INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
MIRAMARE - P.O.B. 586 - 34100 TRIESTE (ITALY) - TELEPHONES: 224281/2/3/4/5/6 - CABLE: CENTRATOM

SMR/23 /45

AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

22 September - 26 November 1976

DISCONTINUOUS WAVES

(Part II)

P. CHADWICK
University of East Anglia
Norwich, U.K.

These are preliminary lecture notes intended for participants only.
Copies are available outside the Publications Office (T-floor) or from
room 112.

8. Basic Equations of Non-Linear Elasticity Theory

We shall be concerned with the purely mechanical theory of elasticity in which no account is taken of thermo-mechanical interaction.

(a) Field equations. The field equations associated with the balances of mass and momentum (equations (4·1) to (4·3)) are in turn

$$\dot{\rho} + \rho \operatorname{div} \underline{v} = 0 \quad \left(\text{in components } \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_p}(\rho v_p) = 0 \right), \tag{8·1}$$

$$\rho \underline{f} = \operatorname{div} \underset{\sim}{\sigma} + \rho \underline{b} \quad \left(\rho f_i = \frac{\partial \sigma_{pi}}{\partial x_p} + \rho b_i \right), \tag{8·2}$$

$$\underset{\sim}{\sigma}^T = \underset{\sim}{\sigma} \qquad (\sigma_{ji} = \sigma_{ij}). \tag{8·3}$$

Here $\underset{\sim}{\sigma}$ is the stress tensor, related to the contact force $\underline{t}_{(\underline{n})}$ by

$$\underline{t}_{(\underline{n})} = \underset{\sim}{\sigma}^T \underline{n} \quad (t_{(\underline{n})i} = \sigma_{pi} n_p), \tag{8·4}$$

and, as in Section 7, a superimposed dot denotes a material time derivative, evaluated with the referential position $\underline{X}$ held fixed. The continuity equation (8·1) can be integrated in the form

$$\rho = J^{-1} \rho_r , \tag{8·5}$$

where ρ_r is the density in the reference configuration B_r and

$$J = \det \underset{\sim}{F} \; (> 0). \tag{8·6}$$

An alternative measure of stress is the nominal stress $\underset{\sim}{s}$, defined by whereas the Cauchy stress $\underset{\sim}{\sigma}$ is exclusively concerned with the current configuration B_t, $\underset{\sim}{s}$ involves both B_t and B_r: $\underset{\sim}{s}$ can be interpreted in terms of force per unit area in B_r. With the aid of the definition (8·7) the equation of motion (8·2) can be put into the referential form

$$\rho_r \ddot{\underset{\sim}{x}} = \operatorname{Div} \underset{\sim}{s} + \rho_r \underset{\sim}{b} \qquad \left(\rho_r \frac{\partial^2 x_i}{\partial t^2} = \frac{\partial s_{pi}}{\partial X_p} + \rho_r b_i \right). \tag{8·8}$$

(b) <u>Constitutive equations.</u> The fundamental idea characterizing the property of elasticity in mechanics is that the stress at a particle $\mathcal{P}$ is uniquely determined by the current value of the deformation gradient at $\mathcal{P}$. Thus the constitutive equation of elasticity can be stated in the alternative forms

$$\underset{\sim}{\sigma} = \hat{\underset{\sim}{\sigma}}(\underset{\sim}{F}), \quad \underset{\sim}{s} = \hat{\underset{\sim}{s}}(\underset{\sim}{F}). \tag{8·9}$$

Strictly these equations relate to a single particle, but we shall suppose that every particle has the same constitutive equation. In other words, we deal with <u>homogeneous</u> elastic materials.

In the case of an elastic material possessing a <u>strain-energy function</u>

$$W = \hat{W}(\underset{\sim}{F}) \tag{8·10}$$

(measured per unit volume in the reference configuration B_r), equations (8· can be replaced by

$$\underset{\sim}{\sigma} = J^{-1} \underset{\sim}{F}(\partial_{\underset{\sim}{F}} \hat{W}) \quad \left(\sigma_{ij} = J^{-1} F_{is} \frac{\partial \hat{W}}{\partial F_{js}} \right), \quad \underset{\sim}{s} = (\partial_{\underset{\sim}{F}} \hat{W})^T \quad \left(s_{ij} = \frac{\partial \hat{W}}{\partial F_{ji}} \right). \quad (8·$$

term hyperelastic is sometimes used for an elastic material which has a strain-energy function, but we take the view here that the notion of an elastic material for which no strain-energy function exists is thermodynamically inadmissible. It may be remarked in this connection that equations (8·11) can be given a precise position within the thermo-mechanical theory of elastic materials.

