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SMR/23/44

AUTUMN COURSE ON APPLICATIONS OF ANALYSIS TO MECHANICS

22 September - 26 November 1976

TOPICS IN LINEAR ELASTICITY

Chapter III - Dynamics

Chapter IV - Stress-strain relation

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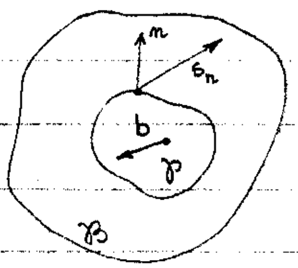
III - DYNAMICS

Let a density field ρ be given on $\overline{\mathcal{B}}$, i.e., a strictly positive scalar field of class $C(\overline{\mathcal{B}})$. Let $\mathcal{T} = (t_0, t_1)$ be a fixed interval of time. We call a motion of $\mathcal{B}$ any smooth vector field $u(x,t)$ defined on the cylinder $\mathcal{B}\times\mathcal{T}$; we also use the fllng terminology:
$\dot{u}$, the velocity; $\ddot{u}$, the acceleration; $\dot{E}$, the strain-rate. Given any part $\mathcal{P}\subset\mathcal{B}$ (part = regular subregion) we call

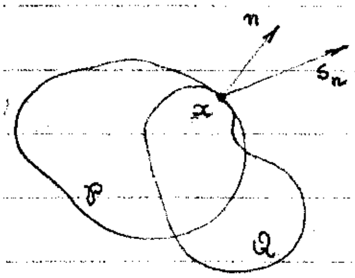
$$q(\mathcal{P}) = \int_{\mathcal{P}} \rho\, \dot{u}\, dV \qquad \text{the momentum of } \mathcal{P}\text{, and}$$

$$k(\mathcal{P}) = \int_{\mathcal{P}} \rho (x - x_0)\times \dot{u}\, dV \qquad \text{the moment of momentum of } \mathcal{P} \text{ (about } x_0).$$

We now introduce the notion of a system of forces for $\mathcal{B}$. For any part $\mathcal{P}\subset\mathcal{B}$ fixed there are two types of forces on $\mathcal{P}$:



(i) body forces per unit volume, $b(x,t)$, of class $C^0(\overline{\mathcal{B}}\times[t_0,t_1))$. These model the long-distance action exerted at x by the external world.



(ii) surface forces per unit area of $\partial\mathcal{P}$, $s_n(x,t)$, of class $C^0(\overline{\mathcal{B}}\times[t_0,t_1)) \cap C^{1,0}(\mathcal{B}\times\mathcal{T})$. These model the contact action exerted at x by material outside of $\mathcal{P}$ on material inside of $\mathcal{P}$. According to Cauchy's stress principle, s_n depends on $\mathcal{P}$ only through the outward unit normal n to $\partial\mathcal{P}$ at x, i.e., there exists a fn. $s_n = s_n(x,t;n)$ defined on $\overline{\mathcal{B}}\times[t_0,t_1)\times\mathcal{S}(1)$ which delivers the so-called stress vector (at (x,t) w.r.t. oriented regular surfaces through x with unit normal n) if $x\in\mathring{\mathcal{B}}$, the so-called surface traction if $x\in\partial\mathcal{B}$. On $\mathcal{P}$ a given system of forces exerts a

total force: $f(\mathcal{P}) = \int_{\mathcal{P}} b\, dV + \int_{\partial\mathcal{P}} s_n\, dS$, and a

total moment (about x_0): $m(\mathcal{P}) = \int_{\mathcal{P}} (x-x_0)\times b\, dV + \int_{\partial\mathcal{P}} (x-x_0)\times s_n\, dS$.

Euler-Cauchy fundamental postulates consist in the law of balance of momentum:

and the law of balance of moment of momentum:

$$\dot{k}(\mathcal{P}) = m(\mathcal{P}), \tag{2}$$

for each part $\mathcal{P} \subset \mathcal{B}$. When appropriate smoothness conditions prevail, these balance laws have local counterparts which have been established by Cauchy, among other things, in the following most celebrated theorem.

