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STRATIFIED FLUID FLOWS
Internal Gravity Waves
(Part II)

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3. The propagation of internal gravity waves.

In Section 2(b) we met internal gravity waves as normal modes; here we shall meet their group velocity, their reflexion, and their production by a source, and shall discuss their relationship to wave fronts and rays and their role in boundary-value problems.

(a) Plane waves

The linearized equation governing internal gravity waves can easily be shown to be

$$\frac{\partial^2}{\partial t^2}\left\{ \frac{\partial}{\partial z}\left(\bar{\rho}\,\frac{\partial w'}{\partial z} \right) + \bar{\rho}\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} \right)\right\} - g\,\frac{d\bar{\rho}}{dz}\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} \right) = 0. \qquad (1)$$

This is essentially equation (2.7) with $U_* \equiv 0$ but without Fourier analysis. For mathematical simplicity we shall take $\bar{\rho} = \rho_o \exp(-z/H)$, and assume that H is much larger than other length scales of the problem (so we neglect density variation except in the buoyancy, as before). Then equation (1) becomes

$$\frac{\partial^2}{\partial t^2}\Delta w' - \frac{g}{H}\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} \right) = 0. \qquad (2)$$

Equation (2) admits plane wave solutions of the form $w' \propto \exp\{i(\alpha x + \beta y + \gamma z - \omega t)\}$ if the dispersion relation

$$\omega^2 = g(\alpha^2 + \beta^2)/H\,\boldsymbol{\alpha}^2 \qquad (3)$$

is satisfied, where the vector wavenumber is given by $\boldsymbol{\alpha} = (\alpha, \beta, \gamma)$. Note that $\omega^2 \le g/H$, with equality if and only if $\gamma = 0$.

These plane wave solutions also exist when α^2, β^2 or γ^2 is negative, giving <u>external gravity waves</u> which grow or decay exponentially with x, y or z. They occur only if there is an appropriate boundary to ensure that w' is finite; for example, an external wave varying like $\exp(-|\gamma|z)$ above the ground $z = 0$ may occur. Such a wave is said to be external because it is appreciable only near the exterior of the domain of flow.

The conditions of incompressibility and mass conservation give div **u** = 0 and thence

$$\boldsymbol{\alpha}\cdot\mathbf{u}' = 0. \qquad (4)$$

Thus internal gravity waves are transverse, motion of the fluid being perpendicular to the direction of phase propagation.

The group velocity is given by

$$\mathbf{c}_g = \left(\frac{\partial\omega}{\partial\alpha}, \frac{\partial\omega}{\partial\beta}, \frac{\partial\omega}{\partial\gamma} \right)$$

$$= \frac{\omega\gamma}{(\alpha^2+\beta^2)\boldsymbol{\alpha}^2}\,(\gamma\alpha, \beta\gamma, -(\alpha^2+\beta^2)). \qquad (5)$$

Note that $\underset{\sim}{\alpha}.\underset{\sim}{c}_g = 0$. It can be shown that the energy flux of the waves is in the direction of the group velocity. Also the phase velocity is given by

$$\underset{\sim}{c} = \omega\underset{\sim}{\alpha}/\underset{\sim}{\alpha}^2. \tag{6}$$

It follows that

$$\underset{\sim}{c} + \underset{\sim}{c}_g = \frac{\omega}{\alpha^2 + \beta^2} \quad (\alpha,\beta,0).$$

These relations can be represented geometrically in an illuminating way, on taking $\underset{\sim}{c}$ and $\underset{\sim}{c}_g$ to represent segments of a circle on the base of a diameter. This is shown in figure 5. Note that the plane of the circle is vertical and contains the direction of phase propagation of the wave, and that the vertical components of the phase and group velocities are in opposite directions.

To solve a boundary-value problem one is likely to need a real Fourier integral of these complex wave components or wave components in other coordinates, for example cylindrical polars. As a simple example, however, consider waves in the rigid rectangular box $0 \le x \le K$, $0 \le z \le M$. This problem admits eigensolutions

$$\left.\begin{aligned} u' &= A\sin(p\pi x/K)\cos(q\pi z/M)\cos\omega t \\ w' &= A(pM/qK)\cos(p\pi x/K)\sin(q\pi z/M)\cos\omega t \end{aligned}\right) \tag{7}$$

for p, q = 1, 2, ... and an arbitrary constant A, where

$$\omega^2 = \frac{gp^2}{HK^2} \Big/ \left(\frac{p^2}{K^2} + \frac{q^2}{M^2}\right). \tag{8}$$

This solution represents cellular standing waves. In a crowded room you may sometimes see such waves made visible as a layer of smoke undulates.

(b) Reflexion of waves at a rigid boundary

The reflexion of internal gravity waves is strange, the angle between the reflected wave and the horizontal or vertical being equal to the angle between the incident wave and the horizontal or vertical. The slope of the rigid boundary does not affect the angle of reflexion! This result seems to be due to Phillips [35], who generalized his earlier result on the analogous reflexion of inertial waves [36].

To illustrate this strange property we shall only take one simple example. Thus we take a wave with velocity equal to the real part of $\underset{\sim}{u}_i = (-\gamma/\alpha,0,1)\varepsilon$ is incident upon the rigid plane z = x, with fluid occupying the region z > x, where we write $\varepsilon = \exp\{i(\alpha x+\gamma z-\omega t)\}$. Then we try the linearized solution in the complex form $\underset{\sim}{u} = \underset{\sim}{u}_i + \underset{\sim}{u}_r$, where $\underset{\sim}{u}_r = R(-\gamma'/\alpha',0,1)\varepsilon'$ and $\varepsilon' = \exp\{i(\alpha'x+\gamma'z-\omega't)\}$ for some constants R,α',γ' and ω'.

The condition that no fluid crosses the rigid plane boundary gives

$$(\underset{\sim}{u}_i + \underset{\sim}{u}_r)\cdot\underset{\sim}{n} = 0 \qquad \text{on} \quad z = x,$$

where $\underset{\sim}{n} = (-1,0,1)/2^{-\frac{1}{2}}$ is the unit normal to the plane directed into the fluid. It follows that

$$\omega' = \omega,\ \alpha' + \gamma' = \alpha + \gamma \quad \text{and} \quad (1 + \gamma/\alpha) + R(1 + \gamma'/\alpha') = 0. \tag{9}$$

In order that the two waves separately satisfy the linearized equations of motion we require that

$$\pm\alpha' \left(\frac{g/H}{\alpha'^2 + \gamma'^2}\right)^{\frac{1}{2}} = \omega' = \omega = \pm\ \alpha\left(\frac{g/H}{\alpha^2 + \gamma^2}\right)^{\frac{1}{2}}, \tag{10}$$

i.e.

$$\cos \underset{\sim}{\alpha}'\hat{O}x = \pm \cos \underset{\sim}{\alpha}'\hat{O}x.$$

Therefore

$$\underset{\sim}{\alpha}'\hat{O}x = \pm\ \underset{\sim}{\alpha}\hat{O}x \quad \text{or} \quad \pm(\pi - \underset{\sim}{\alpha}\hat{O}x). \tag{11}$$

Here we must be careful with the signs and directions, and it may help to look at the one case shown in figure 6. We require that the incident wave directs its energy (and hence its group velocity) from the fluid towards the boundary. Similarly $\underset{\sim}{c}_g'$ is directed from the boundary into the fluid. Also note that the vertical components of $\underset{\sim}{c}$ and $\underset{\sim}{c}_g$ have opposite signs. The equality of angles now follows, a typical case with $\gamma > \alpha > 0$ being shown in figure 6.

It only remains to find the reflexion coefficient R. Squaring (10), eliminating ω and then γ', we find

$$\alpha' = -\ \alpha(\alpha + \gamma)/(\gamma - \alpha),\ \gamma' = \gamma(\alpha + \gamma/(\gamma - \alpha),$$

and thence

$$R = -(\alpha + \gamma)/(\gamma - \alpha). \tag{12}$$

It is interesting to consider an extension of this problem where waves propagate in the wedge between two rigid planes, one being horizontal. At each reflexion the group velocity is directed back into the fluid, making equal angles with the horizontal or vertical (see figure 7). It can be seen that all waves will travel from the source towards the vertex of an acute-angled wedge and never return [37]. What happens to them at the vertex?

(c) Forced oscillations

We shall now consider waves that are not plane in a generally stratified medium. We suppose that they are forced at a given frequency, e.g. by an oscillating wavemaker. Thus we suppose only that $w' \propto e^{-i\omega t}$, and equation (1) gives [38]

$$(1 - N^2/\omega^2)\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2}\right) + \frac{\partial^2 w'}{\partial z^2} + \frac{d\bar{\rho}}{\bar{\rho}dz}\ \frac{\partial w'}{\partial z} = 0, \tag{13}$$

where $N^2(z) = -g d\bar{\rho}/dz$. We suppose that $N^2 > 0$ everywhere to exclude the certainty of instability. It can be seen that equation (13) is locally elliptic where $\omega^2 > N^2$, parabolic where $\omega^2 = N^2$ and hyperbolic where $\omega^2 < N$. The elliptic regime is qualitatively similar to the potential flow of a homogeneous fluid, but the hyperbolic regime, with characteristics given by

$$dz^2 = (N^2/\omega^2 - 1)(dx^2 + dy^2), \tag{14}$$

is very different. Indeed, we find that in the limit as $\omega^2/N^2 \to 0$ the characteristics are horizontal, so that 'information' is propagated only horizontally. This is an aspect of blocking (see, e.g., Turner's book [39]).

Mowbray and Rarity [40] did some experiments which vividly illustrate the elliptic and hyperbolic regimes.

The characteristics may also be interpreted by ray methods, by regarding the disturbance locally as a superposition of plane waves of the kind we discussed in Subsections (a) and (b). This may be justified and used extensively [cf. 39] if N^2 varies slowly over a wavelength and we identify $H = g/N^2$ locally.

