INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
MIRAMARE - P.O.B. 586 - 34100 TRIESTE (ITALY) - TELEPHONES: 224281/2/3/4/5/6 - CABLE: CENTRATOM

SMR/23 /46

AUTUMN COURSE ON APPLICATIONS OF ANALYSIS TO MECHANICS

22 September - 26 November 1976

MATHEMATICAL RESULTS FOR BINGHAM FLUIDS

(Part V)

D. Cioranescu
Université de Paris
France

These are preliminary lecture notes intended for participants only.
Copies are available outside the Publications Office (T-floor) or from room 112.

§9. Compactness theorem and Gronwall inequality.

Compactness theorem

Let $B \subset B_1 \subset B_2$ be three Banach spaces with compact injection from B_0 into B_1 and continuous one from B_1 into B_2. Suppose $\{u_n\} \in$ bounded set of $L^{p_0}(0,T;B_0)$ and $\{\frac{du_n}{dt}\} \in$ bounded set of $L^{p_1}(0,T;B_2)$: $T<+\infty$; $1<p_0, p_1<+\infty \Rightarrow$ we can extract from u_n a subsequence strongly convergent in $L^{p_0}(0,T;B_1)$.

We need for the proof of this theorem the following result:

Lemma. Let $B_0 \subset B_1 \subset B_2$ be three Banach spaces satisfying the hypothesis of the theorem. $\Rightarrow$

$\forall \varepsilon>0 \quad \exists\, c_\varepsilon > 0$ such that for $\forall u \in B_0$ we have

$$\|u\|_{B_1} \le \varepsilon \|u\|_{B_0} + c_\varepsilon \|u\|_{B_2}$$

Proof.

We shall suppose that the lemma is not true, so that $\exists \varepsilon_0>0$ with the property that for $\forall n$ integer one can find $u_n \in B_0 \ni$:

$$\|u_n\|_{B_1} > \varepsilon_0 \|u_n\|_{B_0} + n\|u_n\|_{B_2} \tag{9.1}$$

We may take

$\|u_n\|_{B_1} = 1$ so that u_n is bounded set in B_0. By (9.1) we must have

The injection $B_0 \subset B_1$ being compact and $\{u_n\}$ belonging to a bounded set of B_1 $\Rightarrow$ we can extract a subsequence u_{n_k} such that $u_{n_k} \to u$ in B_1 strongly. But $\|u_{n_k}\|_{B_1} = 1$ and so $\|u\|_{B_1} = 1$, or by (9.2) as $u_{n_k} \to u$ strongly in B_2 too we must have $u = 0$. Contradiction.

Proof.

The proof will be done in two steps. We shall see in the first step that if we can extract from u_n a subsequence, Cauchy in $L^{p_0}(0,T; B_2)$, applying the lemma to this subsequence the statement of the theorem follows. In the second step we shall prove that in fact we can extract from u_n a Cauch subsequence in $L^{p_0}(0,T; B_1)$.

1. Suppose we have extract a subsequence - denoted by u_n equally - which is Cauchy in $L^{p_0}(0,T; B_2)$. We shall apply the lemma to $u_n - u_m$ and we get

$$\|u_n - u_m\|_{B_1} \leq \varepsilon \|u_n - u_m\|_{B_0} + C(\varepsilon) \|u_n - u_m\|_{B_2}$$

it follows that

$$\|u_n - u_m\|_{B_1}^{p_0} \leq \bar{\varepsilon} \|u_n - u_m\|_{B_0}^{p_0} + \bar{C}(\bar{\varepsilon}) \|u_n - u_m\|_{B_2}^{p_0}$$

and therefore

$$\|u_n - u_m\|_{L^{p_0}(0,T; B_1)} \leq \bar{\varepsilon} \|u_n - u_m\|_{L^{p_0}(0,T; B_0)} + \bar{C}(\bar{\varepsilon}) \|u_n - u_m\|_{L^{p_0}(0,T; B_2)}$$

If u_n is Cauchy in $L^{p_0}(0,T; B_1)$ $\Rightarrow$

$$\overline{\lim_{n,m\to\infty}} \|u_n - u_m\|_{L^{p_0}(0,T; B_1)} \leq \varepsilon \times \text{Const.} \quad \forall \varepsilon > 0$$

as $\|u_n - u_m\|_{\ldots} \xrightarrow[n,m\to\infty]{} 0$ and u_n is bounded

ε is arbitrary small $\Rightarrow$ u_n is a Cauchy sequence in $L^{p_0}(0,T;B_1)$ so that the theorem is proved.