Theorem 1 – Let a system of forces (b, s_n) be given. Then, a motion u obeys (1)&(2) if and only if the fllng two conds are satisfied:

(i) $\exists$ a symmetric tensor field S, the stress field, of class $C^{1,0}(\mathcal{B}\times\mathcal{T})$ s.th.

$$s_n = S(x,t)\,n \quad, \forall n \in \mathcal{S}(1)\ ; \tag{3}$$

(ii) u, S, b satisfy the eq. of motion

$$\operatorname{div} S + b = \rho\, \ddot{u}\ . \tag{4}$$

Remark 1 – Under the hypotheses, (1) $\sim$ (4) and (2) $\sim$ $S \in \mathrm{Sym}$.

Remark 2 – In the finite theory, balance law (1) remains formally unaltered. However, in the definitions on $k(\mathcal{P})$ and $m(\mathcal{P})$ x has to be replaced by $\mathrm{x} = \mathrm{x}(x,t)$, the place occupied by x at time t, so that the balance of moment of momentum reads

$$\int_{\mathcal{P}} \rho(\mathrm{x}-x_0)\times\ddot{u}\,dV = \int_{\mathcal{P}}(\mathrm{x}-x_0)\times b\,dV + \int_{\partial\mathcal{P}}(\mathrm{x}-x_0)\times s_n\,dS \tag{2'}$$

Let $\varepsilon = \sup_{\mathcal{B}\times\mathcal{T}} |u(x,t)|$. Then,

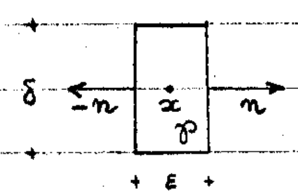
$$\mathrm{x}(x,t) - x_0 = x - x_0 + u(x,t) = x - x_0 + O(\varepsilon) \quad \text{as } \varepsilon \to 0,$$

and (2)' reduces to (2) in the limit. Thus, the statement fllng formula (6), Ch. is now more completely motivated.

Sketch of the proof of Thm 1[1] – (i) the surface force is volume bounded (follows from (1) and the assumption that $b, \ddot{u}$ are bounded):

$$\left|\int_{\partial\mathcal{P}} s_n\,dS\right| \le \text{const.}\ \mathrm{vol}(\mathcal{P}) \quad ; \tag{*}$$

(ii) for the sake of simplicity, consider only the 2-dim.al case. Choose x, n arbitrary and apply (*) to the (δ,ε)-part $\mathcal{P}$ to show that



$$s_n = -s_{-n} \quad \text{(Cauchy reciprocal lemma)}\ ; \tag{**}$$

(iii) for any $(x,t) \in \mathcal{B}\times\mathcal{T}$ fixed, it suffices to show that the

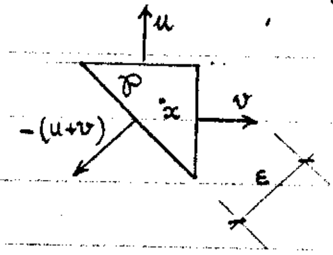
(1) This sketch was furnished by M.E. GURTIN when he was lecturing in Bari, Italy, Sept 1976 and

mapping $n \longmapsto s_n(x,t;n)$ is the restriction to $\mathcal{S}(1)$ of a linear fn on $\mathcal{V}$. Then, extend s_n to all of $\mathcal{V}$ as follows (we omit the first two arguments)

$$\begin{aligned} s(v) &= |v|\ s_n(v/|v|) && \text{if } v \neq 0, \\ &= 0 && \text{if } v = 0; \end{aligned}$$

show that $s(\alpha v) = \alpha\, s(v)$, when α is any real number (if $\alpha \geq 0$, trivial; if $\alpha < 0$, $s(\alpha v) = s(|\alpha|(-v)) = |\alpha|\, s(-v) = \alpha\, s(v)$, the last step following from $(\ast\ast)$);

show that $s(u+v) = s(u) + s(v)$ whenever u and v are linearly independent



(observe that

$$\int_{\partial P} s_n\, dS = \int_{S_u} s_n\left(\frac{u}{|u|}\right) dS_u + \int_{S_v} s_n\left(\frac{v}{|v|}\right) dS_v - \int_{S} s_n\left(\frac{u+v}{|u+v|}\right) dS$$