4. Propagation of internal gravity waves with basic shear.

(a) Airflow over a mountain

An important application of the theory of forced internal gravity waves is to the airflow over a mountain. This application has been treated by many authors [cf. 34] since the first work of Lyra in 1940. Here, trying to capture the essence of the problem with a minimum of detail, we shall neglect the rotation of the earth, compressibility of air, unsteadiness of flow, nonlinearity, three-dimensionality, and non-hydrostatic effects after Drazin and Su [41].

We accordingly suppose that $\underset{\sim}{u} \to U(z)\underset{\sim}{i}$, $\rho \to \bar{\rho}(z)$ as $x \to -\infty$ far upstream. Then we put $\underset{\sim}{u} = U(z)\underset{\sim}{i} + \underset{\sim}{u}'(x,z)$ and $\rho = \bar{\rho}(z) + \rho'(x,z)$ as in equation (2.1) and prepare to linearize the equations of motion. It is convenient first to introduce the dependent variable $\zeta(x,z)$, defined as the height of the streamline through the point (x,z) above its level far upstream. (It may help to look at figure 8.) Thus ζ is a Lagrangian vertical displacement such that

$$w' = \frac{D\zeta}{Dt} = U\frac{\partial \zeta}{\partial x} \tag{1}$$

on linearization. The equation of incompressiblity gives

$$0 = \frac{D\rho}{Dt} = U\frac{\partial \rho'}{\partial x} + w'\frac{d\bar{\rho}}{dz} = U\frac{\partial}{\partial x}\left(\rho' + \zeta\frac{d\bar{\rho}}{dz}\right).$$

Therefore, on integration along a streamline, we find

$$\rho' = -\zeta \frac{d\bar{\rho}}{dz} . \tag{2}$$

Now mass conservation and hydrostatic balance give

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0; \quad \text{and} \quad \frac{\partial p'}{\partial z} = -g\rho' \tag{3}$$

respectively. Therefore the linearized equation of horizontal momentum gives

$$-\frac{\partial p'}{\partial x} = \bar{\rho}(U \frac{\partial u'}{\partial x} + w' \frac{dU}{dz}) = \bar{\rho}(-U \frac{\partial w'}{\partial z} + w' \frac{dU}{dz})$$

$$= -\bar{\rho}U^2 \frac{\partial}{\partial z} (\frac{w'}{U}) = -\bar{\rho}U^2 \frac{\partial^2 \zeta}{\partial z \, \partial x}$$

(if we assume $U > 0$), and thence

$$p' = \bar{\rho}U^2 \partial\zeta/\partial z. \tag{4}$$

Therefore (2), (3) and (4) give

$$\frac{\partial}{\partial z} (\bar{\rho}U^2 \frac{\partial \zeta}{\partial z}) = g \frac{d\bar{\rho}}{dz} \zeta . \tag{5}$$

This is essentially equation (2.7) with $c = 0$, $\beta = 0$, $\alpha^2 << D^2$ and $\zeta \propto \hat{w}/U$; thus the hydrostatic approximation is seen to be equivalent to that of long waves. The general solution is of the form

$$\zeta(x,z) = F(x)\, f(z) + G(x)\, g(z), \tag{6}$$

where F and G are arbitrary functions, and f and g are any two independent solutions of the ordinary differential equation (5).

The upper boundary condition can be shown to be that each wave component radiates energy upwards into the upper atmosphere and away from the source, namely the mountain [42]. To investigate this we shall suppose, to be both specific and simple, that $U \to U_\infty$ and $\bar{\rho} \sim \rho_0 e^{-z/H}$ as $z \to \infty$, although the velocity of the stratosphere in fact usually varies with height quite strongly. Then we define

$$\gamma = + (g/HU_\infty^2 - \tfrac{1}{4}H^{-2})^{\frac{1}{2}} , \tag{7}$$

supposing that $\gamma^2 > 0$ (which is almost always true in the stratosphere). Then solution (6) gives

$$\bar{\zeta}(\alpha,z) \sim e^{z/2H} \, (\bar{F}(\alpha)\, e^{i(\alpha x + \gamma z)} + \bar{G}(\alpha)\, e^{i(\alpha x - \gamma z)} \quad \text{as} \quad z \to \infty,$$

where $\bar{\zeta}$, $\bar{F}$ and $\bar{G}$ are the Fourier transforms of ζ, F and G respectively. Now look at the flow in a frame moving with velocity $U_\infty \underset{\sim}{i}$ relative to the mountain. By this Galilean transformation the mountain and the wave appear to move upstream with speed U but the upper atmosphere is reduced to rest, so we may use formula (3.5) with $\omega = -\alpha U_\infty$. That gives us the group velocity with a positive vertical component, and hence upward propagation of energy, if and only if $\gamma\alpha\, U_\infty > 0$. That implies here, where we choose U_∞ and γ to be positive, that

$$\bar{\zeta} \sim \begin{cases} \bar{F}(\alpha)e^{z/2H + i(\alpha x + \gamma z)} & \text{for } \alpha > 0 \\ \bar{G}(\alpha)e^{z/2H + i(\alpha x - \gamma z)} & \text{for } \alpha < 0 \end{cases} \quad \text{as } z \to \infty. \tag{9}$$

The boundary condition that the mountain, with equation $z = \zeta_0(x)$, say, is a streamline gives

$$\zeta(0,z) = \zeta_0(x). \tag{10}$$

Putting together (9) and (10) with Fourier analysis, we find

$$\zeta = \frac{f(z)}{f(0)} \int_0^\infty \bar{\zeta}_0(\alpha)e^{i\alpha x}\, d\alpha + \frac{f^*(z)}{f^*(0)} \int_{-\infty}^0 \bar{\zeta}_0(\alpha)e^{i\alpha x} d\alpha\ , \tag{11}$$

where $f(z)$ is the solution of equation (5) which behaves like $e^{(i\gamma+1/2H)z}$ as $z \to \infty$ and $\bar{\zeta}_0$ is the Fourier transform of ζ_0. It follows that

$$\zeta = \text{Re}\left\{\frac{f(z)}{f(0)}\right\} \int_{-\infty}^\infty \bar{\zeta}_0(\alpha)e^{i\alpha x} d\alpha + i\,\text{Im}\left\{\frac{f(z)}{f(0)}\right\} \left(\int_0^\infty - \int_{-\infty}^0\right) \bar{\zeta}_0(\alpha)e^{i\alpha x} d\alpha$$

$$= \zeta_0(x)\ \text{Re}\left\{\frac{f(z)}{f(0)}\right\} + \pi^{-1}\, P\int_{-\infty}^\infty \frac{\zeta_0(t)}{t-x}\, dt\ \ \text{Im}\left\{\frac{f(z)}{f(0)}\right\}. \tag{12}$$

The Cauchy principal part of the integral in the latter term may be recognised as the Hilbert transform of $\pi\zeta_0$ with many well known properties (see, e.g., [43]).

To illustrate this theory we take one simple example with

$$\bar{\rho} = \rho_0\, e^{-z/H}, \quad U = \text{constant} \quad \text{for } z \geq 0, \tag{13}$$

and

$$\zeta_0(x) = b^2h/(b^2 + x^2) \quad \text{for } -\infty < x < \infty. \tag{14}$$

Then it can be shown that the Hilbert transform of ζ_0 is given by

$$\pi^{-1}\, P\int_{-\infty}^\infty \frac{\zeta_0(t)}{t-x}\, dt = -bhx/(b^2 + x^2) \tag{15}$$

and thence that

$$\zeta(x,z) = bhe^{z/2H}(b \cos\gamma z - x \sin\gamma z)/(b^2 + x^2) \,. \tag{16}$$

The streamlines of this flow are illustrated in figure 9. Note that the crests of the waves tilt upstream as one rises.

(b) Energy and momentum

Hitherto we have mentioned energy only in asserting that the energy of waves is propagated with the group velocity when the fluid is in a basic state of rest, but there is more to be said.

The linearized equations of motion (2)-(5) give

$$\frac{DE}{Dt} = U\,\frac{\partial E}{\partial x} = \frac{\partial W_x}{\partial x} + \frac{\partial W_z}{\partial z} - \bar{\rho}\,\frac{dU}{dz}\,u'w', \tag{17}$$

where we define

$$E = \tfrac{1}{2}(\bar{\rho}\,u'^2 - g\frac{d\bar{\rho}}{dz}\,\zeta^2),\ W_x = p'u' \quad\text{and}\quad W_z = p'w' \,. \tag{18}$$

We identify E as the energy density of the waves, $\frac{1}{2}\bar{\rho}u'^2$ being the kinetic energy (note that we neglect the kinetic energy of the vertical motion in order to be consistent with the hydrostatic approximation) and $-\frac{1}{2}g\,\frac{d\bar{\rho}}{dz}\,\zeta^2$ being the potential energy. Similarly we identify W_x and W_z as the components of the energy flux and $-\bar{\rho}\,\frac{dv}{dz}u'w'$ as the rate of transfer of energy density from the basic shear flow (this is essentially a Reynolds stress).