2. Let be $h>0$. We shall consider the decomposition

$$u_n(t) = a_n(t) + b_n(t) \quad \text{where}$$

$$a_n(t) = \frac{1}{h}\int_t^{t+h} u_n(s)\,ds$$

$$b_n(t) = \frac{1}{h}\int_t^{t+h} (u_n(t) - u_n(s))\,ds$$

for $t \in [0, \frac{T}{2}]$ (for $[\frac{T}{2}, T]$ we use a symmetric argument) Now we have:

$$\|u_n(t) - u_n(s)\|_{B_2} = \left\| \int_s^t \frac{du_n}{dt}(\tau)\,d\tau \right\|_{B_2} \leq$$

$$\leq \int_s^t \left\|\frac{du_n}{dt}\right\|_{B_2} d\tau \leq \text{by Hölder's inequality}$$

$$\leq |s-t|^{\frac{1}{p_1'}} \left(\int_s^t \left\|\frac{du_n}{dt}\right\|_{B_2}^{p_1} \right)^{1/p_1} \qquad \frac{1}{p_1'} + \frac{1}{p_1} = 1$$

But $\left(\int_s^t \left\|\frac{du_n}{dt}\right\|_{B_2}^{p_1}\right)^{1/p_1} \leq$ const. because $\frac{du_n}{dt} \in$ bounded set of $L^{p_1}(0,T;B_2) \Rightarrow$

$$\|b_n(t)\|_{B_2} \leq \frac{1}{h}\int_t^{t+h} \|u_n(t) - u_n(s)\|_{B_2}\,ds \leq$$

$$\leq C\frac{1}{h}\int_t^{t+h} |s-t|^{1/p_1'}\,ds \leq C\frac{1}{h}\cdot |h|^{\frac{1}{p_1'}} \cdot \int_t^{t+h} ds = C|h|^{1/p_1'}$$

Now it would be sufficient to extract a strong convergent subsequence from a_n. Suppose we have extract it, let denote it by a_n too. If a_n is a Cauchy subsequence ($\Leftrightarrow$ strong convergent subsequence)

$$\overline{\lim} \| u_n - u_m \|_{L^{p_0}(0,T;B_2)} \leq \overline{\lim} \| b_n - b_m \|_{L^{p_0}(0,T;B_2)} \leq$$
$$\leq c\, h^{1/p_1'}$$

(as $\| a_n - a_m \|_{L^{p_0}(0,T;B_2)} \to 0$)

and so letting $h \to 0$ we obtain a Cauchy sequence $\{u_n\}$ in $L^{p_0}(0,T;B_2)$)

Let now prove that we can extract a Cauchy sequence from a_n (in $L^{p_0}(0,T;B_2)$). We have

$$\frac{da_n}{dt} = \frac{1}{h}\left(u_n(t+h) - u_n(t)\right) = \frac{1}{h}\int_t^{t+h} \frac{du_n(\sigma)}{dt}\, d\sigma \text{ so}$$

$$\left\| \frac{da_n}{dt} \right\|_{B_2} \leq \frac{1}{h} \int_t^{t+h} \left\| \frac{du_n(\sigma)}{dt} \right\|_{B_2} d\sigma$$

By hypothesis $\frac{du_n}{dt} \in$ bounded set of $L^{p_1}(0,T;B_2)$ so that

(9.3) $$\left\| \frac{da_n}{dt} \right\|_{L^{p_1}(0,T;B_2)} \leq \text{const.}$$

And now we can prove that $a_n \in$ equicontinuo set of $C^0(0,T;B_2)$. Indeed

$$a_n(t) - a_n(s) = \int_s^t \frac{da_n(\sigma)}{dt}\, d\sigma$$

and then we use (9.3) and we get easily that

$$\| a_n(t) - a_n(s) \|_{B_2} \leq \text{const.}$$

(exactly by the same way as for b_n)