$$= \int_S \frac{1}{|u+v|}\left(s(u) + s(v) - s(u+v)\right) dS$$

and pass to the limit for $\varepsilon \to 0$ taking into account $(\ast)$). ∎

<u>Remark 3</u> – In the proof of Thm 1, continuity of s_n as a fn of x played a central rôle. In answer to the question: are there any physically reasonable hypotheses on the total contact force

$$f_c(P) = \int_{\partial P} s_n\, dS$$

that yield as a consequence the continuity and the linearity (as a fn of n) of the area density s_n of f_c? Gurtin & Martins (2) have shown that s_n is continuous if f_c has uniform average density (in a sense that they make precise), and, further, that s_n is linear a.e. if f_c is weakly balanced, i.e. (cf. $(\ast)$), if

$$|f_c(P)| \leq \text{const.}\ vol(P).$$

– If u is independent of time (4) reduces to the <u>equation of equilibrium</u>

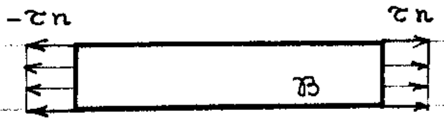
$$\operatorname{div} S + b = 0. \tag{5}$$

As counterparts of pure strains in kinematics, we consider some simple solutions of (5) (with vanishing body force) in order to elucidate the meaning of the stress tensor.

M. E. GURTIN & L. C. MARTINS

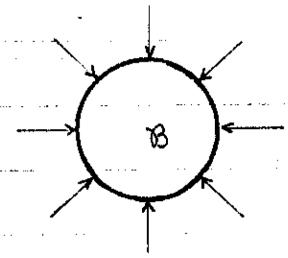
Let S be a symmetric constant stress field of the three following types:

(i) pure tension (with tensile stress τ in the direction n)



$$S = \tau\, n \otimes n \quad , \tau \in \mathbb{R} \quad ;$$

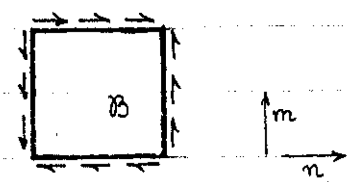
(ii) uniform pressure (of amount π)



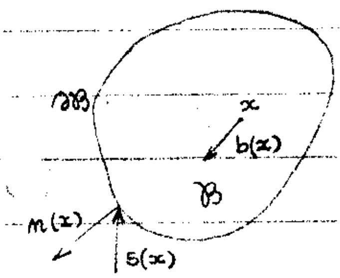
$$S = -\pi\, \mathbf{1} \quad ;$$

(iii) pure shear (of amount σ w.r.t. the direction pair n, m with $n \cdot m = 0$)

$$S = 2\sigma \operatorname{sym}(n \otimes m) \quad .$$



ⓒ – Let now a body force field $b(x)$ and surface force field $s(x)$ be given, respectively, for $x \in \overline{\mathcal{B}}$ and $x \in \partial\mathcal{B}$. Let further $S(x) \in \mathcal{S}$ satisfy the eq.s of equilibrium



$$\operatorname{div} S + b = 0 \qquad \text{in } \mathcal{B} \, , \qquad (5$$

$$S n = s \qquad \text{in } \partial\mathcal{B} \, .$$

Finally, if $\Psi(x)$, $\Phi(x)$ are smooth tensor fields defined on $\overline{\mathcal{B}}$ (Ψ of any order, Φ of 2nd order), consider the follg identity:

$$\int_{\mathcal{B}} (\nabla \Psi)\, \Phi^T \, dV = - \int_{\mathcal{B}} \Psi \otimes \operatorname{div} \Phi \, dV + \int_{\partial\mathcal{B}} \Psi \otimes \Phi n \, dS \quad . \qquad (6$$

On identifying Φ with S in (6), and taking into account eq.s (5), we obtain the follg trivial but important theorem of Signorini[3] –

