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# BODIES WITH MICROSTRUCTURE

(Part II)

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We use below also the set of reciprocal directors $\underset{\sim}{d}^{(K)}$, which are defined by the relations

$$\underset{\sim}{d}^{(K)} \cdot \underset{\sim}{d}_{(H)} = \delta_H^{\ K} \quad , \quad \sum_{1}^{3}{}_K \underset{\sim}{d}^{(K)} \otimes \underset{\sim}{d}_{(K)} = \underset{\sim}{1} .$$

Cartesian components of these directors also appear later, such as $d_a^{(K)}$ (and $d^b_{(H)}$, $\overset{*}{d}{}^{K}_{(H)}$, etc.).

We remark finally that, when the local placements are all changed through a modification of all equivalence classes by a common apparent displacement $\underset{\sim}{p}$, then the directors change according to the rule

$$\tilde{\underset{\sim}{d}}_{(K)} = \underset{\sim}{P}\ \underset{\sim}{d}_{(K)} \quad , \text{where } \underset{\sim}{P} = \text{grad}\, \underset{\sim}{p} \ ,$$

and $\underset{\sim}{H}$ as follows

$$\tilde{\underset{\sim}{H}} = \underset{\sim}{H}\, \underset{\sim}{P}^{-1} ;$$

in terms of components

$$\tilde{\underset{\sim}{d}}{}^{i}_{(K)} = \frac{\partial p^i}{\partial x^r}\, d^r_{(K)} \quad , \quad \tilde{H}^i_{\ c} = H^i_{\ d}\, \frac{\partial x^d}{\partial p^c} \ . \qquad (2.6)$$

A number of important tensors can be defined easily in terms of directors. There is first of all the metric tensor

$$g_{HK} = \underset{\sim}{d}_{(H)} \cdot d_{(K)} \ ,$$

and then the <u>wryness</u> tensor $\underset{\sim}{W}$ (see [4], Sect. 61)

$$\underset{\sim}{\mathbb{W}} = \sum_{1}^{3} {}_R \, \underset{\sim}{d}^{(R)} \otimes \left[ (\operatorname{grad} \underset{\sim}{d}_{(R)}) \underset{\sim}{H} \right] , \qquad (2.7)$$

a third order tensor with components

$$\mathbb{W}^a{}_{bc} = \sum_{1}^{3} {}_R \, d_b^{(R)} \, \frac{\partial d^a{}_{(R)}}{\partial x_f^j} \, H^j{}_c \, , \qquad (2.8)$$

and the strictly related notion of linear connection

$$\Gamma^L{}_{HK} = \underset{\sim}{d}^{(L)} \cdot \frac{\partial \underset{\sim}{d}_{(K)}}{\partial X^H} \, . \qquad (2.9)$$

Recourse to (1.1) and (2.2) leads to the relation

$$\Gamma^L{}_{HK} = \left[ ( \underset{\sim}{\mathbb{W}} \, \underset{\sim}{d}_{(H)} ) \, d_{(K)} \right] \cdot d^{(L)} \qquad (2.10)$$

It is often useful to ~~define~~ define an object which does not change when the changes of local placements are effected, which are mentioned in the paragraph containing formulae (2.6). It is easily found that

$$\mathfrak{S}^a{}_{bc} = \tfrac{1}{2} H^a{}_m \, \overset{-1}{H}{}_b{}^n \, \overset{-1}{H}{}_c{}^p \left( \mathbb{W}^m{}_{np} - \mathbb{W}^m{}_{pn} \right) \qquad (2.11)$$

is such an object. $\underset{\sim}{\mathfrak{S}}$ characterizes intrinsic properties of $\mathcal{B}$, being a measure of anholonomy of local relaxed references. It is called the <u>inhomogeneity</u> of $\mathcal{B}$.

Because of the autisymmetry of $\mathfrak{S}$ with respect to the lower indices we can introduce also a tensor of the second order, using the Ricci comakutator:

$$A^{pq} = e^{pbc}\, \mathfrak{S}^{q}_{\ bc} \, . \qquad (2.12)$$

$\underset{\sim}{A}$ is called the tensor density of dislocations. It can be used to form the Burgers vector $\underset{\sim}{b}$ relative to a plane of unit normal $\underset{\sim}{n}$:

$$\underset{\sim}{b} = \underset{\sim}{A}\, \underset{\sim}{n} \, . \qquad (2.13)$$