Also the vertical flux of horizontal momentum is given by

$$M_x = \bar{\rho}u'w' = -W_z/U. \tag{19}$$

This is related to the drag exerted on the mountain by the wind,

$$D = \int_{-\infty}^{\infty} [p']_{z=0}\,\frac{\partial\zeta_0}{\partial x}\,dx = -\int_{-\infty}^{\infty}\left[\bar{\rho}\ U^2\,\frac{\partial\zeta}{\partial x}\right]_{z=0}\frac{\partial\zeta_0}{\partial x}\,dx$$

$$= \frac{1}{U(0)}\int_{-\infty}^{\infty}[p'w']_{z=0}\,dx\ , \tag{20}$$

on resolving the force due to the pressure perturbation in the horizontal direction, because the x-integral of M_x must be independent of height to conserve momentum. For our example (16) we find

$$D = \tfrac{1}{4}\pi\ell h^2\,\rho_0U^2. \tag{21}$$

More insight into these properties can be gained from the model of Section 3(a), with $U \equiv 0$ and negligible inertial effects of density variation. It gives waves with dispersion relation (3.3) and

$$\alpha v' = \beta u', \quad w' = -(\alpha^2 + \beta^2)u'/\gamma\alpha, \quad p' = \omega\bar{\rho}u'/\alpha . \tag{22}$$

The energy density is then

$$\begin{aligned} E &= \tfrac{1}{2}\bar{\rho}(\underset{\sim}{u}'^2 + N^2 w'^2/\omega^2) \\ &= \tfrac{1}{2}\bar{\rho}(u'^2 + v'^2 + w'^2 + gw'^2/H\omega^2) \\ &= \bar{\rho}(\alpha^2 + \beta^2)\underset{\sim}{\alpha}^2 \, u'^2/\gamma^2 \, \alpha^2 , \end{aligned} \tag{23}$$

and the wave flux vector

$$\begin{aligned} \underset{\sim}{W} &= (W_x, W_y, W_z) = (p'u', p'v', p'w') \\ &= E\underset{\sim}{c}_g , \end{aligned} \tag{24}$$

as we asserted earlier.

Going further back to the model of the Taylor-Goldstein equation (2.10), we find that the average vertical flux of horizontal momentum is given by

$$\begin{aligned} \bar{M}_x = \bar{\rho}\,\overline{u'w'} &= \frac{\alpha\bar{\rho}}{2\pi}\int_0^{2\pi/\alpha} u'w' \, dx \\ &= \frac{\alpha\bar{\rho}}{2\pi}\int_0^{2\pi/\alpha} \mathrm{Re}\left\{D\phi \, e^{i\alpha(x-ct)}\right\} \ \mathrm{Re}\left\{-i\alpha\phi e^{i\alpha(x-ct)}\right\} dx \\ &= \tfrac{1}{4}i\alpha\bar{\rho}(\phi^* D\phi - \phi D\phi^*)e^{2\alpha c_i t} , \end{aligned} \tag{25}$$

and of vertical momentum by

$$\begin{aligned} \bar{W}_z &= \frac{\alpha}{2\pi}\int_0^{2\pi} p'w' \, dx \\ &= -\tfrac{1}{4}\,\alpha\bar{\rho}\,\{i(U-c_r)(\phi^* D\phi - \phi D\phi^*) + c_i D|\phi|^2\}. \end{aligned} \tag{26}$$

When $c_i = 0$ we deduce that

$$\bar{W}_z = -(U-c)\bar{M}_x , \tag{27}$$

in agreement with (19). Also the Taylor-Goldstein equation gives

$$\frac{d\bar{M}_x}{dz} = \tfrac{1}{2}\alpha \, c_i \, \bar{\rho} \left\{\frac{JN^2(U-c_r)}{|U-c|^4} - \frac{U''}{|U-c|^2}\right\}|\phi|^2, \tag{28}$$

as in the proof of (2.28). This also shows that if $c_i = 0$ then $\bar{M}_x$ is constant except possibly where $U = c$, and

$$\frac{d\bar{W}_z}{dz} = -U'\bar{M}_x . \tag{29}$$

(b) Critical layers

Hitherto we have shunned the singularity of the Taylor-Goldstein equation at its critical layer, say $z = z_c$ where $U(z) = c$. This singularity in fact gives rise to the continuous spectrum of neutral modes and to the interpretation of the branch-points of eigenfunctions of marginally stable modes, both of which are mentioned briefly in Section 2. Also in this Section we implicitly assumed that $U \neq 0$ in order that our theory of lee waves would be nonsingular. But what happens if $U = 0$?

Here we shall treat a problem of wave propagation after Booker and Bretherton [44] to illustrate both the mathematical ideas and physical importance of the critical layer. We consider a monochromatic two-dimensional internal gravity wave of numbers α and $\beta = 0$ and fixed angular frequency ω propagating in a stratified shear flow, governed by the Taylor-Goldstein equation (2.10).

If we make the approximation that U and N^2 vary very slowly over a vertical wavelength, then we may use the ideas of ray theory and a local Galilean transformation to deduce from (3.3) that

$$\{\omega - U(z)/\alpha\}^2 = \alpha^2 N^2(z)/\{\alpha^2 + \gamma^2(z)\} . \tag{30}$$

Sometimes $\omega - U(z)/\alpha$ is called the Doppler-shifted frequency. Formula (30) shows how γ varies as the wave propagates upwards. It can be seen that as the wave approaches a critical layer $\gamma^2 \to \infty$; thus the wave fronts become nearly horizontal and get closer together vertically (but not horizontally). The group velocity of the wave relative to the basic velocity of the fluid is zero and so the energy density E tends to infinity at the critical level. In fact [45] the wave action $E/(\omega - U/\alpha)$ is a conserved quantity, i.e. independent of height in this approximation of ray theory, not E itself.

To understand this better we shall work out one example in some detail. The structure of the solution near a critical layer is determined by the gradient of U, so we cannot take a flow which has piecewise constant U. We take a stratified shear layer in dimensionless form

$$N^2 = \text{constant}, \quad U = \begin{cases} 1 & \text{for } 1 < z \\ 1 - (1-z)/s & \text{for } -1 < z < 1 \\ 1 - 2/s & \text{for } z < -1 . \end{cases} \tag{31}$$

We suppose that an incident wave propagates upwards (positive vertical group velocity) from $z = -\infty$ and a reflected wave goes back. It is also convenient to take $\omega = 0$ (this can be effected without loss of generality by a Galilean transformation for any given value of ω). Thus we take the solution of (2.10) as

$$\phi = e^{-i\gamma(z+1)} + R\, e^{i\gamma(z+1)} \quad \text{for} \quad z < -1, \tag{32}$$

where $\gamma = (N - \alpha^2)^{\frac{1}{2}}$ and R is a reflexion coefficient to be determined. Similarly we take

$$\phi = T\, e^{-i\gamma(z+1)} \qquad \text{for} \quad z > 1, \qquad (33)$$

where T is a transmission coefficient. Equations (2.10) and (31) also give

$$\phi'' + \left\{\frac{\mu^2 + \frac{1}{4}}{(z-z_c)^2} - \alpha^2\right\}\phi = 0 \quad \text{for} \quad -1 < z < 1, \qquad (34)$$

where $\mu = +(s^2 N^2 - \frac{1}{4})^{\frac{1}{2}}$ and $z_c = 1 - s$. We take $0 < s < 2$ in order that the critical layer lies in the shear layer. The nature of the critical layer enters the calculation through the choice of the branches of the solution of (34).

The solutions of (34) give

$$\phi \sim (z - z_c)^{\frac{1}{2}} \; \{A\,(z - z_c)^{i\mu} + B(z - z_c)^{-i\mu}\} \quad \text{as} \quad z \to z_c$$

for some constants A and B. Now the powers of $(z - z_c)$ arise from the factor

$$\begin{aligned} U - c &= \left(1 - \frac{1-z}{s} - \alpha\omega\right) \\ &= \lim_{\omega_i \downarrow 0} (z - z_c - i\alpha s\omega_i)/s\,. \end{aligned}$$

So we interpret

$$\begin{aligned} \log(z - z_c) &= \lim_{\omega_i \downarrow 0} \log\,(z - z_c - i\alpha s\omega_i) \\ &= \begin{cases} \log\,(z - z_c) & \text{for} \quad z > z_c \\ \log\,(z_c - z) - i\pi & \text{for} \quad z < z_c, \end{cases} \end{aligned} \qquad (35)$$

the singularity being at $z_c + i\alpha s\omega_i$, just <u>above</u> the real z-axis when $\alpha s\omega_i > 0$. Thus we have

$$\phi \sim \begin{cases} (z-z_c)^{\frac{1}{2}}\{A\, e^{i\mu\log\,(z-z_c)} + B\, e^{-i\mu\log\,(z-z_c)}\} & \text{for} \quad z > z_c \\ -i(z_c-z)^{\frac{1}{2}}\;\{A\, e^{\pi\mu+i\mu\log(z-z_c)} + B\, e^{-\pi\mu-i\mu\log\,(z-z_c)}\} & \text{for} \quad z < z_c \end{cases}$$

$$\text{as} \quad z \to z_c\,. \qquad (36)$$

With this interpretation of the singularity we solve (34) in terms of modified Bessel functions to get

$$\phi = (z-z_c)^{\frac{1}{2}} \;\; \{D\, I_{i\mu}(\alpha(z-z_c)) + E\, I_{-i\mu}(\alpha(z-z_c))\} \quad \text{for} \quad -1 < z < 1, \qquad (37)$$

where D and E are some constants to be determined. Then conditions (2.35) at $z = \pm 1$ give four linear inhomogeneous equations in the four unknowns R, T, A and B. The method of solution is complicated and uninformative, but not difficult, giving R and T as functions of α, N^2 and s, whether there is a critical layer or not. It becomes simpler, however, in the limit as $N^2 \to \infty$ (this is effectively large overall Richardson number in our present choice of dimensionless variables) and much simpler as $\alpha \to 0$. It so happens that the dominant behaviour of R and T as $N^2 \to \infty$ is independent of α, so it suffices to solve the problem for $\alpha=0$ and general values of $N > 0$.