Theorem 2. Let S be a (symmetric, smooth) stress field that satisfies the eq.s of equil., and let Ψ be any smooth tensor field. Then

$$\int_{\mathcal{B}} (\nabla \Psi)\, S \, dV = \int_{\mathcal{B}} \Psi \otimes b \, dV + \int_{\partial\mathcal{B}} \Psi \otimes s \, dS \quad . \qquad (7$$

(3) [illegible]

Signorini's thm has many interesting consequences. The most important one is

<u>Corollary 1</u> (<u>Thm. of virtual work</u>). Let u be a smooth displacement field defined on $\overline{\mathcal{B}}$. Then

$$\int_{\mathcal{B}} S \cdot E(u)\, dV = \int_{\mathcal{B}} b \cdot u \, dV + \int_{\partial\mathcal{B}} s \cdot u \, dS \,. \tag{8}$$

In words, the work done by the surface and body forces over the displacement field u equals the work done by the stress field over the strain field $E(u)$ corresponding to u.

<u>Pf. of Cor. 1</u>. In (7), put $\mathbf{\Psi} \equiv u$, and take the trace of both members. ∎

Define the <u>mean stress</u> $\overline{S}(\mathcal{B})$ as follows

$$\overline{S}(\mathcal{B}) = \frac{1}{\mathrm{vol}(\mathcal{B})} \int_{\mathcal{B}} S \, dV \,. \tag{9}$$

<u>Corollary 2</u>. The mean stress at equilibrium depends only on the assigned body and surface forces:

$$\overline{S}(\mathcal{B}) = \frac{1}{\mathrm{vol}(\mathcal{B})} \left(\int_{\mathcal{B}} (x - x_0) \otimes b \, dV + \int_{\partial\mathcal{B}} (x - x_0) \otimes s \, dS \right). \tag{10}$$

<u>Pf.</u> In (7), put $\mathbf{\Psi} \equiv (x - x_0)$, with x_0 any fixed point, and use the identity $\nabla(x - x_0) = \mathbf{1}$. ∎

<u>Remark 4</u>. Clearly, as $S(x) \in \mathrm{Sym}$, (10) can be equivalently written as

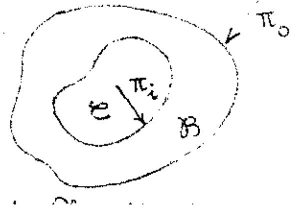
$$\overline{S}(\mathcal{B}) = \frac{1}{\mathrm{vol}(\mathcal{B})} \, \mathrm{sym} \left(\int_{\mathcal{B}} (x - x_0) \otimes b \, dV + \int_{\partial\mathcal{B}} (x - x_0) \otimes s \, dS \right); \tag{11}$$

for the same reasons,

$$\mathrm{skw}\left(\int_{\mathcal{B}} (x - x_0) \otimes b \, dV + \int_{\partial\mathcal{B}} (x - x_0) \otimes s \, dS \right) = 0 \quad \text{, i.e., } \; m(\mathcal{B}) = 0 \,.$$

We now give two examples, due to Signorini, of application of (10)[4].

(i) (<u>hydrostatic loading</u>). Let $\mathcal{B}$ have a cavity $\mathcal{C}$, and let the body be subject to hydrostatic pressures



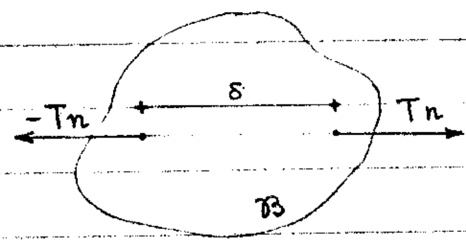
π_o, π_i on the outer and inner surfaces, respectively. Neglect body forces. Then, using also the divergence thm recalled in the proof of Thm 3, Chap. II with $u \equiv (x - x_0)$, we arrive at

$$\bar{S}(\mathcal{B}) = -\left(\pi_o + \frac{\mathrm{vol}(\mathcal{C})}{\mathrm{vol}(\mathcal{B})}(\pi_o - \pi_i)\right) 1 ,$$

a result which shows that hydrostatic loading always gives rise to an hydrostatic mean stress.