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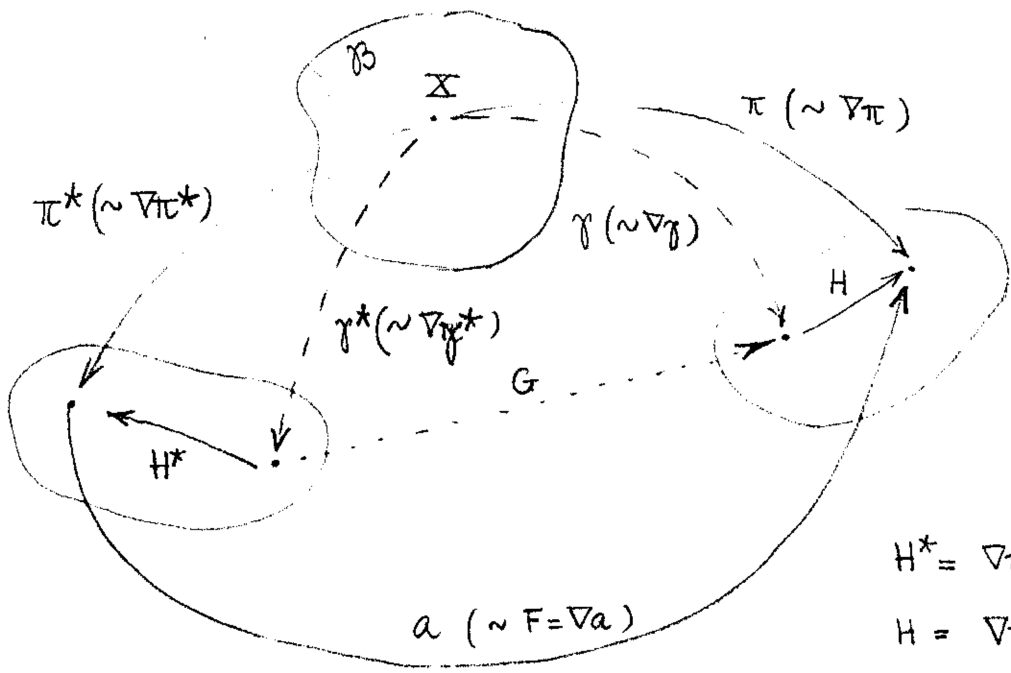


$$H^* = \nabla\pi^* \circ (\nabla\gamma^*)^{-1}$$

$$H = \nabla\pi \circ (\nabla\gamma)^{-1}$$

$$\boxed{F = H\,G\,H^{*-1}}$$

3. Strain, Kinematics.

We are ready now to compare placements, using for this purpose the objects defined in the previous section. One of the placements will be the primary placement and all quantities relating to it will be marked with an asterisk.

Obviously relevant quantities are ~~[illegible]~~ the displacement gradient $\underset{\sim}{F} = \nabla \underset{\sim}{a}$, and $\underset{\sim}{G}$. The corresponding strain tensor are the classical tensors

$$\underset{\sim}{E} = \underset{\sim}{F}^T \underset{\sim}{F} - \underset{\sim}{1}, \tag{3.1}$$

and a new one

$$\underset{\sim}{B} = \underset{\sim}{G}^T \underset{\sim}{F} - \underset{\sim}{1}. \tag{3.2}$$

As an alternative to $\underset{\sim}{B}$ one could use the symmetric tensor

$$\underset{\sim}{E}^{(G)} = \underset{\sim}{G}^T \underset{\sim}{G} - \underset{\sim}{1} \tag{3.3}$$

and ~~[illegible]~~ a second tensor

$$\underset{\sim}{E}^{(R)} = [\underset{\sim}{R}^{(G)}]^T \underset{\sim}{R}^{(F)} - \underset{\sim}{1} \tag{3.4}$$

which involves the orthogonal tensors appearing in the polar decomposition

$$\underset{\sim}{F} = \underset{\sim}{R}^{(F)} (\underset{\sim}{F}^T \underset{\sim}{F})^{1/2} \quad , \quad \underset{\sim}{G} = \underset{\sim}{R}^{(G)} (\underset{\sim}{G}^T \underset{\sim}{G})^{1/2}.$$

A necessary and sufficient condition for the rigidity of the complete displacement $(\underset{\sim}{a}, \underset{\sim}{G})$ is that $\underset{\sim}{E}$ and $\underset{\sim}{B}$, or $\underset{\sim}{E}$ and both $\underset{\sim}{E}^{(G)}$, $\underset{\sim}{E}^{(R)}$ be zero ~~[illegible]~~ over ~~[illegible]~~ $\pi^*(\mathcal{B})$. In fact, if $\underset{\sim}{E}$ vanishes, then the apparent displacement is rigid and $\underset{\sim}{F}$ is orthogonal; furthermore, if $\underset{\sim}{B} = 0$, then $\underset{\sim}{G}$ coincides with $\underset{\sim}{F}$.