9. Acceleration Waves in an Elastic Material. The Propagation Condition

The analysis of acceleration waves within the framework of the mechanical theory of elasticity is one of the few instances in non-linear elastodynamics in which a tolerably complete treatment can be given without recourse to approximation. This is, in itself, a good reason for studying the subject: exact solutions in non-linear mechanics are rare and they invariably shed considerable light on the behaviour of the class of materials concerned. In the present case the theory of acceleration waves serves as an epitome for non-linear elastic wave propagation phenomena in general. No theory of elastic shock waves of comparable scope has yet been constructed. The mathematical difficulties are formidable and a physically realistic approach looking beyond the possibility of very weak shocks would necessarily have to take account of thermo-mechanical interaction. This is not to say that the analysis which follows is and show some promise of providing a useful diagnostic tool in the study of macroscopic material behaviour. And close parallels exist between the relations governing the propagation of acceleration waves and progressive harmonic waves in an elastic medium previously subjected to a finite static strain.

We start by substituting the constitutive equation $(8.11)_4$ into the referential equation of motion $(8.8)_2$, multiplying through by J^{-1} and using equation (8.5). The result can be expressed as

$$J^{-1}\frac{\partial \hat{s}_{ti}}{\partial F_{su}}\frac{\partial F_{su}}{\partial x_p}\frac{\partial x_p}{\partial X_t} + \rho b_i = \rho f_i . \qquad (9.1)$$

It has been noted in Section 3 that the deformation gradient $\underset{\sim}{F}$ is continuous on an acceleration wave. When the material transmitting the wave is elastic it follows from $(8.9)_2$ that the nominal stress $\underset{\sim}{s}$ and its derivatives with respect to $\underset{\sim}{F}$ are also continuous. Assuming that the extrinsic force $\underset{\sim}{b}$ is continuous (as is usually the case for the fields of force encountered in classical physics), the result of forming the jump on an acceleration wave Σ of each term in (9.1) is hence

$$J^{-1}F_{pt}\frac{\partial \hat{s}_{ti}}{\partial F_{su}}\left[\frac{\partial F_{su}}{\partial x_p}\right] = \rho a_i , \qquad (9.2)$$

use having been made of the definitions (3.3) and (7.14).

The next step is to substitute from the formula (7.17) for the jump appearing on the left-hand side of (9.2).

$$J^{-1} F_{pt} F_{ru} \frac{\partial \hat{s}_{ti}}{\partial F_{su}} n_p n_r a_s = \rho V^2 a_i , \tag{9.3}$$

which can be cast into the more concise form

$$B_{pirs} n_p n_r a_s = \rho V^2 a_i , \tag{9.4}$$

where

$$B_{ijkl} = J^{-1} F_{ip} F_{kq} \frac{\partial \hat{s}_{pj}}{\partial F_{lq}} = J^{-1} F_{ip} F_{kq} \frac{\partial^2 \hat{W}}{\partial F_{jp} \partial F_{lq}} . \tag{9.5}$$

In the final step use has been made of $(8.11)_4$. B_{ijkl} are the components of a fourth-order <u>linear elasticity tensor</u> $\underset{\sim}{B}$, evaluated at the state of deformation prevailing at the current location of the wave. There are many such elasticity tensors, depending on the choice of stress and deformation measures. $\underset{\sim}{B}$ can be interpreted as the instantaneous rate of change of the nominal stress with the transposed deformation gradient as an elastic body passes through its current configuration. The definition (9.5) implies the symmetry property

$$B_{klij} = B_{ijkl} , \tag{9.6}$$

indicating that $\underset{\sim}{B}$ generally has 45 independent components.