(ii) (<u>tensile loading</u>)

Let the body be subject to two equilibrated concentrated loads, a distance δ apart. Then,



$$\tau_N(\bar{S}(\mathcal{B}); n) = \frac{T}{\mathrm{vol}(\mathcal{B})/\delta}$$ is the only non vanishing component of $\bar{S}(\mathcal{B})$,

and this shows that tensile loading gives rise to pure tension in mean.

IV - STRESS-STRAIN RELATION

ⓐ - We want to make formal our intuitive notion of a linearly elastic body as a body for which the stress S is a linear function of the displacement gradient ∇u such as to vanish when the displacement is rigid.

<u>Definition 1</u> (Cauchy, 1829) - A point $x \in \mathcal{B}$ is <u>linearly elastic</u> if there exists $\mathbb{C}(x) \in \mathrm{Lin}(\mathrm{Lin})$ s. th.

(i) (<u>minor symmetries</u>) $\quad \mathbb{C}V = \mathbb{C}(\mathrm{sym}V) \in \mathrm{Sym} \quad , \forall V \in \mathrm{Lin}$; (1)

(ii) (<u>stress-strain relation</u>) $\quad S(x) = \mathbb{C}(x)\nabla u(x)$. (2)

<u>Remark 1</u> - In view of $(1)_1$, (2)

$$S = \mathbb{C}E \, , \qquad (3)$$

whence the importance of the concept of strain.

<u>Remark 2</u> - In view of $(1)_2$, and of a result established in Chap. II,

$$\nabla u(x) \in \mathrm{Skw} \Leftrightarrow u \text{ is an i.r.d. loc.} \Rightarrow S(x) = 0 \, .$$

<u>Remark 3</u> - It follows from (1) and the properties of the i.p. of Lin that

$$V \cdot \mathbb{C}W = (\mathrm{sym}V) \cdot \mathbb{C}(\mathrm{sym}W) \quad , \forall V, W \in \mathrm{Lin} \, . \qquad (4)$$

Henceforth we identify $\mathbb{C}$ with its restriction to Sym : $\mathbb{C} \in \mathrm{Lin}(\mathrm{Sym}, \mathrm{Sym})$.

Most frequently, $\mathbb{C}$ has the fllwg <u>qualitative</u> property :

(ms) (<u>major symmetry</u>) $\quad V \cdot \mathbb{C}W = W \cdot \mathbb{C}V \quad , \forall V, W \in \mathrm{Sym}$ (5)

(in components : $\mathbb{C}_{ijhk} = \mathbb{C}_{hkij}$) and the fllwg <u>quantitative</u> property :

(p) (<u>positiveness</u>) $\quad V \cdot \mathbb{C}V > 0 \quad , \forall V \in \mathrm{Sym} - \{0\}$. (6)

To see the importance and the meaning of properties (ms) and (p), we firstly introduce the notion of a <u>strain path</u> Σ connecting E_0 and E_1, i.e., of a smooth curve $E = E(\alpha)$, $\alpha \in [\alpha_0, \alpha_1]$ in Sym, such that $E_0 = E(\alpha_0)$ and $E_1 = E(\alpha_1)$. Given $\mathbb{C}$ and Σ, we call (<u>density of</u>) <u>work done on going from E_0 to E_1</u> along the strain path Σ the scalar

$$\lambda(\Sigma) = \int_{\alpha_0}^{\alpha_1} S(\alpha) \cdot \frac{d}{d\alpha}E(\alpha)\, d\alpha \, , \quad \text{where } S(\alpha) = \mathbb{C}E(\alpha) \, , \ \alpha \in [\alpha_0, \alpha_1] \, . \qquad (7)$$

Secondly we make another relevant definition

<u>Definition 2</u> (Green, 1839). Let $x \in \mathcal{B}$ be linearly elastic with elasticity tensor $\mathbb{C}$. A smooth function $\sigma : \mathrm{Sym} \to \mathbb{R}$ is called a (density of) <u>stored energy function</u> for x if σ is s.th.