~~The conclusion~~ If $\underset{\sim}{E}^{(R)}=0$ , then $\underset{\sim}{R}^{(G)} \equiv \underset{\sim}{F}$ and finally also $\underset{\sim}{G} \equiv \underset{\sim}{F}$ .

If we return now to formula (2.8) and we express ~~[illegible]~~ the quantities appearing in it in terms of $\underset{\sim}{G}$ and of elements of the primary placement we obtain the relation

$$\mathbb{W}^{a}_{\ bc} = (\overset{-1}{G})^{A}_{b} \ \frac{\partial G^{a}_{A}}{\partial X^{H}} \ \overset{*}{H}^{H}_{\ K} \ (\overset{-1}{G})^{K}_{c} \ + $$
$$+ (\overset{-1}{G})^{A}_{b} \ G^{a}_{B} \ (\overset{-1}{G})^{K}_{c} \ \overset{*}{\mathbb{W}}^{B}_{\ AK} .$$

This formula suggests the introduction of the following tensor $\underset{\sim}{\mathbb{F}}$ of the <u>strain of orientation</u>

$$\mathbb{F}^{Q}_{\ RS} = (\overset{-1}{G})^{Q}_{a} \ G^{b}_{R} \ G^{c}_{S} \ \mathbb{W}^{a}_{\ bc} - \overset{*}{\mathbb{W}}^{Q}_{\ RS} .$$

Because

$$\mathbb{F}^{Q}_{\ RS} = \frac{\partial G^{a}_{R}}{\partial X^{H}} \ \overset{*}{H}^{H}_{\ S} \ (\overset{-1}{G})^{Q}_{a} \ ,$$

the vanishing of $\underset{\sim}{\mathbb{F}}$ over ~~[illegible]~~ $\pi^{*}(\mathcal{B})$ implies that $\underset{\sim}{G}$ is a constant tensor there. If furthermore $\underset{\sim}{E} = 0$ in $\pi^{*}(\mathcal{B})$ and $\underset{\sim}{B}$ vanishes at least in one place, then the complete displacement is rigid.

It is perhaps interesting to note that

$$\mathbb{F}^{\;q}_{[RS]} = (\underset{\sim}{F}\,\overset{*}{\underset{\sim}{H}})^{b}_{R}\,(\underset{\sim}{F}\,\overset{*}{\underset{\sim}{H}})^{c}_{S}\,(\overline{\underset{\sim}{F}\,\overset{*}{\underset{\sim}{H}}}^{-1})^{q}_{a}\;\mathbb{S}^{a}_{\;bc} - \overset{*}{\mathbb{W}}{}^{q}_{[RS]},$$

so ~~and~~ that the vanishing of this tensor implies ~~to~~ the equality

$$\overset{-1}{F}{}^{V}_{a}\,F^{b}_{T}\,F^{c}_{U}\,S^{a}_{\;bc} = \overset{*}{\mathbb{S}}{}^{V}_{\;TU}\,.$$

We consider now a <u>motion</u> of $\mathcal{B}$, i.e. a one-parameter family of ~~p~~ complete placements; it can be specified by ~~giving~~ assigning the complete displacement as a function of the parameter $t$ over an interval $(0, d)$

$$\underset{\sim}{x} = \underset{\sim}{a}(\underset{\sim}{X}, t)\quad,\quad \underset{\sim}{G} = \underset{\sim}{G}(\underset{\sim}{X}, t).$$

Beside the classical velocity gradient

$$\underset{\sim}{V} = \dot{\underset{\sim}{F}}\,\overset{-1}{\underset{\sim}{F}}$$

another field over $\overset{*}{\pi}(\mathcal{B})$ becomes important: the <u>wrenching</u>

$$\underset{\sim}{W} = \dot{\underset{\sim}{G}}\,\overset{-1}{\underset{\sim}{G}}\,.$$