The long-wave ($\alpha=0$ or hydrostatic) calculation is simple, because (32), (33), (36), (37) give

$$\phi = \begin{cases} Te^{-iN(z+1)} & \text{for } 1 < z \\ (z-z_c)^{\frac{1}{2}}\{A(z-z_c)^{i\mu} + B(z-z_c)^{-i\mu}\} & \text{for } z_c < z < 1 \\ -i(z_c-z)^{\frac{1}{2}}\{Ae^{\pi\mu}(z_c-z)^{i\mu} + Be^{-\pi\mu}(z_c-z)^{-i\mu}\} & \text{for } -1 < z < z_c \\ e^{-iN(z+1)} + Re^{iN(z+1)} & \text{for } z < -1\,. \end{cases} \tag{38}$$

Now conditions (2.35) at $z = \pm 1$ give

$$T = (1-z_c)^{\frac{1}{2}}\{A(1-z_c)^{i\mu} + B(1-z_c)^{-i\mu}\},$$

$$-iNT = A\{\tfrac{1}{2}+i\mu-(1-z_c)/s\}(1-z_c)^{-\frac{1}{2}+i\mu} + B\{\tfrac{1}{2}-i\mu-(1-z_c)/s\}(1-z_c)^{-\frac{1}{2}-i\mu},$$

$$1+R = -ie^{\pi\mu}A(z_c+1)^{\frac{1}{2}+i\mu}-ie^{-\pi\mu}B(z_c+1)^{\frac{1}{2}-i\mu},$$

$$(1-2/s)(-iN)(1-R) = -ie^{\pi\mu}A\{(1-2/s)(-\tfrac{1}{2}-i\mu)-(z_c+1)/s\}(z_c+1)^{-\frac{1}{2}+i\mu}$$
$$-ie^{-\pi\mu}B\{(1-2/s)(-\tfrac{1}{2}+i\mu)-(z_c+1)/s\}(z_c+1)^{-\frac{1}{2}-i\mu};$$

i.e.

$$-T + As^{\frac{1}{2}+i\mu}+Bs^{\frac{1}{2}-i\mu} = 0,$$

$$-iNT + A(\tfrac{1}{2}-i\mu)s^{-\frac{1}{2}+i\mu} + B(\tfrac{1}{2}+i\mu)s^{-\frac{1}{2}-i\mu} = 0,$$

$$R + ie^{\pi\mu}A(2-s)^{\frac{1}{2}+i\mu} + ie^{-\pi\mu}B(2-s)e^{\frac{1}{2}-i\mu} = -1,$$

$$NR + e^{\pi\mu}(\tfrac{1}{2}-i\mu)A(2-s)^{-\frac{1}{2}+i\mu}+ e^{-\pi\mu}(\tfrac{1}{2}+i\mu)B(2-s)^{-\frac{1}{2}-i\mu} = N.$$

Even with all the simplifications, this system requires a lot of tedious algebra to solve, so we shall make the further simplification that N is large. Then it can be quite easily shown that

$$R \to \begin{cases} (s-1) & \text{if } s \neq 1 \\ \tfrac{1}{4}i/\mu & \text{if } s = 1 \end{cases} \quad \text{as } N \to \infty \tag{39}$$

and that

$$T \sim 4\mu e^{-\pi\mu} \{s(2-s)\}^{\frac{1}{2}} \left(\frac{s}{2-s}\right)^{i\mu} \quad \text{as } N \to \infty \quad . \tag{40}$$

It can in fact be shown that (39) and (40) are true for all values of α, not just $\alpha=0$.

The important physical consequence of this analysis is that an exponentially small transmission coefficient arises when N is large, even though the reflexion coefficient may not be close to unity. One may consider where the energy of the incident wave goes when both T and R are small, and conclude that it must be absorbed by the basic flow near the critical layer.

5. Some nonlinear problems

So far we have considered only linear problems. They lead on to many nonlinear problems, but here we shall follow up only one type of nonlinear instability. We have found linear instability with exponential growth in time, although it is intuitively obvious that this growth will not continue indefinitely but will be modified by nonlinearity. We shall enquire into how nonlinearity modifies the growth and seek the flow into which the instability ultimately develops.

(a) The Landau equation

Weakly nonlinear instability can be seen at its simplest in the model of the elementary ordinary differential equation,

$$dA/dt = kA - \ell A^3 \quad . \tag{1}$$

This has the general solution

$$A^2 = A_o^2/\{(1-A_o^2/A_e^2)e^{-2kt} + A_o^2/A_e^2\} \quad , \tag{2}$$

where we suppose that $A = A_o$ at $t=0$ and define $A_e^2 = k/\ell$. This gives $A \sim A_o e^{kt}$ as $A_o \to 0$ for any fixed value of t, as in linear instability; if $k > 0$, however, this limit is not uniformly valid as $t \to \infty$. The limit as $t \to \infty$ depends crucially upon the sign of the Landau constant ℓ and of k.

If $k > 0$ and $\ell > 0$, then $A^2 \to A_e^2 > 0$ as $t \to \infty$, whatever the value of A_o. The solution is said to equilibrate; and the process is called supercritical stability, because above the critical condition for linear stability a nonlinear disturbance approaches the finite equilibrium amplitude A_e (see figure 10(a)).

If $k > 0$ and $\ell < 0$, then $A_e^2 < 0$ and $A^2 \to \infty$ as $t \to (2k)^{-1} \log(1-A_e^2/A_o^2)$. Thus the solution grows without bound in a finite time, although the presence of higher-order nonlinear terms on the right-hand side of (1) could remove this singularity.

If $k < 0$ and $\ell > 0$ then $A^2 \to 0$ as $t \to \infty$, much as in linear stability.

If $k < 0$ and $\ell < 0$ then $A_e^2 > 0$, and $A^2 \to \infty$ as $t \to \infty$ if $A_o^2 > A_e^2$ but $A^2 \to 0$ as $t \to \infty$ if $A_o^2 < A_e^2$. This is called subcritical instability, becuase although all disturbances are linearly stable those with amplitudes greater than the 'threshold' value A_e are nonlinearly unstable (see figure 10(b)).

In practice we are interested in Landau equations (1) for which k and ℓ depend upon the parameters of the basic flow, e.g the Richardson number. To illustrate this, suppose that $k \gtrless 0$ for $J \gtrless J_c$, there being a simple zero of $k(J)$ at $J=J_c$, where J_c is some value such as one quarter. Then the equilibrium solutions of equation (1) as $J \to J_c$ can easily be pictured, as in figure 11. If $\ell > 0$ at $J=J_c$, then the basic solution $A=0$ bifurcates at $J=J_c$ with real solutions $A = \pm A_e(J_c)$ for $J > J_c$; this is the case of supercritical stability. If $\ell < 0$ at $J=J_c$ then the basic solution $A=0$ again bifurcates but has real solutions $A = \pm A_e$ for $J < J_c$; this is subcritical instability. Note that

$$A_e \sim \{(dk/\ell dJ)_c (J-J_c)\}^{\frac{1}{2}} \quad \text{as} \quad J \to J_c \quad ,$$

so the curves of bifurcation are locally parabolae near the critical point $A=0$, $J=J_c$.

In a visionary paper of 1944, Landau related these simple ideas to nonlinear hydrodynamic stability and the onset of turbulence [cf.46,§27]. These ideas have been subsequently applied to many problems involving partial differential equations and developed in many fundamental ways, notably by Stuart [47] and Watson [48]. We shall apply the ideas just to one of the linear problems we have met.

(b) Nonlinear Kelvin-Helmholtz instability

If surface tension T is present in our model of Kelvin-Helmholtz instability, then to our original equations (1.2) - (1.6) we must add

$$\rho_1\left(\frac{\partial\phi_1}{\partial t} + U_1\frac{\partial\phi_1}{\partial x} + g\zeta\right) = \rho_2\left(\frac{\partial\phi_2}{\partial t} + U_2\frac{\partial\phi_2}{\partial x} + g\zeta\right)$$

$$+ T[\Delta\zeta\{1+(\nabla\zeta)^2\} - \tfrac{1}{2}\nabla\zeta.\nabla(\nabla\zeta)^2][1+(\nabla\zeta)^2]^{-3/2} \text{ at } z = \zeta \qquad (3)$$

instead of the pressure boundary condition (1.7). The linearized forms of (1.6) and (3) at the interface can be reduced to

$$\begin{pmatrix} -\partial/\partial z & 0 & \partial/\partial t+U_1\partial/\partial x \\ 0 & \partial/\partial z & -(\partial/\partial t+U_2\partial/\partial x) \\ \rho_1(\partial/\partial t+U_1\partial/\partial x) & -\rho_2(\partial/\partial t+U_2\partial/\partial x) & g(\rho_1-\rho_2)-T\Delta \end{pmatrix} \underset{\sim}{\phi}=0 \text{ at } z=0, \qquad (4)$$

where $\underset{\sim}{\phi}$ is the column vector with rows ϕ_1,ϕ_2 and ζ. The solution of the linear problem (1.4), (1.5) and (4) for a general two-dimensional normal mode is the real part of

$$\underset{\sim}{\phi} = \begin{pmatrix} \alpha^{-1}(s+i\alpha U_1)\exp(i\alpha x + \alpha z) \\ -\alpha^{-1}(s+i\alpha U_2)\exp(i\alpha x - \alpha z) \\ \exp(i\alpha x) \end{pmatrix} A(t) \quad , \qquad (5)$$

where we take $\alpha > 0$ and $A \propto e^{st}$, and find

$$s = -\frac{i\alpha(\rho_1U_1+\rho_2U_2)}{\rho_1+\rho_2} \pm\left\{\frac{\alpha^2\rho_1\rho_2(U_1-U_2)^2}{(\rho_1+\rho_2)^2} - \frac{\alpha g(\rho_1-\rho_2)}{\rho_1+\rho_2} - \frac{\alpha^3T}{\rho_1+\rho_2}\right\}^{\frac{1}{2}}. \qquad (6)$$

This is a classical result of 1871 due to Kelvin [2] himself. Kelvin deduced that the <u>mode</u> is unstable if and only if

$$\rho_1\rho_2(U_1-U_2)^2 > (\rho_1+\rho_2)\{\alpha T + \alpha^{-1}g(\rho_1-\rho_2)\} , \qquad (7)$$

buoyancy stabilizing the long and surface tension the short waves, and that the flow is unstable if and only if (7) is satisfied for some value of $\alpha > 0$, i.e.

$$\rho_1^2\rho_2^2(U_1-U_2)^4 > 4gT(\rho_1-\rho_2)(\rho_1+\rho_2)^2. \qquad (8)$$

For simplicity it is convenient to make a Galilean transformation in order that $U_2 = -U_1$ and to use dimensionless variables based upon the length scale $\ell=\{T/g(\rho_1-\rho_2)\}^{\frac{1}{2}}$ and the velocity scale $V=\{gT(\rho_1-\rho_2)/(\rho_1+\rho_2)^2\}^{\frac{1}{4}}$. We shall also assume that $(\rho_1-\rho_2)/(\rho_1+\rho_2)<<1$ for fixed ℓ and V, which is essentially equivalent to neglecting the effects of the density difference in the inertia but not the buoyancy.