(i) (cond. of normalization) $\sigma(0) = 0$;

(ii) $\nabla\sigma(E) = \mathbb{C}\,E \quad , \forall E \in \mathrm{Sym}$ (8)

<u>Theorem 1</u>. If a stored energy fn. exists, then it is unique and

$$\lambda(\Sigma) = \sigma(E_1) - \sigma(E_0)$$

for each strain path Σ going from E_0 to E_1.

<u>Pf.</u> A direct consequence of Def. 2. ■

<u>Theorem 2</u> (<u>Characterization of property (ms)</u>). The fllng statements are equivalent.

(i) $\mathbb{C}$ has property (ms).

(ii) $\sigma(E) = \frac{1}{2} E \cdot \mathbb{C} E$. (9)

(iii) The work done along any <u>closed</u> strain path vanishes.

We omit the proof of this thm, that is not instructive within the limits of our present purposes.

<u>Theorem 3</u> (<u>Characterization of property (p)</u>). Let $\mathbb{C}$ have the property (ms). Then, the fllng statements are equivalent.

(i) $\mathbb{C}$ has property (p).

(ii) The work done along any strain path starting from $E_0 = 0$ is positive.

<u>Remark 4</u>. We omit the proof of Thm 3 as well, for the same reason as before. The interest of this thm is twofold: first, it supplies a physically reasonable motivation for accepting (6) as an a priori restriction on $\mathbb{C}$; moreover, the hypotheses of major symmetry and positiveness are quite standard in establishing all of the variational principles of elastostatics.

) We now make precise the notion of a linearly elastic solid with maximal symmetry of response.

<u>Definition 3</u> – The <u>symmetry group</u> of a l.e.p. $x \in \mathcal{B}$ is the group of all $Q \in \mathrm{Orth}$ s.th.

$$Q(\mathbb{C}E)Q^T = \mathbb{C}(QEQ^T) \quad , \ \forall E \in \mathrm{Sym}. \tag{10}$$

<u>Definition 4</u> – A l.e.p. $x \in \mathcal{B}$ is <u>isotropic</u> if the symmetry group is the full orthogonal group Orth.

The stress-strain relation is especially simple in the isotropic case, which is by far the most studied one for this obvious reason. The aspect of $\mathbb{C}$ when the body is isotropic at x is given by the following <u>representation theorem</u>.

<u>Theorem 4</u> (1) – Let a l.e.p. with elasticity tensor $\mathbb{C}$ be isotropic. Then there exist scalars μ and λ s.th.

$$\mathbb{C} = 2\mu\, \mathbb{1} + \lambda\, 1 \otimes 1 \, . \tag{11}$$

In order to prove Thm 4, we need an auxillary result.

<u>Lemma 1</u> – Let $\mathbb{S} : \mathrm{Sym} \to \mathrm{Sym}$ be isotropic, i.e., s.th.

$$Q\,\mathbb{S}(E)\,Q^T = \mathbb{S}(QEQ^T) \quad , \ \forall E \in \mathrm{Sym},\ Q \in \mathrm{Orth}. \tag{12}$$

Then each proper vector of $E \in \mathrm{Sym}$ is a proper vector of $\mathbb{S}(E)$.

<u>Pf.</u> Given $e \in \mathcal{S}(1)$, denote by $R_e \in \mathrm{Orth}$ the reflection across the plane perpendicular to e, i.e., an orthogonal transf. s.th.:

(i) $R_e v = -v \iff v$ is parallel to e;

(ii) $R_e v = v \iff v$ is orthogonal to e.

Now let e be a proper vector of E. In view of the spectral thm (see Chap. I)

$$E = \varepsilon\, e \otimes e + \varepsilon_2\, e_2 \otimes e_2 + \varepsilon_3\, e_3 \otimes e_3 \, ,$$

where e_2 and e_3 are orthogonal to e. Also, in view of the definition of R_e,

$$R_e E R_e^T = E \, ,$$

and (12) implies that

$$R_e\, \mathbb{S}(E)\, R_e^T = \mathbb{S}(E) \text{, i.e., } R_e \text{ commutes with } \mathbb{S}(E). \text{ Thus}$$