Its gradient is ~~[illegible]~~ simply related with the time-derivative of the strain of orientation

$$\dot{\mathbb{F}}{}^{T}_{\;UV} = \overset{-1}{G}{}^{T}_{a}\,G^{b}_{U}\,V^{a}_{\;b,H}\,\overset{*}{H}{}^{H}_{V}\,.$$

The question arises now of associating with the motion appropriate measures of inertia. There are ~~good~~ reasons to believe (see, for instance [5]) that, beyond the usual momentum per unit volume $\rho \dot{\underset{\sim}{x}}$ ($\rho$, mass density), one should account also for a generalized moment of momentum per unit mass $\rho \underset{\sim}{S}$ of a molecular character, expressed with the use of a tensor $\underset{\sim}{S}$ obtained by composition of the wrenching $\underset{\sim}{W}$ with a Eulerian inertia tensor $\underset{\sim}{I}$ :

$$\rho \underset{\sim}{S} = \rho \underset{\sim}{I}\, \underset{\sim}{W}^T .$$

$\underset{\sim}{I}$ satisfies an evolution ~~condition~~ equation

$$\dot{\underset{\sim}{I}} = 2 \operatorname{sym} \underset{\sim}{S},$$

which, in a sense, parallels the usual equation of continuity

$$\dot{\rho} + \rho \operatorname{div} \dot{\underset{\sim}{x}} = 0 .$$

$\underset{\sim}{S}$ must be added to the usual measure $\underset{\sim}{x} \otimes \dot{\underset{\sim}{x}}$ to form the total generalized moment of momentum per unit mass. Correspondingly ~~the kinetic energy~~ one must attribute to our body ~~the~~ kinetic energy $\tau$ per unit mass expressed by

$$\tau = \frac{1}{2} \dot{\underset{\sim}{x}}^2 + (\underset{\sim}{I}\, \underset{\sim}{W}^T) \cdot \underset{\sim}{W}^T .$$

4. Balance equations

The balance of linear momentum for $\mathcal{B}$ can be expressed as usual, with a supply due to a body force $\underset{\sim}{b}$ and a surface traction ~~deriving~~ derived from a Cauchy stress tensor $\underset{\sim}{T}$, which there is no reason ~~to~~ now to suppose symmetric. Locally we have, as usual, the equation

$$\rho \ddot{x} = \rho \underset{\sim}{b} + \operatorname{div} \underset{\sim}{T} .$$

More delicate appears the question of writing the balance for generalized moment of momentum. On the one hand the g. m. with respect to the origin of inertia forces has a more complex expression; per unit mass it turns out to be

$$\underset{\sim}{x} \otimes \ddot{\underset{\sim}{x}} + \dot{\underset{\sim}{S}} - \underset{\sim}{W}\, \underset{\sim}{S} .$$

We have of course the supply due to body and traction forces (respectively $\rho \underset{\sim}{x} \otimes \underset{\sim}{b}$, $\underset{\sim}{x} \otimes \underset{\sim}{T} \underset{\sim}{n}$; $\underset{\sim}{n}$ unit normal on the boundary) and we must accept that there may be a supply due to generalized body couples $\underset{\sim}{L}$ (per unit mass) and g. surface couples derived from a third order tensor $\underset{\sim}{H}$.

~~Besides~~ On the other hand, we must not forget that, internal forces, although ~~they~~ add up to give a null moment, ~~they~~ they provide a supply of generalized moment, which is not zero. In fact, in ordinary continuous bodies such generalized moment is equal to $-\underset{\sim}{T}$. In the present context one can take into account of both facts by introducing a supply of g. m. of m. due to internal forces which is measured ~~by a~~ per unit mass by a symmetric tensor $\underset{\sim}{Z}$.

In all, the second balance equation becomes

$$\rho(\dot{S} - WS) = \rho \underset{\sim}{L} + \underset{\sim}{T}^T + \operatorname{div} \underset{\sim}{\mathbb{H}} + \underset{\sim}{Z},$$

$$\underset{\sim}{Z} \in Sym.$$

There is finally the balance of energy, which involves internal energy $\varepsilon$, heat flux $\underset{\sim}{h}$ and heat supply $r$:

$$\rho \dot{\varepsilon} = \rho q + \operatorname{div} \underset{\sim}{h} + \underset{\sim}{T} \cdot (\underset{\sim}{V} - \underset{\sim}{W}) +$$

$$+ \underset{\sim}{\mathbb{H}} \cdot \operatorname{grad} \underset{\sim}{W}^T - \underset{\sim}{Z} \cdot \underset{\sim}{W}.$$

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