Now we can tackle the problem [49] of the weakly nonlinear instability of slightly unstable modes. We first reduce the nonlinear system (1.5) and (3) to the form

$$L\,\underset{\sim}{\phi} = N\,\underset{\sim}{\phi} \qquad \text{at} \quad z=0\ , \tag{9}$$

where $U_1=W$, $U_2=-W$ in dimensionless form,

$$L = \begin{pmatrix} -\partial/\partial z & 0 & \partial/\partial t+W\partial/\partial x \\ 0 & \partial/\partial z & -(\partial/\partial t-W\partial/\partial x) \\ \partial/\partial t+W\partial/\partial x & -(\partial/\partial t-W\partial/\partial x) & 2(1-\partial^2/\partial x^2) \end{pmatrix}, \tag{10}$$

and the nonlinear operator N can easily be found at length.

The solution of the linearized problem gives $s=\pm\alpha(W^2-W_c^2)^{\frac{1}{2}}$, where $W_c(\alpha)=\{(1+\alpha^2)/\alpha\}^{\frac{1}{2}}$; this is equivalent to (6) in the present case and notation. It gives marginal stability when $W=W_c$ (or $-W_c$). If W is slightly greater than W_c then the mode of wavenumber α grows exponentially in time so long as its amplitude is small. We shall trace the nonlinear development of such a mode. It can be seen that $W-W_c$, $\partial/\partial t$ and the amplitude A will all be 'small', and that we have to find their relationship. The most interesting mode is the most unstable one, for which $\alpha=1$ and $W_c=2^{\frac{1}{2}}$, and it is this mode which will grow when the flow is just unstable, i.e. when $0 < W - 2^{\frac{1}{2}} << 1$.

For these reasons we seek to perturb the marginally stable solution

$$\underset{\sim}{\phi}_0 = A(t) \begin{pmatrix} W_c \cos\alpha x\ e^{\alpha z} \\ W_c \cos\alpha x\ e^{-\alpha z} \\ \sin\alpha x \end{pmatrix}, \tag{11}$$

which we know from the above is a solution with A=constant when $U_1=W_c$. It satisfies (1.4), (1.5) and the equation

$$D\underset{\sim}{\phi}_0 = 0 \qquad \text{at} \qquad z=0, \tag{12}$$

where

$$D = \begin{pmatrix} -\partial/\partial z & 0 & W_c\partial/\partial x \\ 0 & \partial/\partial z & W_c\partial/\partial x \\ W_c\partial/\partial x & W_c\partial/\partial x & 2(1-\partial^2/\partial x^2) \end{pmatrix}. \tag{13}$$

Putting all the small terms on the right-hand side, we rewrite (9) without further approximation as

$$D\underset{\sim}{\phi} = (N+D-L)\underset{\sim}{\phi} \quad , \quad \text{at} \quad z=0 \tag{14}$$

and seek to expand

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}_o + \underset{\sim}{\phi}_1 + \underset{\sim}{\phi}_2 + \cdots \quad , \tag{15}$$

where $\underset{\sim}{\phi}_n = O(A^n)$ as $A \to 0$, $W \downarrow W_c$ and $\partial/\partial t \to 0$.

Retaining terms of order A, we find

$$D\,\underset{\sim}{\phi}_1 = \{(N+D-L)\underset{\sim}{\phi}_o\}, \quad \text{at} \quad z=0 \,. \tag{16}$$

The solvability condition for (1.4), (1.5), and (16) can be found by use of the adjoint operator of D, and it gives $W-W_c = o(A^2)$ as $A \to 0$. One can then easily solve the equations to find $\underset{\sim}{\phi}_1$. The next approximation to (14) gives

$$D\underset{\sim}{\phi}_2 = \{(N+D-L)(\underset{\sim}{\phi}_o+\underset{\sim}{\phi}_1)\}_2 \quad \text{at} \quad z=0 \,. \tag{17}$$

The solvability condition for (17) can be shown to give the nonlinear ordinary differential equation

$$d^2A/dt^2 = 2\alpha^2(W-W_c)W_cA - \tfrac{1}{8}\alpha^3(4+\alpha^2)A^3 + O(A^3). \tag{18}$$

This equation is not the Landau equation, but it can be solved in explicit terms of elliptic functions. It is more informative, however, to plot its trajectories in the phase plane for $W > W_c$ (see figure 12). The basic flow, represented by the point $A=0$ and $dA/dt = 0$, is unstable. Small perturbations grow according to the linear approximation to (18) which is consistent with $s^2 = \alpha^2(W^2-W_c^2)$ when $W \downarrow W_c$. There are also steady nonlinear waves $A = \pm A_e$, where

$$A_e^2 = 16(W-W_c)W_c/\alpha(4+\alpha^2).$$

These finite amplitude waves are stable. The phase plane also illustrates the other solutions in which there are closed trajectories and therefore periodic solutions $A(t)$.

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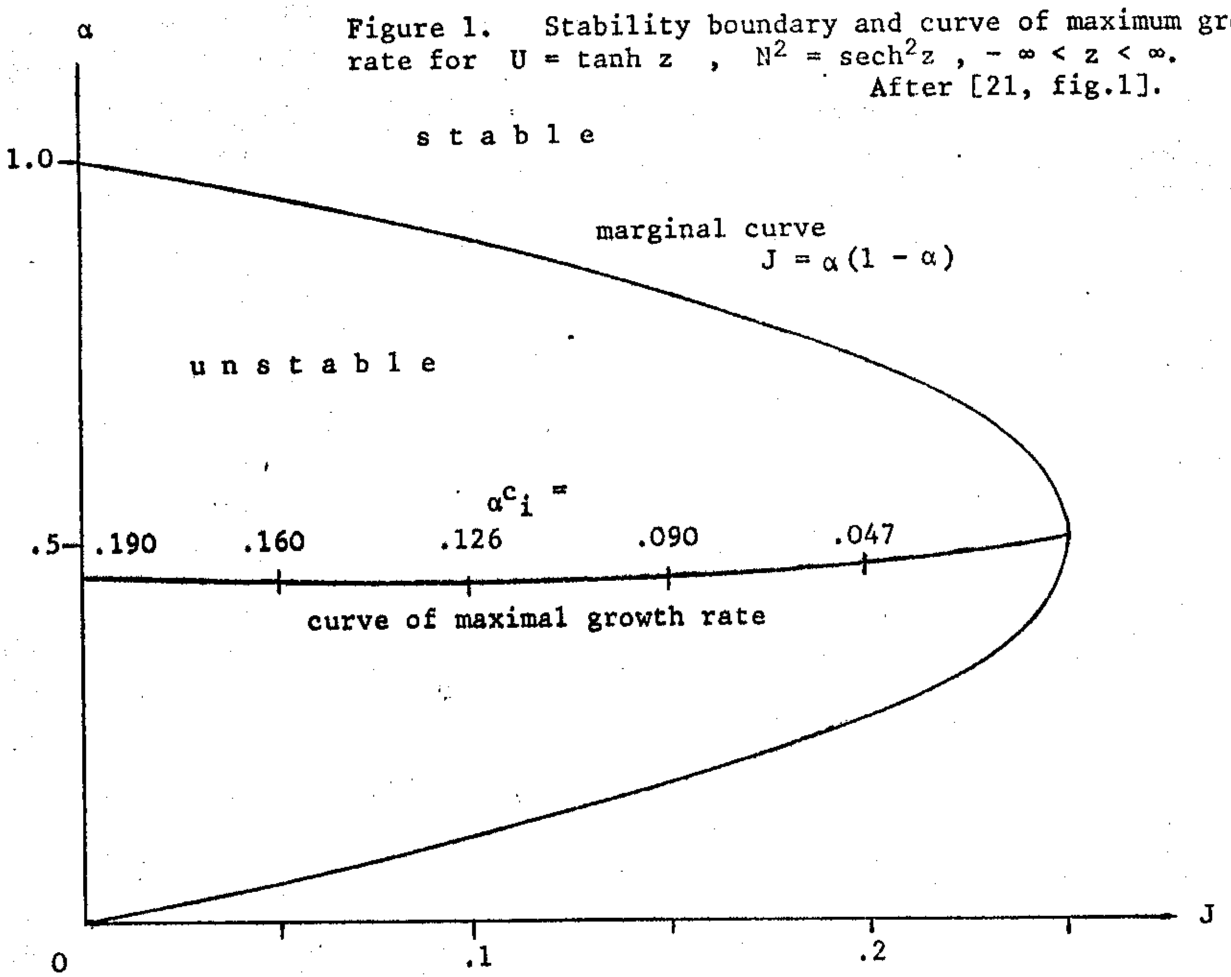
Figure 1. Stability boundary and curve of maximum growth rate for U = tanh z , $N^2 = \text{sech}^2 z$, $-\infty < z < \infty$. After [21, fig.1].