[illegible]

$$R_e \mathbb{S}(E)\, e = \mathbb{S}(E) R_e e = -\mathbb{S}(E)\, e$$

and we conclude, again from the definition of R_e, that $\mathbb{S}(E)e$ is parallel to e, i.e., e is a proper vector of $\mathbb{S}(E)$. ■

Pf. of Thm. 4 - Notice that, for $e \in \mathcal{S}(1)$, the dyad $e\otimes e$ has for proper vectors e and every unit vector orthogonal to e. Thus, in view of Lemma 1, $\mathbb{C}(e\otimes e)$ has the same proper vectors, so that, by applying the spectral thm,

$$\mathbb{C}(e\otimes e) = 2\mu(e)\, e\otimes e + \lambda(e)\, 1 \quad , \forall e \in \mathcal{S}(1) \quad . \tag{13}$$

Now, for any choice of e and f in $\mathcal{S}(1)$, there exists $Q \in \mathrm{Orth}$ s.t. $Qe = f$. As $\mathbb{C}$ is isotropic, using such e, f and Q we have

$$0 = Q\,\mathbb{C}(e\otimes e)Q^T - \mathbb{C}(f\otimes f) = 2\left(\mu(e)-\mu(f)\right) f\otimes f + \left(\lambda(e)-\lambda(f)\right) 1,$$

or rather, as $f\otimes f$ and 1 are linearly independent, μ and λ must be scalar constants, and (13) reduces to

$$\mathbb{C}(e\otimes e) = 2\mu\; e\otimes e + \lambda\, 1 \quad . \tag{14}$$

Then the desired result follows easily from the linearity of $\mathbb{C}$, the spectral thm, and the definition of trace. ■

Remark 5 - Notice that $\mathbb{S}$ in Lemma 1 is not assumed to be linear.

Remark 6 - The stress-strain relation for an isotropic material reads

$$S = 2\mu E + \lambda\,(1\cdot E)\, 1 \quad , \forall E \in \mathrm{Sym}\,; \tag{15}$$

μ, λ are called the Lamé moduli of the material.

Remark 7 - $\mathbb{C}$, as given by (11), has property (ms), so that for isotropic materials Cauchy and Green notions of elasticity lead to the same results.

Recalling some definitions of Chap. I, we write

$$E = \mathrm{sph}\, E + \mathrm{dev}\, E \tag{16}$$

and

$$\mathbb{C}E = (2\mu+3\lambda)\,\mathrm{sph}\, E + 2\mu\,\mathrm{dev}\, E \quad . \tag{17}$$

Then the fllng interesting thm becomes obvious, if one recalls that Dev is the orthogonal complement of Sph w.r.t. Sym.

<u>Theorem 5</u>[2] – Let $\mathbb{C}$ isotropic, and let $\lambda \neq 0$. Then, $\mathbb{C}$ has exactly two pairs (proper value, characteristic space), and these are:
$(2\mu + 3\lambda\ ,\ \mathrm{Sph})$ and $(2\mu\ ,\ \mathrm{Dev})$.

Moreover, from (ii) of Thm 2 and the fact that
$$|E|^2 = |\mathrm{sph}\,E|^2 + |\mathrm{dev}\,E|^2 ,$$
we easily obtain the formula
$$\sigma(E) = \frac{1}{2}\left((2\mu + 3\lambda)\,|\mathrm{sph}\,E|^2 + 2\mu\,|\mathrm{dev}\,E|^2 \right) \tag{18}$$
and the majorizations
$$\max\left\{\mu\,,\ \frac{2\mu+3\lambda}{2}\right\} |E|^2 \geqslant \sigma(E) \geqslant \min\left\{\mu\,,\ \frac{2\mu+3\lambda}{2}\right\} |E|^2 \quad, \forall E \in \mathrm{Sym}. \tag{19}$$
Thus, the fllng thm is also obviously true.

<u>Theorem 6</u>[2] – Let $\mathbb{C}$ isotropic. Then, the fllng statements are equivalent –
(i) $\mathbb{C}$ has the property (p).
(ii) $\mu > 0$ & $2\mu + 3\lambda > 0$. (20)