* + k\delta U_*)^2$$
$$- \tfrac{1}{2}(\bar{\rho}_* + \delta\bar{\rho}_*)(U_* + \{1-k\}\delta U_*)^2$$
$$= k(1-k)\bar{\rho}_*(\delta U_*)^2 + U_*\,\delta U_*\delta\bar{\rho}_*$$
$$\leq \tfrac{1}{4}\,\bar{\rho}_*(\delta U_*)^2 + U_*\delta U_*\delta\bar{\rho}_* \ ,$$

equality holding for $k = \frac{1}{2}$. Now a necessary condition for this interchange, and thus for instability, is that

$$\delta W \leq \delta T$$

and therefore that somewhere in the field of flow

$$-\,g\,\frac{d\,\bar{\rho}_*}{dz_*} \leq \tfrac{1}{4}\,\bar{\rho}_*\left(\frac{dU_*}{dz_*}\right)^2 + U_*\,\frac{dU_*}{dz_*}\,\frac{d\bar{\rho}_*}{dz_*}\ , \tag{19}$$

i.e.

$$-\,g\,\frac{d\bar{\rho}_*}{dz_*}\Bigg/\bar{\rho}_*\left(\frac{dU_*}{dz_*}\right)^2 \leq \tfrac{1}{4} \tag{20}$$

if the inertial effects of the variation of density are negligible. The essential idea of this argument is due to Richardson, who in 1920 applied it to turbulence. However, it has been recast by Prandtl, Taylor and many others since. The above form of the argument is essentially that of Chandrasekhar [15, p.491]. The argument is heuristic in the sense that only energetics are considered, the detailed kinematics and dynamics of the interchange of the particles being ignored.

A rigorous form of the argument comes on assuming $c_i \neq 0$, defining H by

$$H = \phi/(U-c)^{\frac{1}{2}}, \tag{21}$$

and substituting H for ϕ in the Taylor-Goldstein equation. This gives

$$D\{(U-c)DH\} - \{\alpha^2(U-c) + \tfrac{1}{2}U'' + (\tfrac{1}{4}U'^2 - JN^2)/(U-c)\}\,H = 0\ . \tag{22}$$

Multiplying this equation by H^* and integrating, we find

$$\int_{z_1}^{z_2} (U-c)\{|DH|^2 + \alpha^2|H|^2\} + \tfrac{1}{2}\,U''|H|^2 + \frac{\tfrac{1}{4}U'^2-JN^2}{U-c}\,|H|^2\,dz = 0\ . \tag{23}$$

The imaginary part of this equation gives

$$-c_i \int_{z_1}^{z_2} |DH|^2 + \alpha^2|H|^2 + (JN^2 - \tfrac{1}{4}\,U'^2)|H|^2/|U-c|^2 dz = 0\ . \tag{24}$$

Therefore

$$0 > -\int_{z_1}^{z_2} |DH|^2 dz$$

$$= \int_{z_1}^{z_2} \{(JN^2 - \tfrac{1}{4}U'^2) + \alpha^2|U-c|^2\}|H|^2/|U-c|^2 \, dz \tag{25}$$

if $c_i \neq 0$. Therefore the local Richardson number satisfies the inequality

$$JN^2/U'^2 < \tfrac{1}{4} \tag{26}$$

somewhere in the field of flow. This is Howard's [16] general proof of the necessary condition of instability Miles [17] proved otherwise for a special class of flows. It can be stated in the form that there is stability if the local Richardson number is everywhere greater than or equal to one quarter. Howard [18] also showed that inequality (25) gives

$$\alpha^2 c_i^2 \leq \max_{z_1 \leq z \leq z_2}(\tfrac{1}{4}U'^2 - JN^2) \; . \tag{27}$$

A similar integral of the Taylor-Goldstein equation can be used to show that instability implies that

$$U'' = 2(U-c_r)JN^2/\{(U-c_r)^2 + c_i^2\} \tag{28}$$

somewhere in the field of flow [18]. When $J=0$ this yields Rayleigh's criterion that there is instability of homogeneous fluid only if the basic velocity profile has a point of inflexion. Unfortunately condition (28) for stratified fluid involves the unknowns c_r and c_i, and so does not give a simple criterion like Rayleigh's. However, it implies that if $U'' \neq 0$ then

$$c_i \leq \max_{z_1 \leq z \leq z_2} |2JN^2/U''| \; ,$$

because $c_i \leq |U-c| = |2(U-c_r)JN^2/U''(U-c)|$ somewhere.