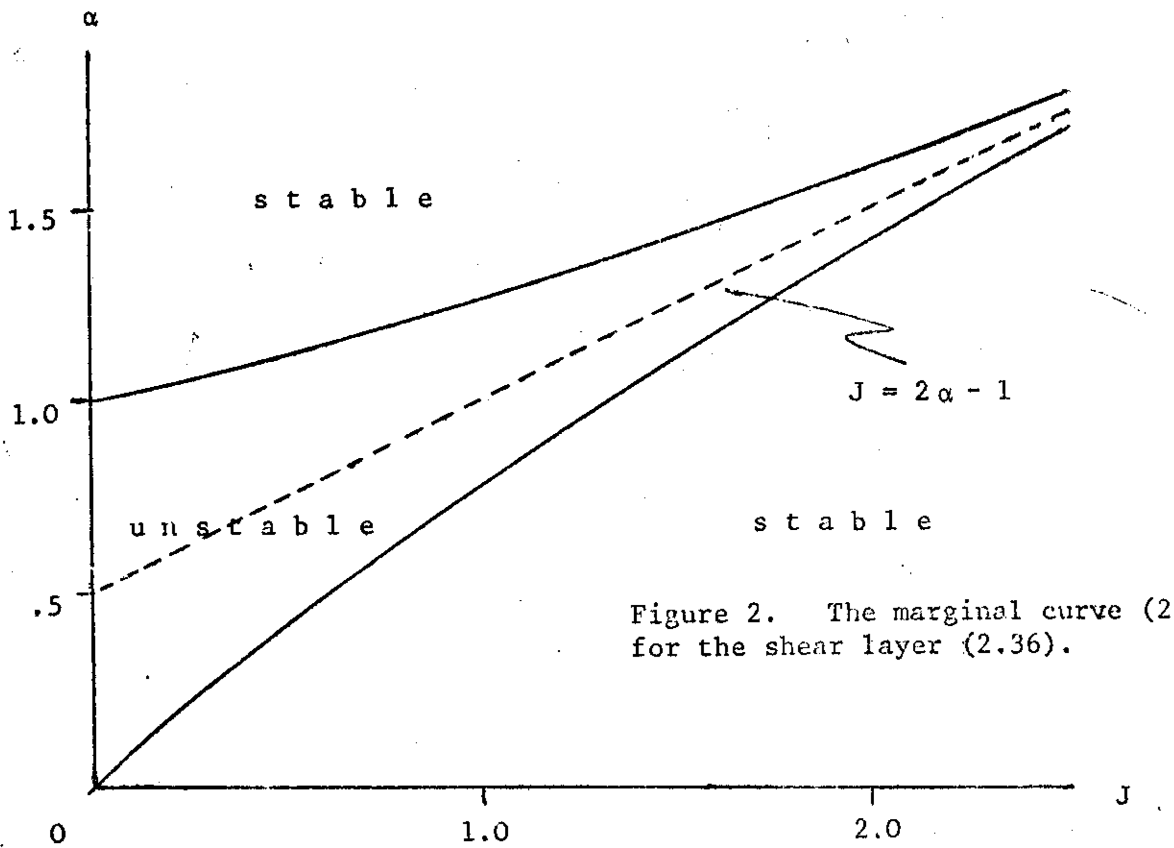


Figure 2. The marginal curve (2.37) for the shear layer (2.36).

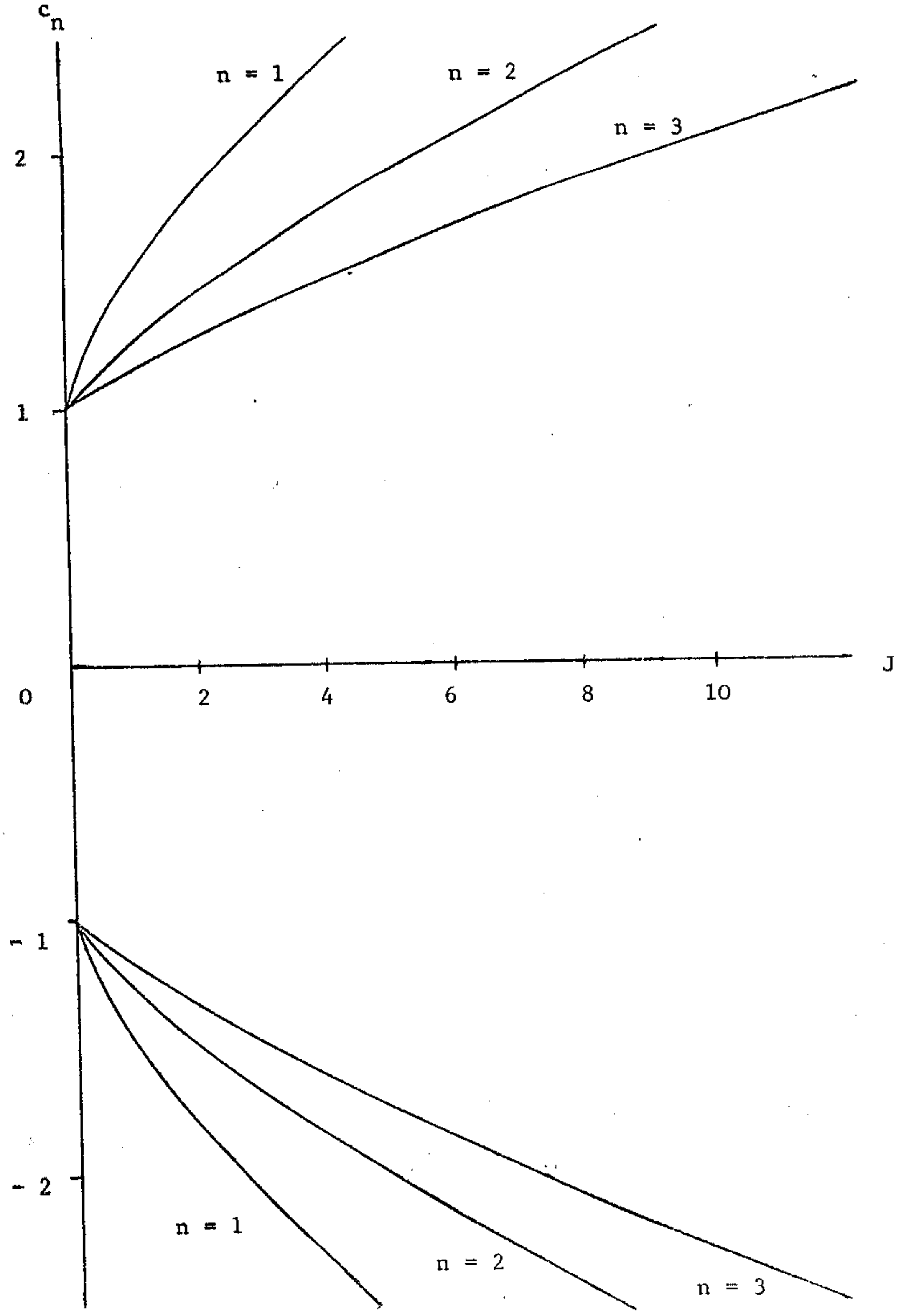


Figure 3. Sinusoidal flow $U = \sin z$, $N^2 = 1$ for $-\pi < z < 1$: c vs J for $n = 1, 2$, and 3, and $\alpha^2 = \frac{3}{4}$. Note that $U_{max} = 1$ and $U_{min} = -1$. After [25, fig. 2].

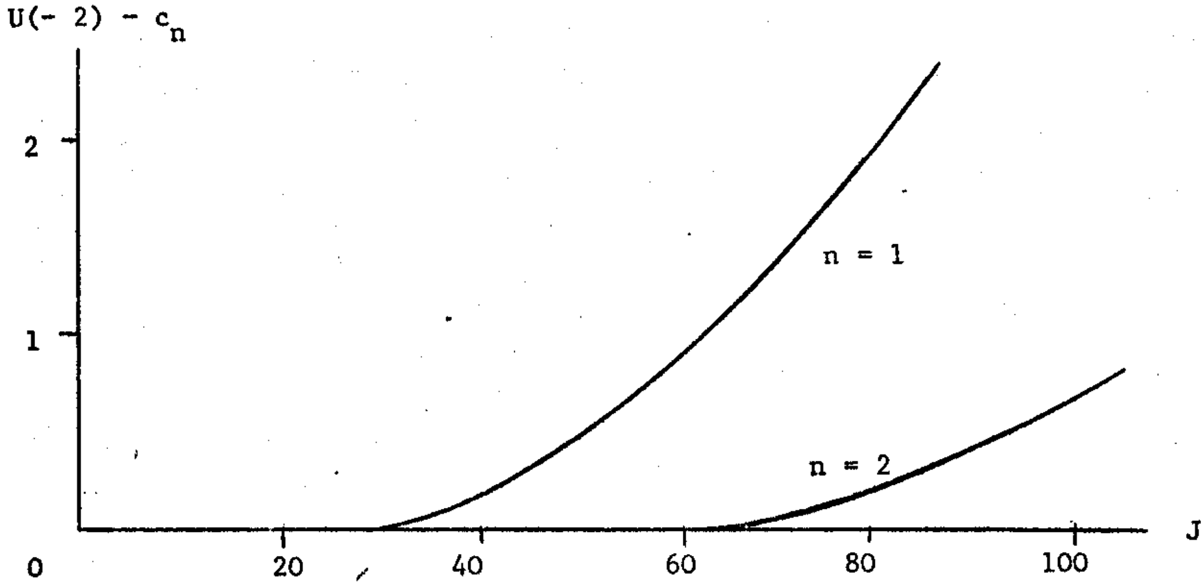
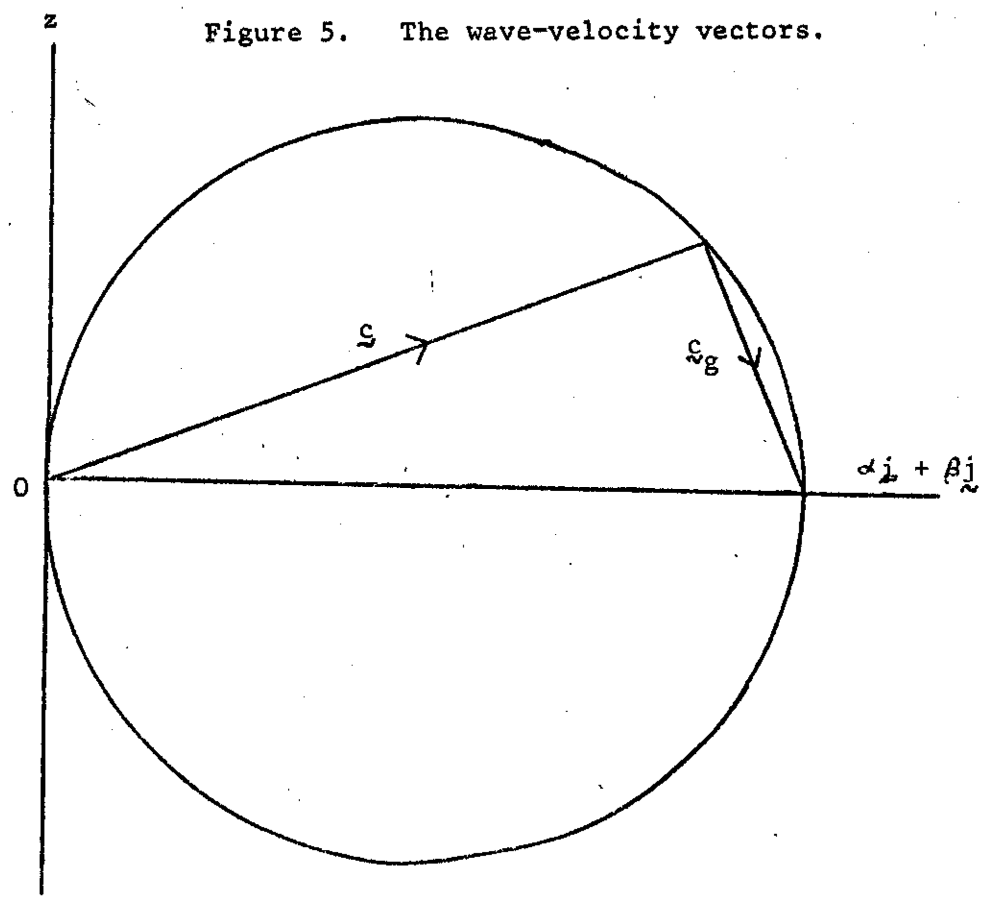