At this point we introduce the tensor $\underset{\sim}{Q}(\underline{n})$ defined by the component relation

$$Q_{ij}(\underline{n}) = B_{piqj} n_p n_q \quad \forall \ \underline{n} \in S , \tag{9.7}$$

where S is the set of all unit vectors. $\underset{\sim}{Q}(\underline{n})$ is called the <u>acoustical tensor</u>

$$\underset{\sim}{Q}(-\underline{n}) = \underset{\sim}{Q}(\underline{n}), \quad \underset{\sim}{Q}^T(\underline{n}) = \underset{\sim}{Q}(\underline{n}) \quad \forall \; \underline{n} \in S. \tag{9.8}$$

Equation (9.4) can now be expressed in direct notation as

$$\underset{\sim}{Q}(\underline{n})\,\underline{a} = \rho V^2 \underline{a}, \tag{9.9}$$

and this is the propagation condition for an acceleration wave in an elastic material. The purely local character of this result should be carefully noted: it applies to a single point on the acceleration wave at a single instant of time. As the wave propagates it will generally encounter a varying state of deformation in the transmitting material and the speed V, the amplitude $\underline{a}$ and the shape of the wave can be expected to change progressively. As we shall now show, the propagation condition determines the local speed of propagation V and the direction of the amplitude $\underline{a}$. The variation of the magnitude of $\underline{a}$ as the wave propagates is considered later in Section 11.

It follows from the symmetry property $(9.8)_2$ that the acoustical tensor has three proper numbers and an associated set of orthonormal proper vectors, this conclusion holding for all $\underline{n} \in S$ and regardless of the state of deformation (as specified by $\underset{\sim}{F}$). A wave corresponding to a proper number q of $\underset{\sim}{Q}(\underline{n})$ can actually propagate if and only if $q > 0$ and then its speed is $V = (q/\rho)^{\frac{1}{2}}$ and its amplitude is aligned with a proper vector associated with q. Thus if $\underset{\sim}{Q}(\underline{n})$ has positive proper numbers, three acceleration waves can travel in the direction defined by $\underline{n}$, their amplitudes being mutually orthogonal. A sufficient condition for $\underset{\sim}{Q}(\underline{n})$ to be positive definite (and therefore to have positive proper numbers) for all $\underline{n} \in S$ is the following <u>strong ellipticity condition</u> on $\underset{\sim}{B}$:

$$B_{pqrs}\, a_p b_q a_r b_s > 0 \quad \forall \text{ non-zero real vectors } \underline{a}, \underline{b}. \qquad (9 \cdot 10)$$

Whether or not an elastic material behaving in a physically realistic manner necessarily meets this requirement is still an open question.

It is clear from equation (9·9) that the propagation condition places no restriction on $|\underline{a}|$. Also, because of the relation

$$\rho V^2 = \frac{\underline{a} \cdot \{\underset{\sim}{Q}(\underline{n})\underline{a}\}}{\underline{a} \cdot \underline{a}}, \qquad (9 \cdot 11)$$

implied by (9·9), the propagation speed V does not depend upon $|\underline{a}|$. In this sense acceleration waves in elastic materials are non-dispersive, a fact otherwise evident from the absence of a characteristic length in the constitutive equations (8·9).

10. <u>Longitudinal and Transverse Acceleration Waves</u>

The results in this section are somewhat peripheral to the main elementary topological ideas.

In the classical (linearized) theory of elastic waves for isotropic materials the acoustical tensor has the simple form

$$\underset{\sim}{Q}(\underline{n}) = \mu \underset{\sim}{1} + (\lambda+\mu)\underline{n}\otimes\underline{n}, \qquad (10.1)$$

where $\underset{\sim}{1}$ is the identity tensor and λ, μ are the Lamé elastic constants of the material. The right-hand side of this expression is recognized as a spectral representation and we can read off the proper numbers of $\underset{\sim}{Q}(\underline{n})$ as $\lambda+2\mu$ and μ (twice), and associated unit proper vectors as $\underline{n}$ and any pair of unit vectors forming with $\underline{n}$ an orthonormal triad. Of the three possible waves in the direction of propagation defined by $\underline{n}$ one is therefore <u>longitudinal</u>, in the sense that its amplitude is aligned with $\underline{n}$, and the other two are <u>transverse</u> in the sense that their amplitudes are orthogonal to $\underline{n}$.

