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TO MECHANICS

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FREE SURFACE PROBLEMS FOR VISCOUS AND VISCOELASTIC FLUIDS

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Free surface problem for viscous and viscoelastic fluids

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I am going discuss my mapping method for studying problems with free surfaces. I am interested in the applications of this method in the theory of rheological fluid mechanics. It is important to distinguish between the method and the applications. For this reason I advocate the following program of study:

1.) Study the mapping method for a simple problem in which the deformation of the domain is prescribed. The eigenvalue problem of my first lecture (in these notes) is one such simple problem.

2.) Study the mapping method for a simple problem in which the deformation is an unknown. I recommend that you study the "made-up" problem in the appendix of the paper by Joseph and Sturges (1974).

3.) Study the mapping method a realistic problem problem of viscous fluid mechanics. "The free surface on a liquid filling a trench heated from its side" by Joseph and Sturges (1974) is good for this

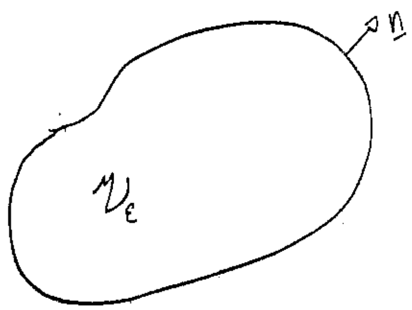
4.) You may be interested in the application of the mapping method to unsteady problems. A theory for such problems is given the paper "The free surface on simple fluid between cylinders undergoing torsional oscillations" by D. D. Joseph. (Part I; there are two other parts, one of them, by Beavers, reports experimental results). This paper will appear in ARMA and a preprint has been left with Prof. Landon.

Unfortunately the paper just mentioned is complicated by the difficulties of constitutive theory for materials with memory. I had to solve a free surface problem and a problem of rheology simultaneously. A simple theory for unsteady problems of free surfaces on Navier-Stokes fluids is possible. Somebody could write a good research paper on this topic.

I left some reprints and preprints with Prof. Landon. You will need to look at these papers if your interest in this subject is more than cultural. I have included in these notes a full account of my first lecture. The other two lectures were summaries of the results reported in my papers. The summaries are perhaps good for motivation but not for study.

One advantage of the mapping method is ~~that~~ applies to many problems. For example, it applies to many problems (more than a million) in which variational inequalities do not apply. But the mapping method is a perturbation theory and unlike variational inequalities, which ~~tolo~~ give global results when they apply, it leads to local results, for small ε. We might say that variational methods have a small domain of applicability for global results and that mapping methods have a large domain of applicability for local results.

Lecture 1: Domain dependence of eigenvalues of the membrane equation



Suppose

$F(\underline{x}(\varepsilon), \varepsilon) = 0 \quad , \quad \underline{x} \in \mathcal{V}_\varepsilon$

$f(\underline{x}(\varepsilon), \varepsilon)$ is defined in $\mathcal{V}_\varepsilon$ and

$f(\underline{x}(\varepsilon), \varepsilon) = 0 \quad , \quad \underline{x} \in \partial\mathcal{V}_\varepsilon$

For the moment we merely assume that $\underline{x} = \underline{x}(\varepsilon)$. Later, we will show precisely how x depends on ε. The equations

(1) $$G(\varepsilon) = F(\underline{x}(\varepsilon), \varepsilon) = 0 \qquad \underline{x} \in \mathcal{V}_\varepsilon$$

(2) $$g(\varepsilon) = f(\underline{x}(\varepsilon), \varepsilon) = 0 \qquad \underline{x} \in \partial\mathcal{V}_\varepsilon$$

hold on an open interval $I(\varepsilon)$ containing $\varepsilon = 0$. Of course $\mathcal{V}_0$ is arbitrary. We may compute derivatives of $G(\varepsilon)$ and $g(\varepsilon)$ at the point $\varepsilon = 0$:

(3) $$G^{[1]}(0) \equiv \left.\frac{dG}{d\varepsilon}\right|_{\varepsilon=0} = F^{[1]} = F^{\langle 1\rangle} + \underline{x}^{[1]} \cdot \nabla F^{\langle 0\rangle} = 0$$

(4) $$G^{[2]}(0) \equiv \left.\frac{d^2 G}{d\varepsilon^2}\right|_{\varepsilon=0} = F^{[2]} = F^{\langle 2\rangle} + 2\,\underline{X}^{[1]}\cdot\nabla F^{\langle 1\rangle} + \underline{X}^{[2]}\cdot\nabla F^{\langle 0\rangle} + (\underline{X}^{[1]}\cdot\nabla)^2 F^{\langle 0\rangle} = 0$$

and

(5) $$G^{[n]} = F^{[n]} = \left[\left(\frac{\partial}{\partial\varepsilon} + \frac{d\underline{x}}{d\varepsilon}\cdot\nabla\right)^n F\right]_{\varepsilon=0} = 0$$

where

(6) $$F^{[n]} = \left(\frac{d^n}{d\varepsilon^n} F(\underline{X}(\varepsilon),\varepsilon)\right)_{\varepsilon=0}$$

and

(7) $$F^{\langle n\rangle} = \left(\frac{\partial^n F(\underline{x},\varepsilon)}{\partial\varepsilon^n}\right)_{\varepsilon=0} \quad \text{holding } \underline{x} \text{ fixed.}$$

Since, from (1),

(8) $$F^{\langle 0\rangle} = F(X(0),0) \equiv 0 \qquad \underline{x} \in \mathcal{V}_0$$

we find that $\nabla F^{\langle 0\rangle}$ and, from (3), that

(9) $$F^{\langle 1\rangle} = 0, \qquad \underline{x} \in \mathcal{V}_0 .$$

In the same way, using (8) and (9) with (4), we find that

(10) $$F^{\langle 2\rangle} = 0, \qquad \underline{x} \in \mathcal{V}_0$$

By induction $F^{\langle n\rangle} = 0$. Hence

$$F^{[n]} = 0 \quad \text{and} \quad F^{\langle n\rangle} = 0 \qquad \text{for } \underline{X} \in \partial\mathcal{V}_0$$

Since (2) is an identity in $I(\varepsilon)$ we find, using (2), th

(11) $$g^{[n]} = \left[\left(\frac{\partial}{\partial\varepsilon} + \frac{d\underline{X}}{d\varepsilon}\cdot\nabla\right)^n f\right]_{\varepsilon=0} = 0.$$

For example

(12) $$g^{[1]} = f^{\langle 1\rangle} + \underline{X}^{[1]}\cdot\nabla f^{\langle 0\rangle}$$

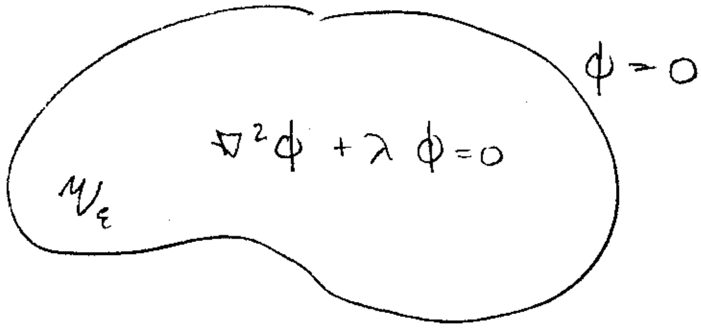
Now $f^{\langle 0\rangle} = 0$ on $\partial\mathcal{V}_0$ but the normal derivative of $f^{\langle 0 \rangle}$, $\underline{n} \cdot \nabla f^{\langle 0 \rangle} \neq 0$. Hence, we may not conclude that $f^{\langle 1 \rangle} = 0$; in general,

$$f^{\langle n \rangle} \neq 0 \qquad \underline{x} \in \partial \mathcal{V}_0$$

Let

$$(13) \qquad F(x, \varepsilon) = \nabla^2 \phi + \lambda \phi = 0, \quad \underline{x} \in \mathcal{V}_\varepsilon$$

$$(14) \qquad f(\underline{x}, \varepsilon) = \phi(\underline{x}, \varepsilon) = 0, \quad \underline{x} \in \partial \mathcal{V}_\varepsilon$$