We have therefore that $\{a_n\}$ is a uniformly bounded equicontinuous set of $C^0(0,T;B_2)$; it takes values in a bounded set in B_0, which is compact in B_2 ($\| a_n(t) \|_{B} \leq \frac{1}{h} h^{1/p_0} \| u_n \|$. $\leq$

$\Rightarrow$ we can apply Arzela–Ascoli's theorem and thus extract a strongly convergent subsequence in $C^0(0,T;B_0)$ and so in $L^{p_0}(0,T;B_2)$.

Gronwall inequality.

If $\varphi(t) \leq A + \int_0^t B(s)\varphi(s)$ with $\varphi(t) \geq 0$. $\varphi(t) \in L^\infty(0,T)$, $B(t) \geq 0$, $B(t) \in L^1(0,T) \Rightarrow$

$$(9.4) \quad \varphi(t) \leq A + \int_0^t B(s)\varphi(s)\,ds \leq A e^{\int_0^t B(s)\,ds}$$

Proof.

Let $\psi(t) = A + \int_0^t B(s)\varphi(s)\,ds$; ψ is absolutely continuous in t and $\psi'(t) = B(t)\varphi(t)$, by hypothesis

$$\varphi(t) \leq A + \int_0^t B(s)\varphi(s) = \psi(t) \quad \text{so that}$$

$$\psi'(t) \leq B(t)\psi(t)$$

$$\Rightarrow \psi'(t) - B(t)\psi(t) \leq 0$$

or

$$\left(\psi(t)\, e^{-\int_0^t B(s)ds}\right)' \leq 0$$

and so

$$\psi(t) \leq \psi(0)\, e^{\int_0^t B(s)\,ds} \quad \text{after integration} \blacksquare$$

Exactly by the same way one can prove a variant of (9.4):

If $\varphi(t)^2 \leq A + \int_0^t B(s)\varphi(s)\,ds$ with the same hypothesis as in theorem 1 $\Rightarrow$

$$\varphi(t)^2 \leq A + \int^t B(s)\varphi(s)\,ds \leq \left(\sqrt{A} + \frac{1}{2}\int_0^t B(s)\,ds\right)^2 \leq$$

§10. Remarks.

1. Let X be a Banach space.

$L^p(0,T;X)$ = the space of functions $t \to f(t)$ measurable of $[0,T] \longrightarrow X$ (for the measure $\frac{d}{dt}$) such that:

$$\left(\int_0^T \| f(t) \|_X^p \, dt \right)^{1/p} = \| f \|_{L^p(0,T;X)} < \infty \quad p \neq \infty$$

$L^p(0,T;X)$ is a Banach space.

$\mathcal{D}'(]0,T[;X)$ - the space of distributions on $]0,T[\longrightarrow X$, defined by

$$\mathcal{D}'(]0,T[;X) = \mathcal{L}(\mathcal{D}(]0,T[);X)$$

($\mathcal{L}(Y;X)$ = the space of continuous linear applications from $Y \longrightarrow X$)

To $f \in L^p(0,T;X)$ corresponds a distribution $\tilde{f} :]0,T[\longrightarrow X$ defined by

$$\tilde{f}(\varphi) = \int_0^T f(t)\varphi(t)\, dt \qquad \varphi \in \mathcal{D}(]0,T[)$$

($\varphi \longrightarrow \tilde{f}(\varphi)$ is indeed continuous & linear from $\mathcal{D}(]0,T[) \longrightarrow X$)

For $f \in \mathcal{D}'(]0,T[;X)$ on defines $\frac{d^k f}{dt^k} = f^{(k)}$ ($\frac{df}{dt} = f'$) by

$$f^{(k)}(\varphi) = (-1)^k f(\varphi^{(k)}) \quad \forall \varphi \in \mathcal{D}(]0,$$

which defines $f^{(k)} \in \mathcal{D}'(]0,T[;X)$

In all variational formulations we have used this notation $\frac{df}{dt}$ (and this has nothing to do with the notation $\frac{d}{dt}$ used in mechanics as the material derivative!)