Another simple integral gives Howard's semicircle theorem, that

$$\{c_r - \tfrac{1}{2}(U_{max} + U_{min})\}^2 + c_i^2 < \{\tfrac{1}{2}(U_{max} + U_{min})^2\} \tag{29}$$

provided that $c_i \neq 0$ and $N^2 \geq 0$ everywhere in the field of flow. The method also shows that if $N^2 \leq 0$ everywhere then no non-singular neutral mode exists; unfortunately this does not imply instability because a continuous spectrum of singular stable modes with velocity c within the range of $U(z)$ may exist [cf.19].

For any given basic flow the marginally stable modes are modifications of the 's-modes' for the case $J=0$, although the significance of a point of inflexion is lost when $J=0$. For illustration take the example

$$U = \tanh z, \quad N^2 = \text{sech}^2 z \quad \text{for} \quad -\infty<z<\infty \; . \tag{30}$$

Note that this Brunt-Väisälä frequency comes from taking $\bar{\rho} = \exp(-\tanh z)$, so that $\rho_\infty = \bar{\rho}(\infty) = 1/e$ and $\bar{\rho}_{-\infty} = \bar{\rho}(-\infty) = e$. Holmboe [cf. 20] verified that then a neutral eigensolution of the system (10), (11) is given by

$$c = 0, \quad J = \alpha(1-\alpha), \quad \phi = (\operatorname{sech} z)^{\alpha}(\tanh z)^{1-\alpha} \quad \text{for} \quad 0 \leq \alpha \leq 1. \tag{31}$$

Miles [20] examined the branch point at the critical layer where $z = 0$ in the limit as $c_i \downarrow 0$, finding that the solution (31) was the unique marginally unstable solution provided that one interprets $(\tanh z)^{1-\alpha} > 0$ for $z > 0$ but

$$(\tanh z)^{1-\alpha} = e^{-i\pi(1-\alpha)}|\tanh z|^{1-\alpha} \quad \text{for} \quad z < 0.$$

Hazel [21] computed the unstable mode for various values of α and J. These results are shown in Fig. 1. It can be seen that $\alpha c_i \leq (\frac{1}{4} - J)^{\frac{1}{2}}$ in accord with inequality (27), equality being attained only when $J = \frac{1}{4}$, $\alpha = \frac{1}{2}$. Thus the condition $J \geq \frac{1}{4}$ everywhere happens to be both necessary and sufficient for stability of flows (30). The semi-circle theorem is satisfied, it being found that $c_r = 0$ and $c_i \leq 1$ for the unstable mode (with $c_i = 1$ for $\alpha = 0$, $J = 0$). The s-solution can be seen to arise from (31) when $J = 0$ and $\alpha = \alpha_s = 1$.

Howard [22] showed that if the marginal curve for a given class of flows has equation

$$J = J_s(\alpha) \tag{32}$$

in the (J, α)-plane, then the generalization of formula (4.3.21) gives

$$\left(\frac{\partial c}{\partial \alpha^2}\right)_J = \lim_{c_i \downarrow 0} \int_{z_1}^{z_1} \phi^2 dz \Big/ \int_{z_1}^{z_2} \{2JN^2(U-c)^{-1} - U''\}\phi^2/(U-c)^2 dz. \tag{33}$$

It also follows by partial differentiation that

$$\left(\frac{\partial c}{\partial J}\right)_\alpha = -\left(\frac{\partial c}{\partial \alpha}\right)_J \Big/ \frac{dJ_s(\alpha_s)}{d\alpha_s} \tag{34}$$

at each value α_s of α on the marginal curve (32). Care is needed to evaluate the integrals (33) in the limit as $c_i \downarrow 0$. Incorrect results arise sometimes, for reasons not yet fully understood [23]. If formula (33) gives $(\partial c_i/\partial \alpha)_J = \infty$, the infinity can be resolved when the true behaviour is such that $c \sim (\alpha^2 - \alpha_s^2)^{\frac{1}{2}} \times$ constant as $\alpha \to \alpha_s$ for a fixed value of $J = J_s(\alpha_s)$ [24]. A case with $(\partial c_i/\partial \alpha)_J = 0$ was resolved by Banks, Drazin & Zaturska [25]. However, Howard successfully applied his formulae (33), (34) to Holmboe's solution (31), finding that on the inside of the marginal curve