Figure 4. $U = z^3 - z$, $N^2 = 1$ for $-2 < z < 2$: $U(-2) - c_n$ versus J for n = 1, 2 and $\alpha = 1$. After [25, fig. 5]. Note that J = 25.6 , p = 1, and $\mu = 0$ at $J = 30\frac{1}{4}$.

Figure 5. The wave-velocity vectors.



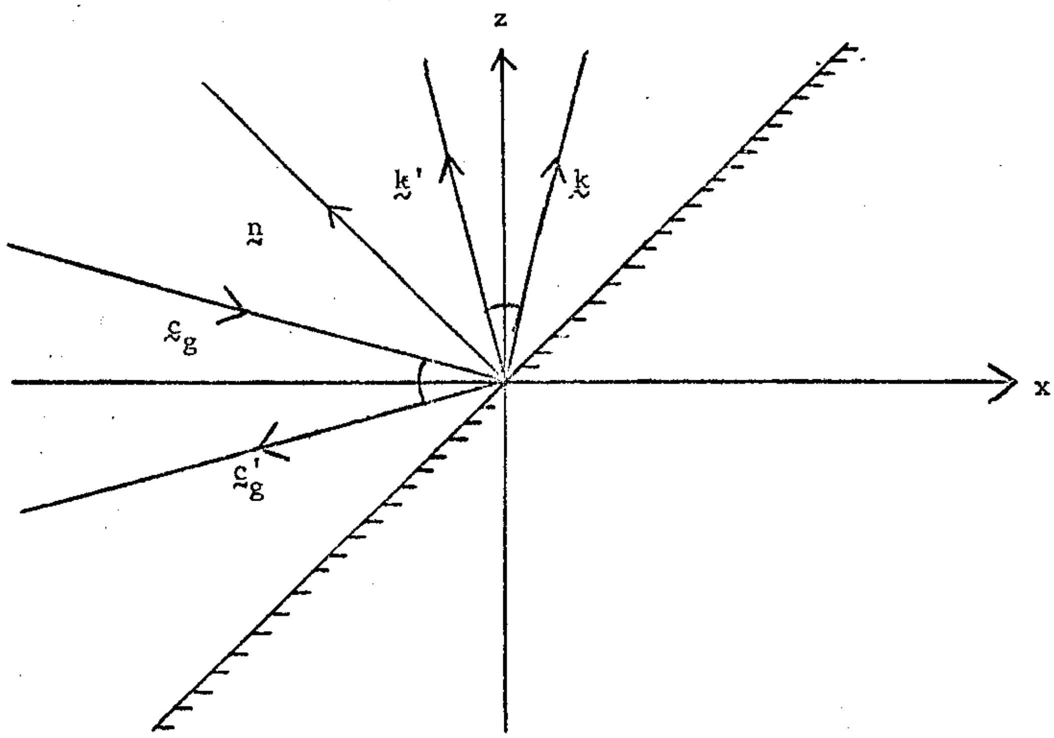
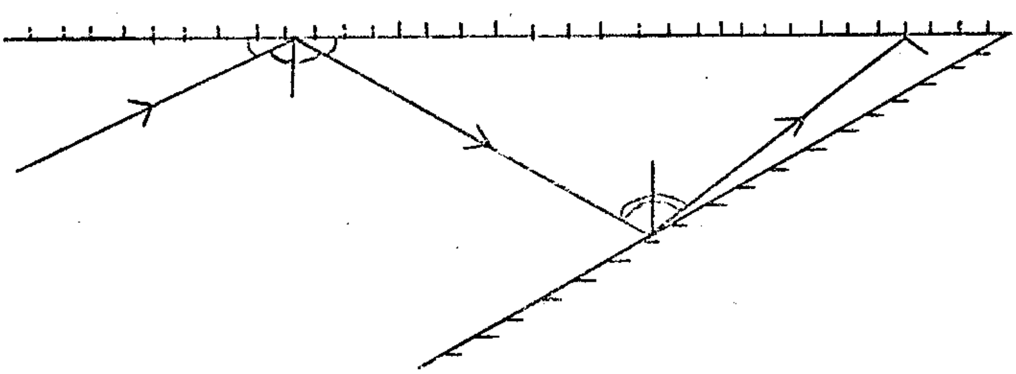


Figure 6. Reflexion of a two-dimensional internal gravity wave by a rigid plane.

Figure 7. Reflexion of a two-dimensional internal gravity wave in a wedge.



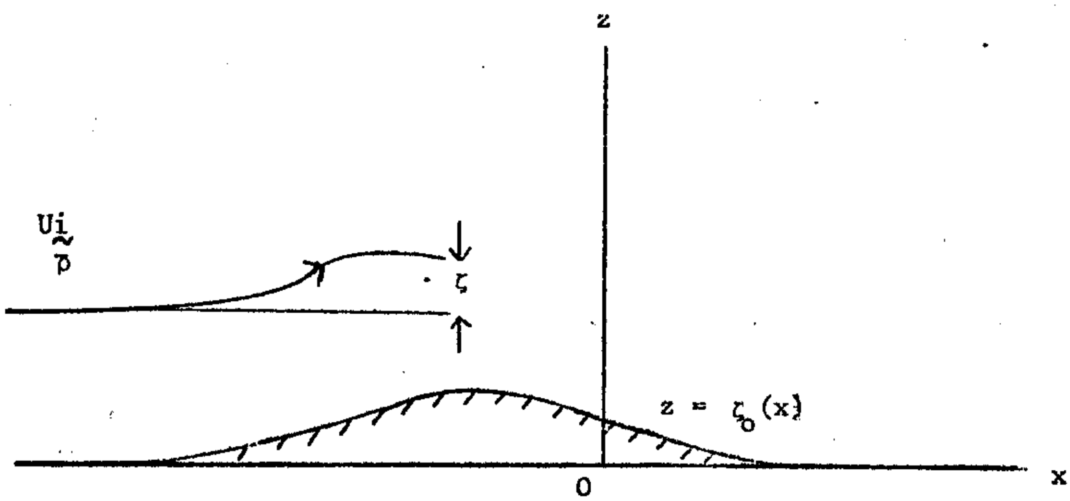
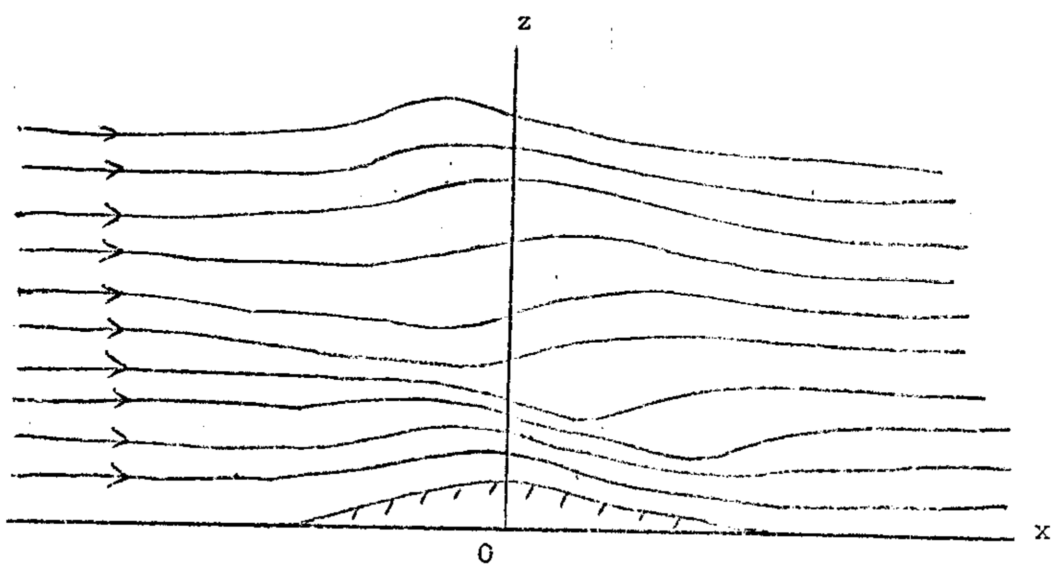
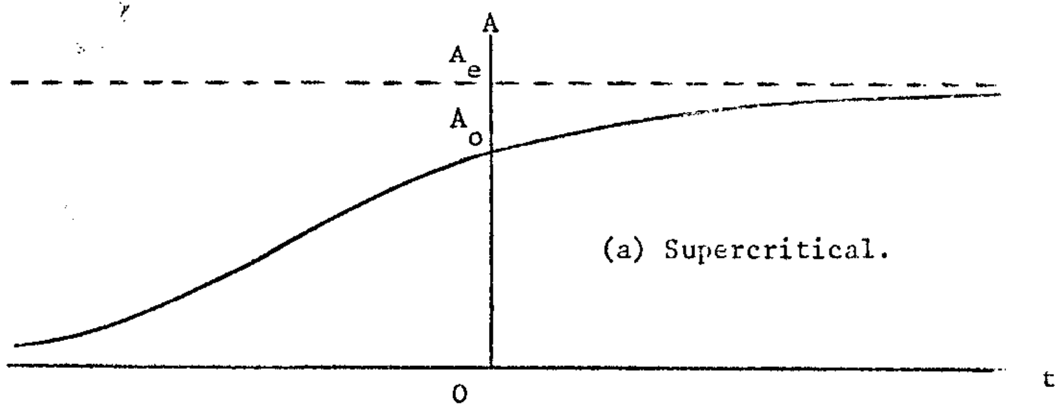