We now ask if, in the general situation governed by the propagation condition (9.9), an acceleration wave can be longitudinal or transverse. We say that $\underline{n}$ defines a <u>specific direction</u> for a longitudinal (resp. transverse) acceleration wave if there is a non-zero vector $\underline{a}$ satisfying (9.9) and such that $\underline{a}\times\underline{n} = \underline{0}$ (resp. $\underline{a}\cdot\underline{n} = 0$). The existence of such directions is established in the two theorems which now follow.

regardless of the state of deformation, at least one specific direction $\underline{n}$ for a longitudinal acceleration wave. The wave in question is actually able to propagate if and only if

$$\underline{n} \cdot \{ \underset{\sim}{Q}(\underline{n}) \underline{n} \} > 0 . \qquad (10 \cdot 2)$$

Proof. Consider the vector-valued function

$$\underline{m}(\underline{l}) = \underset{\sim}{Q}(\underline{l}) \underline{l} - [\underline{l} \cdot \{ \underset{\sim}{Q}(\underline{l}) \underline{l} \}] \underline{l} \quad \forall \ \underline{l} \in S \qquad (10 \cdot 3)$$

defined on the set of all unit vectors. We notice that (a) $\underline{l} \cdot \underline{m}(\underline{l}) = 0$ $\forall \ \underline{l} \in S$, and (b) $\underline{m}$ is a continuous function of $\underline{l}$ owing to the quadratic dependence of $\underset{\sim}{Q}(\underline{l})$ on $\underline{l}$ (see equation (9·7)). S may be regarded as a unit sphere (in E) and properties (a) and (b) then assert that $\underline{m}$ is a continuous tangent field on S. By a well-known topological theorem a continuous tangent field on a sphere in E has at least one zero. Hence there exists a unit vector $\underline{n}$ such that $\underline{m}(\underline{n}) = 0$. From (10·3) this means that $\underset{\sim}{Q}(\underline{n}) \underline{n}$ is a scalar multible of $\underline{n}$, i.e. that $\underline{n}$ is a proper vector of $\underset{\sim}{Q}(\underline{n})$. Thus $\underline{n}$ defines a specific direction for a longitudinal wave.

The proper number of $\underset{\sim}{Q}(\underline{n})$ associated with $\underline{n}$ is $\underline{n} \cdot \{ \underset{\sim}{Q}(\underline{n}) \underline{n} \}$ and, from (9·11), with $\underline{a} = \underline{n}$, this equals ρV^2 where V is the local speed of propagation of the wave. Propagation is possible if and only if $V^2 > 0$ which is true if and only if (10·2) holds. □

wave guarantees that there is an orthogonal pair of transverse wave amplitudes associated with the same direction. We now ask what can be said about the existence of specific directions for <u>a single</u> transverse acceleration wave. By looking to the possibility of there being just one transverse wave we might expect to find an enlarged set of directions and this turns out to be the case.

<u>Theorem 2</u>. At an arbitrary point p in an elastic body there exists in every plane, regardless of the state of deformation, at least one specific direction $\underline{n}$ for a transverse acceleration wave. The wave in question can actually propagate if and only if

$$[\underline{n}\cdot\{\underset{\sim}{Q}(\underline{n})\,\underline{n}\}][\underline{n}\cdot\{\underset{\sim}{Q}^*(\underline{n})\,\underline{n}\}] > \det \underset{\sim}{Q}(\underline{n}), \qquad (10.4)$$

$\underset{\sim}{Q}^*(\underline{n})$ being the adjugate of $\underset{\sim}{Q}(\underline{n})$.

<u>Proof</u>. Let $\underline{k}$ be a unit vector chosen arbitrarily from S and let

$$C_k = \{\underline{l} : \underline{l} \in S,\ \underline{k}\cdot\underline{l} = 0\} \qquad (10.5)$$

be the set of all unit vectors orthogonal to $\underline{k}$. C_k may be regarded as a great circle on the unit sphere S. Our task is to show that there is a unit vector $\underline{n} \in C_k$ which is orthogonal to a proper vector of $\underset{\sim}{Q}(\underline{n})$.