We are going to study how $\lambda(\varepsilon)$ and $\phi(\underline{x}, \varepsilon)$ depend on ε (actually Rellich showed that they are analytic in ε; a good reference for this is the book by Kato on analytic perturbation). First we give a good proof of <u>Hadamards formula</u> (a less good proof was given by Hadamard and is repeated, for example, in the book on partial differential equations by Garabedian)

$$(15) \qquad \lambda^{\langle 1 \rangle} = -\int_{\partial \mathcal{V}_0} V_n \left(\frac{\partial \phi^{\langle 0 \rangle}}{\partial n}\right)^2 \Big/ \int_{\mathcal{V}_0} (\phi^{\langle 0 \rangle})^2$$

where

$$(16) \qquad V_n = \underline{n} \cdot \underline{X}^{[1]} \qquad\qquad \underline{X}^{[1]} = \underline{V}$$

... of the deformation

(17) $F^{\langle 0\rangle} = \nabla^2\phi^{\langle 0\rangle} + \lambda^{\langle 0\rangle}\phi^{\langle 0\rangle} = 0 \qquad \underline{x} \in \mathcal{V}_0$

(18) $f^{\langle 0\rangle} = \phi^{\langle 0\rangle} = 0 \qquad \underline{x} \in \partial\mathcal{V}_0$

(19) $F^{[1]} = \nabla^2\phi^{\langle 1\rangle} + \lambda^{\langle 0\rangle}\phi^{\langle 1\rangle} + \lambda^{\langle 1\rangle}\phi^{\langle 0\rangle} = 0 \qquad \underline{x} \in \mathcal{V}_0$

(20) $f^{\langle 1\rangle} = \phi^{\langle 1\rangle} + \underline{X}^{[1]}\cdot\nabla\phi^{\langle 0\rangle} \qquad \underline{x} \in \partial\mathcal{V}_0$

Using (18) we may write (20) as

(21) $\phi^{\langle 1\rangle} = -\underline{X}^{[1]}\cdot\nabla\phi^{\langle 0\rangle} = -\underline{X}^{[1]}\cdot\underline{n}\frac{\partial}{\partial n}\phi^{\langle 0\rangle} = -V_n\frac{\partial\phi^{\langle 0\rangle}}{\partial n}$

We compute the integral

(22) $0 = \int_{\mathcal{V}_0}\phi^{\langle 0\rangle}[\nabla^2\phi^{\langle 1\rangle} + \lambda^{\langle 0\rangle}\phi^{\langle 1\rangle} + \lambda^{\langle 1\rangle}\phi^{\langle 0\rangle}] =$

$$\int_{\partial\mathcal{V}_0}[\underbrace{\cancel{\phi^{\langle 0\rangle}\frac{\partial\phi^{\langle 1\rangle}}{\partial n}}}_{0} - \phi^{\langle 1\rangle}\frac{\partial\phi^{\langle 0\rangle}}{\partial n}] + \int_{\mathcal{V}_0}\phi^{\langle 1\rangle}\underbrace{\cancel{[\nabla^2\phi^{\langle 0\rangle} + \lambda^{\langle 0\rangle}\phi^{\langle 0\rangle}]}}_{0}$$

$$+ \lambda^{\langle 1\rangle}\int_{\mathcal{V}_0}(\phi^{\langle 0\rangle})^2 .$$

Eliminating $\phi^{\langle 1\rangle}$ in (22) with (21) we prove (15).

Mathematically, Hadamards formula is a consequence of the Fredholm alternative at a simple eigenvalue.