Figure 8. The configuration of the model of airlow over a mountain.

Figure 9. Sketch of typical streamlines for airflow with $U = U_{\infty}$, $\bar{\rho} = \rho_o e^{-\beta z}$, $\zeta_o = b^2 h/(b^2 + x^2)$.





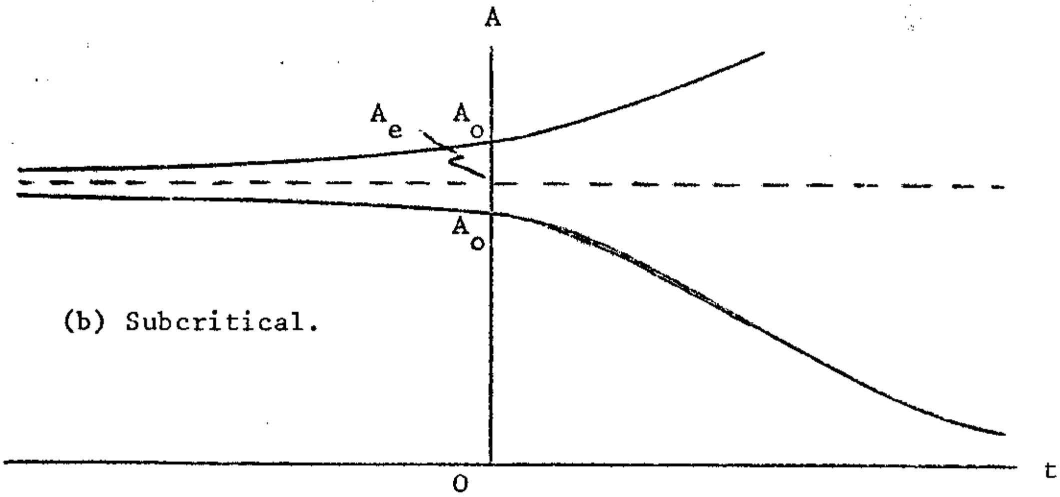
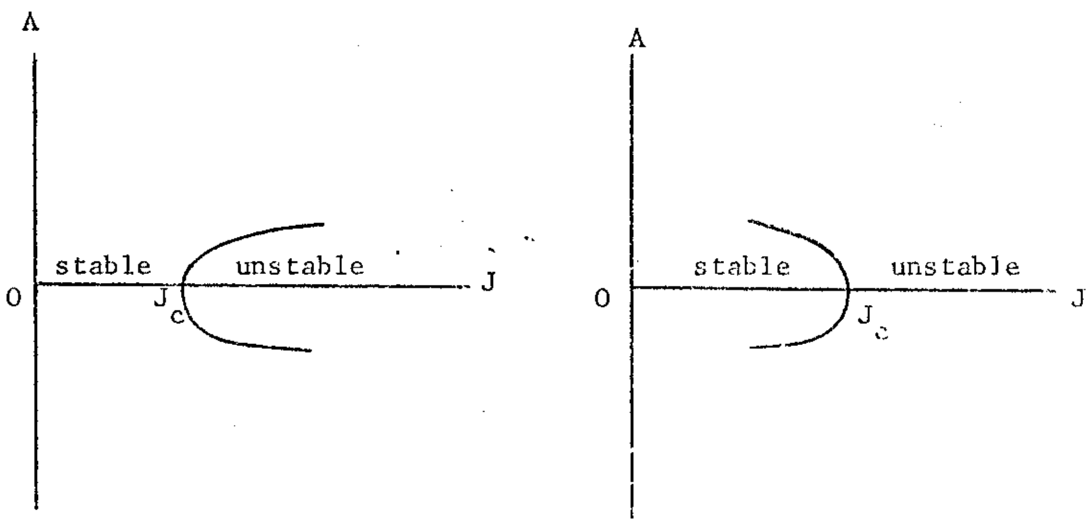


Figure 10. (a) Supercritical instability. (b) Subcritical instability.

Figure 11. (a) Supercritical branching. (b) Subcritical branching.



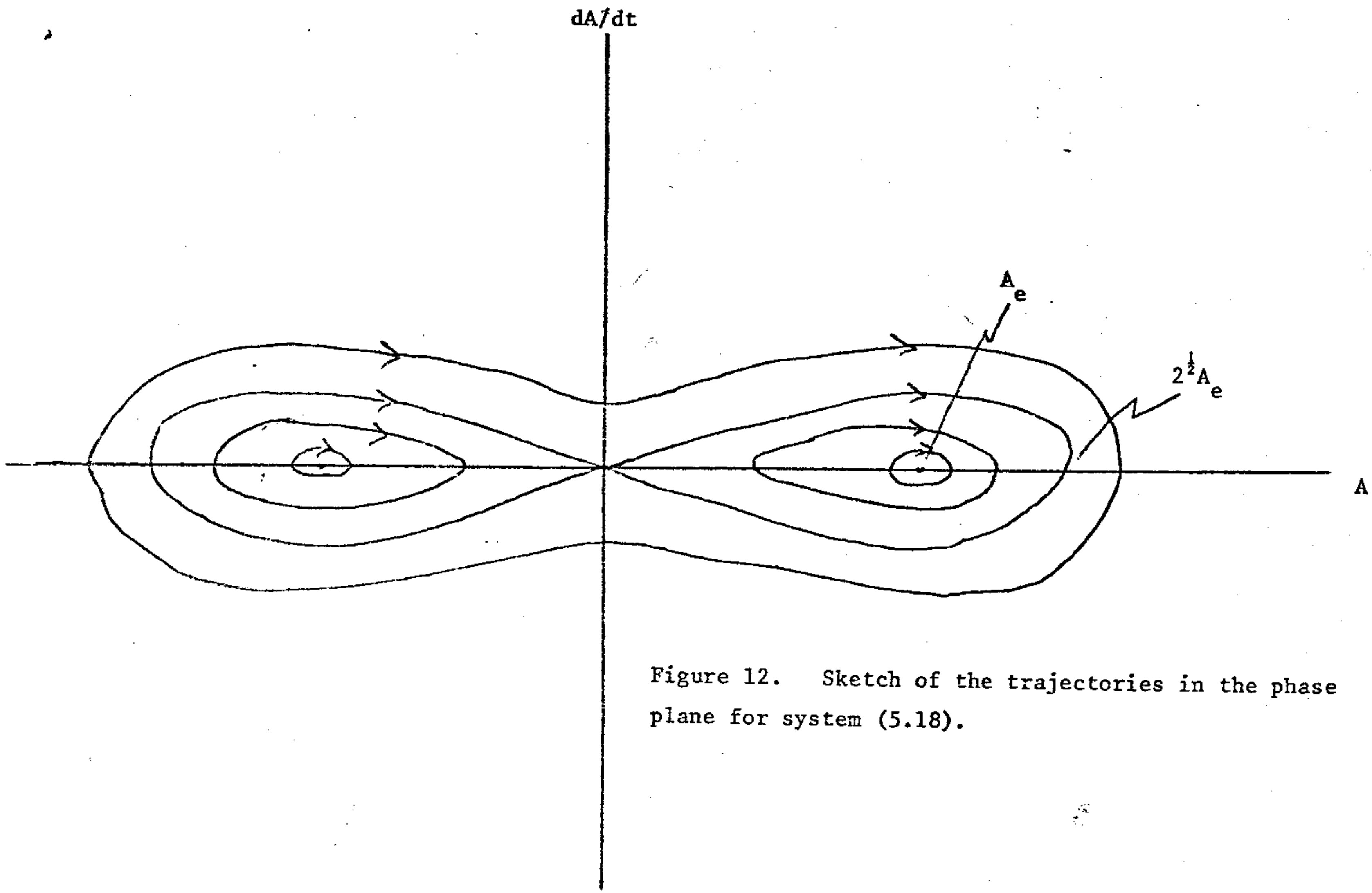


Figure 12. Sketch of the trajectories in the phase plane for system (5.18).