$$\left(\frac{\partial c}{\partial \alpha}\right)_J = \frac{i \cos \pi\alpha_s}{\pi\alpha_s} \; B\left(\tfrac{3}{2} - \alpha_s, \alpha_s\right) ,$$

$$\left(\frac{\partial c}{\partial J}\right)_\alpha = -\frac{i}{\pi\alpha_s} \; B\left(\alpha_s, \tfrac{1}{2}\right) ,$$

in terms of the beta function.

If U, U' or $\bar{\rho}$ is discontinuous, at $z = z_o$ say, one can show that

$$\begin{aligned} &\Delta[\phi/(U-c)] = 0 , \\ &\Delta[\bar{\rho}\{(U-c)D\phi - U'\phi + \phi/F(U-c)\}] = 0, \end{aligned} \tag{35}$$

in order that the normal velocity and the pressure are continuous at the interface with mean position $z = z_o$.

These conditions are useful to work out simple examples for which the Taylor-Goldstein equation can be solved piecewise in terms of elementary functions. Here we take the example with the basic flow

$$U = \begin{cases} 1 \\ z \\ -1 \end{cases} \quad \bar{\rho} = \begin{cases} \rho_\infty & \text{for } z>1 \\ \tfrac{1}{2}(\rho_\infty + \rho_{-\infty}) & \text{for } -1<z<1 \\ \rho_{-\infty} & \text{for } z<-1, \end{cases} \tag{36}$$

after Taylor [10, §3] and Goldstein [11, §§3,5]. (Note that $N^2 = 0$ for $z \neq \pm 1$.) They found that there is instability if and only if

$$2\alpha/(1-e^{-2\alpha}) - 1 < J < 2\alpha/(1-e^{-2\alpha}) - 1 . \tag{37}$$

where $J = gL(\rho_{-\infty}-\rho_\infty)/V^2(\rho_{-\infty}+\rho_\infty)$ and $(\rho_{-\infty}-\rho_\infty) \ll (\rho_\infty+\rho_\infty)$, as shown in Fig. 2.

When $J = 0$ this problem reduces to one of Rayleigh's for homogeneous fluid. It can be seen that for a given value of $\alpha>\alpha_s \doteqdot 0.64$ a wave is stable when $J = 0$, unstable for a narrow band of positive values of J, and stable again for large values of J. Thus buoyancy, which seems a stabilizing influence, can render unstable a wave that would otherwise be stable. Taylor [10, p.500] remarked that the unstable band centres round the particular wave whose length is such that a backward-moving internal gravity wave at $z = 1$, regarded as existing at a discontinuity of density in fluid all of which moves with velocity $U = 1$, has the same velocity as a similar forward-moving wave at $z = -1$, and therefore the instability may be regarded as due to a kind of resonance between these backward- and forward-moving internal gravity waves. Howard and Maslowe [26] have discussed this phenomenon, and also how a flow which is stable when $J = 0$ may be unstable for $J > 0$.

(d) Internal gravity waves with basic shear

Next we consider the qualitative character of the eigensolutions of the Taylor-Goldstein problem (10), (11) as J decreases from infinity for fixed functions $U(z)$ and $N(z)$ and a fixed value of α. When $J = \infty$, a condition which we may regard as expressing either $V = 0$ or $g = \infty$, there are internal gravity waves as described in section (b). There we found an infinity of discrete modes with eigenvalues of the form $c = \pm J^{\frac{1}{2}}\gamma_o(n)$ for $n = 1, 2, \ldots$, and with a complete set of eigengunctions ${}^o\phi_n(z)$; these modes are all stable if $N^2(z) \geq 0$ everywhere. When $J = 0$, which we may regard as expressing either $V = \infty$, $g = 0$ or $N^2(z) \equiv 0$, there arises the case of a homogeneous fluid, for which it is well known that there is a finite number (possibly zero) of unstable modes and a continuous spectrum of singular neutrally stable modes. The change of the pattern of the modes as J decreases from infinity to zero is quite complicated, but briefly one may say that the phase velocities c of the infinite discrete spectrum of stable modes are divided into two classes. For the first class the values of c decrease to the global maximum value of $U(z)$ over the field of flow as J decreases from infinity, and for the second class c increases to the minimum value of $U(z)$ [25]. The eigenfunctions associated with each class form a complete set. As J decreases so much that the local value of the Richardson number is less than a quarter somewhere, either class may be replaced in whole or in part by a finite number of complex eigenvalues or a continuous spectrum of real values.