<u>The mapping</u> : Now we get more precise about $X(\varepsilon)$

Let $\mathcal{V}_\varepsilon \leftrightarrow \mathcal{V}_0$ be a mapping:

(23)
$\underline{x} = \underline{X}(\underline{x}_0, \varepsilon) \in \mathcal{V}_\varepsilon$ one to one
$\underline{x}_0 = \underline{X}(\underline{x}, \varepsilon) \in \mathcal{V}_0$ invertible
$\underline{x} \in \partial\mathcal{V}_\varepsilon \Rightarrow \underline{x}_0 \in \partial\mathcal{V}_0$ boundary points map

$\underline{X}_0 = \underline{X}\ (\underline{X}_0, \varepsilon)$ identity map when $\varepsilon = 0$

$\underline{X}(\underline{X}_0, \varepsilon)$ analytic in ε

<u>The solution</u>: The solution is given by the series

$$(24) \qquad \lambda(\varepsilon) = \sum_0 \frac{1}{n!} \lambda^{\langle n \rangle}$$

$$(25) \qquad \phi(\underline{x}, \varepsilon) = \sum_{n=0} \frac{\varepsilon^n}{n!} \phi^{[n]}(\underline{x}_0) = \sum_0 \frac{\varepsilon^n}{n!} \phi^{[n]}(\underline{x}_0(\underline{x}, \varepsilon))$$

$$= \sum \frac{\varepsilon^n}{n!} \phi^{\langle n \rangle}(\underline{x})$$

Remark: (25) gives two ways to represent the solution in $\mathcal{V}_\varepsilon$. The functions $\phi^{[n]}(\underline{x}_0(\underline{x}, \varepsilon))$ satisfy good boundary conditions $\phi^{[n]} = 0 \quad x_0 \in \partial \mathcal{V}_0$ but $\phi^{\langle n \rangle}(\underline{x}) \neq 0 \quad \underline{x} \in \partial \mathcal{V}_\varepsilon$. Actually we compute $\phi^{\langle n \rangle}(\underline{x}_0)$ so to get $\phi^{\langle n \rangle}(\underline{x})$ we must be able to extend the domain of $\phi^{\langle n \rangle}(\cdot)$ to $\mathcal{V}_\varepsilon$. In general

$$\phi^{\langle n \rangle}(\underline{x}) \neq \phi^{[n]}(\underline{x}_0(\underline{x}, \varepsilon))$$

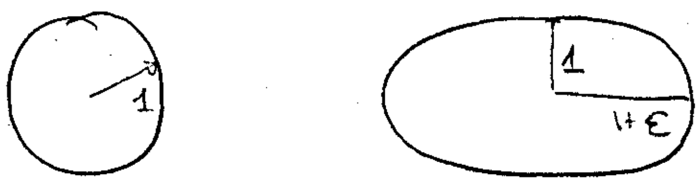
We compute $\phi^{\langle n \rangle}(\underline{x}_0)$ and $\lambda^{\langle n \rangle}$ by perturbations using the differentiation rules already mentioned. For example $\phi^{\langle 0 \rangle}, \lambda^{\langle 0 \rangle}$ are given by (17) and (18); $\phi^{\langle 1 \rangle}$ and $\lambda^{\langle 1 \rangle}$ are given by (19) and (20) and $\phi^{\langle 2 \rangle}$ and $\lambda^{\langle 2 \rangle}$ are governed by

(26) $\quad \nabla^2\phi^{\langle 2\rangle} + \lambda^{\langle 0\rangle}\phi^{\langle 2\rangle} + 2\lambda^{\langle 1\rangle}\phi^{\langle 1\rangle} + \lambda^{\langle 2\rangle}\phi^{\langle 0\rangle} = 0 \qquad \underline{x} \in \mathcal{V}_0$

(27) $\quad \phi^{\langle 2\rangle} + 2\underline{x}^{[1]}\cdot\nabla\phi^{\langle 1\rangle} + \underline{x}^{[2]}\cdot\nabla\phi^{\langle 0\rangle} + (\underline{x}^{[1]}\cdot\nabla)^2\phi^{\langle 0\rangle} = 0 \qquad \underline{x} \in \partial\mathcal{V}_0$

To find $\lambda^{\langle 2\rangle}$, multiply (26) by $\phi^{\langle 0\rangle}$ and integrate, as in the derivation of Hadamards formula.

