INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
MIRAMARE - P.O.B. 586 - 34100 TRIESTE (ITALY) - TELEPHONES: 224281/2/3/4/5/6 - CABLE: CENTRATOM

SMR/23 - 6

AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

22 September - 26 November 1976

MATHEMATICAL ASPECTS OF FLUID FLOW

T.B. Benjamin
University of Essex
Colchester, U.K.

These are preliminary lecture notes intended for participants only.
Copies are available outside the Publications Office (T-floor) or from room 112.

MATHEMATICAL ASPECTS OF FLUID FLOW, T. B. BENJAMIN

The object of these lectures is to point out a few of the mathematical issues presented by the theory of fluid motion. There is sometimes a need for great care in dealing with such issues, because misconceptions can easily arise and incorrect conclusions can be drawn. Also, certain central questions of the subject still remain unanswered: for example, whether the general initial-value problem for the motion of a viscous incompressible fluid has a smooth (i.e. physically acceptable) solution for arbitrarily large times. On the whole, the science of fluid mechanics can be developed very far on an intuitive, non-rigorous basis, and mathematical precision is a luxury that cannot be afforded in the treatment of many practically important problems (e.g. problems of turbulence). Nevertheless, when rigour can be coupled with physical understanding of complex problems, the value of the research is enormously enhanced. In these lectures there will be time only for a brief account of a few well-established results, and the discussion will have to be somewhat superficial. But anybody who notices logical deficiences will, by this fact, have received the intended message. Above all, the intention is to emphasize the need to distinguish carefully between <u>definitions</u>, <u>hypotheses</u> (i.e. arbitrary but scientifically directed assumptions), <u>logical deductions</u> and <u>plausible guesswork</u>.

Part 1. Basic Equations

(Here I closely follow parts of the book by R. E. Meyer, "Introduction to Mathematical Fluid Dynamics".)

Notation: Always D denotes a bounded open set in $\mathbb{R}^3$, with boundary ∂D and closure $\overline{D} = D \cup \partial D$.

What is a fluid motion? Suppose that D_0 is such a domain occupied by a fluid at a certain instant of time, say $t = 0$. We understand a fluid motion to be a transformation $T_t : \overline{D}_0 \to \mathbb{R}^3$ such that $D_t = T_t D_0$ is the domain occupied by the fluid at time $t > 0$. (This statement is, of course, a definition.)

An essential hypothesis of fluid mechanics is that the transformation is defined on $\overline{D}_0$ over some finite time interval. A further essential hypothesis is that T_t has an inverse on $D_t = T_t D_0$, so that if the original transformation is represented by $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{a}, t)$ for $\underset{\sim}{a} \in D_0$ [i.e. $\underset{\sim}{a}$ is the position vector of a 'fluid particle' in D_0 and $\underset{\sim}{x}$ its position vector in D_t], then $\underset{\sim}{a} = \underset{\sim}{a}(\underset{\sim}{x}, t)$ represents an inverse transformation $D_t \to D_0$. The physical idea expressed here is that two different pieces of fluid cannot simultaneously occupy the same part of space. A third hypothesis, required for consistency with the observed behaviour of fluids, is that the function $\underset{\sim}{x}(\underset{\sim}{a}, t)$ is continuous in $\underset{\sim}{a}$ and t. Thus, unlike a solid, a fluid cannot be cut, or broken apart, so that two arbitrarily close points in the fluid domain would be separated.

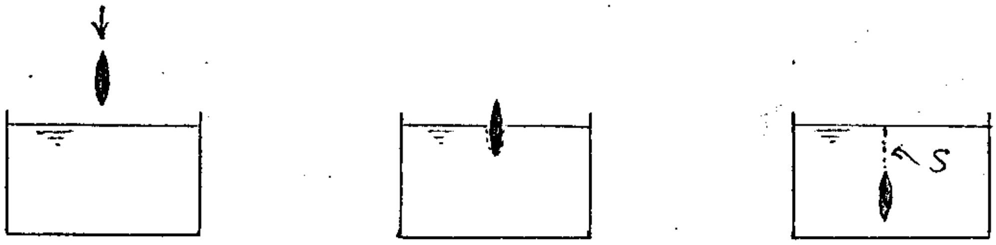
(Here we ignore certain special physical processes, such as evaporation, cavitation and drop formation, to describe which ad hoc assumptions are necessary.)