It can be shown that eigenvalues are given by

$$c = \pm J^{\frac{1}{2}}\gamma_o(n) + O(1) \qquad \text{as } J \uparrow \infty \tag{38}$$

for $n = 1, 2, \ldots$. If z_m is the height of a simple maximum or minimum of $U(z)$ such that $z_1 < z_m < z_2$, then it can be shown that

$$c = U_m - 2JN_m^2/n(n+2)U_m'' + o(J) \qquad \text{as } J \downarrow 0 \tag{39}$$

for $n = 1, 2, \ldots$ and any value of α, where $U_m = U(z_m)$ etc. If, however, the height z_m of the maximum or minimum is at a boundary, say $z_m = z_1$, and the shear does not vanish there, then there is a finite number p (possibly zero) of modes such that

$$U_1 - c \sim U_1'\{A_n(J - J_n)\}^{(1 - 4J_nN_1^2/U_1'^2)^{-\frac{1}{2}}} \qquad \text{as } J \downarrow J_n \tag{40}$$

for $n = 1, 2, \cdots, p$ and some constants $A_n(\alpha)$ and $J_n(\alpha) < \frac{1}{4}U_1'^2/N_1^2$. For the rest of the modes,

$$U_1 - c \sim B_n \exp\{-n\pi(JN_1^2/U_1'^2 - \tfrac{1}{4})^{-\frac{1}{2}}\} \qquad \text{as } J \downarrow \tfrac{1}{4}U_1'^2/N_1^2 \tag{41}$$

for $n = p+1, p+2, \ldots$ and some $B_n(\alpha)$. Here we take $U_1 = U(z_1)$ etc. and may derive similar results if $z_m = U_2$. These results are perhaps most easily understood by seeing figures 3 and 4 for two examples, which also indicate the dependence of c upon n.

(e) Viscous fluid

The instability of plane parallel flow of a viscous stratified fluid leads to a sixth-order eigenvalue problem, it being desirable to allow for the diffusion of density as well as momentum in order to model a real fluid consistently. The problem thus depends upon the Reynolds and Prandtl numbers (if the density variation is due to temperature variation) as well as the Richardson number for each class of dynamically similar basic flows. Flows with points of inflexion, such as unbounded flows, are the most unstable at large values of the Reynolds number; thus the inviscid theory gives a useful criterion of their overall stability. Maslowe and Thompson [27] found numerically some detailed stability characteristics of a hyperbolic-tangent shear layer and Gage and Miller [28] of a jet, which exemplify this. Gage and Reid [29] found the characteristics for plane Poiseuille flow of a stratified viscous fluid with Prandtl number one by asymptotic methods for large Reynolds numbers. Gage [30] generalized these methods to find a universal criterion for stability of flows with a point of inflexion when the fluid has Prandtl number one, namely that

$$JN^2/U'^2 > 0.0554 \qquad (38)$$

everywhere in the field of flow. Davey and Reid [31] have found many detailed properties of the modes as well as critical Reynolds numbers for plane Poiseuille flow with two different density distributions; for one they illuminate the effects of viscosity when light fluid above heavy creates instability.

(f) Experimental results

Taylor's [10] paper contained work originally written for his Adams Prize essay of 1915, but he delayed publication for 16 years in the vain hope of performing experiments to confirm his theory. Even today the technical difficulties of controlled experiments on the flow of stratified fluids are formidable. They have prevented close quantitative comparison with the theory, but Thorpe [32] and Scotti and Corcos [33] have found encouraging agreement between theoretical results and their experiments. The importance of this kind of instability in clear-air turbulence and in generating internal gravity waves in the atmosphere has also led meteorologists to relate their observations to the theory (see, for example, [34]).