<u>an example</u>: $\mathcal{V}_0$ is a circle of radius one and $\mathcal{V}_\varepsilon$ is an ellipse



$X = X_0(1+\varepsilon)$
$Y = Y_0$

satisfies the properties required for the mapping:

$X_0 = \dfrac{X}{1+\varepsilon}$
$Y_0 = Y$

invertible

$X_0 = X$
$Y_0 = Y$

when $\varepsilon = 0$ identity

analytic in ε

$$X_0^2 + Y_0^2 = 1 = \frac{X^2}{1+\varepsilon^2} + Y^2$$

boundary points on the circle go to boundary points on the ellipse

All the perturbation problems are defined on the circle.

$$\nabla^2 = \frac{1}{r_0}\frac{\partial}{\partial r_0}\left(r_0\frac{\partial}{\partial r_0}\right) + \frac{1}{r_0^2}\frac{\partial^2}{\partial\theta^2}$$

$$\underline{X}^{[1]} = \underline{e}_x \, x_0$$

$$\underline{X}^{[n]} = 0$$

$$\underline{X}^{[1]} \cdot \nabla \phi^{\langle 0 \rangle} = x_0 \cos\theta \, \phi^{\langle 0 \rangle}_{,r_0} = \cos^2\theta \, \phi^{\langle 0 \rangle}_{,r_0}$$

(28) $$\nabla^2 \phi^{\langle 0 \rangle} + \lambda^{\langle 0 \rangle} \phi^{\langle 0 \rangle} = 0 \ , \quad \phi^{\langle 0 \rangle}\Big|_{r_0 = 1} = 0$$

(29) $$\nabla^2 \phi^{\langle 1 \rangle} + \lambda^{\langle 0 \rangle} \phi^{\langle 1 \rangle} + \lambda^{\langle 1 \rangle} \phi^{\langle 0 \rangle} = 0 \ , \quad \phi^{\langle 1 \rangle} + \cos^2\theta \, \phi^{\langle 0 \rangle}_{,r_0} = 0 \ \text{ on } r_0 = 1$$

(30) $$\nabla^2 \phi^{\langle 2 \rangle} + \lambda^{\langle 0 \rangle} \phi^{\langle 2 \rangle} + 2\lambda^{\langle 1 \rangle} \phi^{\langle 1 \rangle} + \lambda^{\langle 2 \rangle} \phi^{\langle 0 \rangle} = 0 \ ,$$

$$\phi^{\langle 2 \rangle} + 2 \cos\theta \frac{\partial \phi^{\langle 1 \rangle}}{\partial x_0} + \left(x_0 \frac{\partial}{\partial x_0} \right)^2 \phi^{\langle 0 \rangle} = 0 \ \text{ on } r_0 = 1$$

The solution of the problem in the ~~ex~~ [illegible] may be represented as

) $$\phi(x, y, \varepsilon) = \sum_{n=0} \frac{\varepsilon^n}{n!} \phi^{[n]}(x_0, y_0) = \sum_{n=0} \frac{\varepsilon^n}{n!} \phi^{[n]}\left(\frac{x}{1+\varepsilon}, y\right) = \sum \frac{\varepsilon^n}{n!} \phi^{\langle n \rangle}(x, y)$$

Exercises: Let $\int_{\mathcal{V}_\varepsilon} \phi^2(x, \varepsilon) = 1$. (1) Find the smallest eigenvalue of (28). (2) Derive and solve (28), (29) and (30). (3) Derive the explicit functional form of $\phi^{[1]}\left(\frac{x}{1+\varepsilon}, y\right)$, $\phi^{[2]}\left(\frac{x}{1+\varepsilon}, y\right)$ for the solution found under (2)