2. Let $v \in L^2(0,T;X)$ with $\frac{dv}{dt} \in L^2(0,T;X')$
X is a Hilbert space such that

$X \subset Y$ — Y Hilbert space too with dense injection $X \subset Y$.

Identifying Y to its dual, Y is so identified with a subspace of X', the dual of $X \Rightarrow$

$$X \subset Y \subset X'$$

$$\Rightarrow v \in C^0([0,T]; Y)$$ (see Bibliography [1] Chap 1.)

Conclusion = the solution u obtained in § 5 is such that $u \in C^0([0,T]; H)$ so that we are allowed to consider $u(0,x)$ and $u(T,x)$

Bibliography: J.L. Lions - E. Magenes : Problèmes aux limites non-homogènes et applications Dunod. Paris Vol. 1 (1968)

General bibliography

[1]. J. L. Lions: Quelques méthodes de résolution des problèmes aux limites non-linéaires Dunod. Gauthier - Villars 1969.

[2]. J. L. Lions - E. Magenes: Problèmes aux limites nonhomogènes et applications. Dunod. Paris, Vol 1 & 2 (1968); vol 3 (1970)

[3]. G. Duvaut - J.L. Lions: Les inéquations en mécanique et en physique. Dunod. Paris (1972)

[4]. N. Dunford - J.T.S Schwartz - Linear operators Part I. Interscience Pub. (1958)

[5] R. Finn - On the exterior stationary problem for Navier-Stokes equations. Arch. Rat. Mech Anal. 19 (1965) 363-406

[6]. C. Foias - G. Prodi - Sur le comportement global des solutions nonstationnaires des équations de Navier - Stokes en dimension 2. Rend. Sem. Mat. Padova, XXXIX, 1967, 1-34

[7]. M. Fortin - Résolution numérique de l'écoulement newtoniens et non newtoniens. Thèse. Paris 19..

[8]. P. Germain - Mécanique des milieux continus Masson. (1962)

[9]. R. Hill - Mathematical theory of plasticity University Press, Oxford (1950)

[10] - E. Hille - R. S. Phillips - Functional Analysis & Semi-groups. Amer. Math. Soc. Coll. Publ. XXXI, (1957)

[11] - O. A. Ladyzhenskaya - The Mathematical Theory of Viscous Incompressible Flow - Gordon & Breach Science Publishers, New York, 1963.

[12]. L. Landau - E. Lifchitz - Course of theoretical Physics, Fluid Mechanics. Pergamon Press (1959)

[13] G. Minty - Monotone (non-linear) operators in Hilbert space. Duke Math. Journal, 29 (1962) 341-346

[14]. P.P. Mosolov - V.A. Miasnikov - Variational methods in the theory of the fluidity of a viscous plastic medium. PMM. 29, (1965) 468-492

[15] Muskat - The Flow of homogeneous fluid flow through porous media. Mac Graw-Hill (1937)

[16]. W. Prager - Introduction to mechanics of continua Ginn & Company (1961)

[17]. T. Rockafellar - Duality & stability in extremum problems involving convex functions Pacific Journal of Math. 21 (1967), 167-187

— Integrals which are convex functionals. Pacific J. of Math. 24 (1968)

[18] - L. Schwartz - Distributions à valeurs vectorielles. I. II. Annales Institut Fourier, 7, (1957), 1-141. 8 (1958) 1-209.

[19] - J. Serrin - The initial value problem for the Navier - Stokes equations. In: "Non linear problems" edited by R.E. Langer (1963) 69-

[20] - J. Serrin - Mathematical Principles of Classical Fluid Mechanics. Handbuch der Physik. Band VIII / 1. Springer Verlag 1959

[21]. K. Yosida - Functional Analysis. Grundlehren (B 123) - Springer (1965)

For Non-Newtonian Fluids:
The Flow of Complex Mixtures in Pipes
By Govier & Aziz.
Van Nostrand Reinhold Company 1972.