Consider the scalar-valued function

$$\varphi(\underline{l}) = [\![\, \underline{l},\ \underset{\sim}{Q}(\underline{l})\,\underline{l},\ \underset{\sim}{Q}^2(\underline{l})\,\underline{l} \,]\!] \quad \forall\ \underline{l} \in C_k, \qquad (10.6)$$

vectors. We notice that (a) $\varphi(-\underline{l}) = -\varphi(\underline{l})\ \forall\ \underline{l} \in C_k$, on account of the even property (9·8), of the acoustical tensor, and (b) φ maps C_k continuously into R, the real line. A second well-known topological theorem states that in a continuous mapping of a circle into R the terminal points of at least one diameter of the circle have the same image. Hence the exists $\underline{n} \in C_k$ such that $\varphi(-\underline{n}) = \varphi(\underline{n})$. But φ is an odd function, as noted above, so $\varphi(\underline{n}) = 0$. From (10·6) this means that the three vectors $\underline{n}$, $\underset{\sim}{Q}(\underline{n})\underline{n}$ and $\underset{\sim}{Q}^2(\underline{n})\underline{n}$ are coplanar or, equivalently, that there is a unit vector $\underline{m}$ orthogonal to each of these vectors.

Consider now the two pairs of vectors

$$\underline{m},\ \underset{\sim}{Q}(\underline{n})\underline{m} \quad \text{and} \quad \underline{n},\ \underset{\sim}{Q}(\underline{n})\underline{n}. \tag{10·7}$$

We have

$$\underline{m}\cdot\underline{n} = 0 \quad \text{and} \quad \underline{m}\cdot\{\underset{\sim}{Q}(\underline{n})\underline{n}\} = 0, \tag{10·8}$$

in addition to which

$$\left.\begin{aligned} \{\underset{\sim}{Q}(\underline{n})\underline{m}\}\cdot\underline{n} &= \underline{m}\cdot\{\underset{\sim}{Q}^T(\underline{n})\underline{n}\} = \underline{m}\cdot\{\underset{\sim}{Q}(\underline{n})\underline{n}\} = 0, \\ \{\underset{\sim}{Q}(\underline{n})\underline{m}\}\cdot\{\underset{\sim}{Q}(\underline{n})\underline{n}\} &= \underline{m}\cdot\{\underset{\sim}{Q}^T(\underline{n})\underset{\sim}{Q}(\underline{n})\underline{n}\} = \underline{m}\cdot\{\underset{\sim}{Q}^2(\underline{n})\underline{n}\} = 0. \end{aligned}\right\} \tag{10·9}$$

Thus each member of the first pair in (10·7) is orthogonal to both members of the second pair and it follows that either (a) $\underset{\sim}{Q}(\underline{n})\underline{m}$ is a scalar multiple of $\underline{m}$, or (b) $\underset{\sim}{Q}(\underline{n})\underline{n}$ is a scalar multiple of $\underline{n}$. In case (a) $\underline{m}$

set out to prove. In case (b) $\underline{n}$ is a proper vector of $\underset{\sim}{Q}(\underline{n})$ and, as pointed out in the remark following Theorem 1, the existence of <u>two</u> proper vectors of $\underset{\sim}{Q}(\underline{n})$ orthogonal to $\underline{n}$ is assured.

This completes the main part of the proof. The establishment of the necessary and sufficient condition (10·4) is a matter of fairly simple algebra which is left as an exercise. □

<u>Exercise.</u> For a certain elastic material the strain-energy function is

$$W = \tfrac{1}{2}\mu(I - 3) + f(J),$$

where μ is a positive constant, $I = \mathrm{tr}(\underset{\sim}{F}\underset{\sim}{F}^T)$ and f is a twice differentiable function. Calculate the acoustical tensor $\underset{\sim}{Q}(\underline{n})$. Deduce that if f'' is non-negative there exist, for all $\underset{\sim}{F}$ and for all $\underline{n} \in S$, a longitudinal acceleration wave and two transverse waves with orthogonal amplitudes. Obtain the formulae

$$\rho V_L^2 = \mu(\underset{\sim}{F}^T\underline{n})\cdot(\underset{\sim}{F}^T\underline{n}) + Jf''(J),$$

$$\rho V_T^2 = \mu(\underset{\sim}{F}^T\underline{n})\cdot(\underset{\sim}{F}^T\underline{n}),$$

where V_L and V_T are the local speeds of propagation of the longitudinal and transverse waves respectively.