It follows from the last hypothesis that D_t is also bounded, nd from this and the second hypothesis that D_t is also open. EXERCISE: Verify this statement.) Moreover,

$$\partial D_t = T_t \partial D_o ,$$

.e. the boundary of a fluid domain is always formed by the same et of fluid particles.

NOTE 1. At first sight one might suppose that the transformation T_t is necessarily bijective, because it is surjective by definition and it would appear injective in view of the condition $J = \det[\partial x_i / \partial a_j] \neq 0$ consistent with mass conservation (see below). But this supposition is not correct in general, because T_t must be defined on the closure $\bar{D}_o$ and a single-valued inverse does not always exist on the closure of $T_t D_o$. The following example is instructive, being considered in Meyer's book.



A solid body is moved downwards into a liquid contained in a tank. Because T_t is continuous on $\bar{D}_o$, not only on D_o, the boundary ∂D_o cannot be cut by the body. A subset of ∂D_o is stretched around the body; and, when the body is fully submerged, a subset S indicated by the dotted line in the third diagram is left behind in the fluid. Evidently, in this case, the mapping $T_t : \bar{D}_o \to \bar{D}_t$ is not injective, therefore not bijective, and the inverse $T_t^{-1} : \bar{D}_t \to \bar{D}_o$ is not single-valued.

This idea has an important practical bearing on the generation of aerodynamic lift (see p. 6 and pp. 75, 76 of Meyer's book).

Note 2. Definitions and hypotheses have been referred only to bounded fluid domains, and this restriction is entirely reasonable from a practical point of view. But, in much work on fluid mechanics, theoretical models are used for which the fluid domain is infinite. This is often a convenient idealization; however, it poses various conceptual and analytical difficulties. (For example, try to work out the total momentum in the case of a sphere that moves with constant velocity in an infinite incompressible fluid, generating an irrotational motion of the fluid such that $|\underset{\sim}{u}| \to 0$ as $|\underset{\sim}{x}| \to \infty$. The problem remains indeterminate unless additional, extremely delicate hypotheses are introduced.)

In addition to the hypotheses so far made about bounded fluid motions, another is needed in order to make any progress: namely, the fluid velocity

$$\underset{\sim}{u} \equiv \frac{\partial \underset{\sim}{x}(\underset{\sim}{a},t)}{\partial t}$$

is defined on $\bar{D}_0$. In most theoretical work on fluid mechanics it is conventional to <u>assume</u> that $\underset{\sim}{u}$ and other fluid properties such as density and the stress tensor are C^∞ with respect to t and to $\underset{\sim}{x}$ on $\bar{D}_t$. Exceptional phenomena, such as vortex sheets in inviscid fluids and shock waves in inviscid incompressible fluids, are treated <u>ad hoc</u>.

We call

$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{a},t) \quad \text{a LAGRANGIAN description}$$

and

$$\underset{\sim}{u} = \underset{\sim}{u}(\underset{\sim}{x},t) \quad \text{a EULERIAN description}$$

of fluid motions. By the fundamental theorem in the theory of ordinary differential equations, the assumption that

$\underset{\sim}{u} \in C^\infty(\bar{D}_t \to \mathbb{R}^3)$ for all $t \in \mathbb{R}_+$ ensures that the differential equation

$$\frac{\partial \underset{\sim}{x}}{\partial t} = \underset{\sim}{u}(\underset{\sim}{x}, t)$$

with initial condition $\underset{\sim}{x}(\underset{\sim}{a}, 0) = \underset{\sim}{a}$ has a unique solution $\underset{\sim}{x}(\underset{\sim}{a}, t)$ for each $\underset{\sim}{a} \in \bar{D}_0$, and this solution is C^∞ with respect to $\underset{\sim}{a}$ and t.

<u>A very simple example.</u> Consider the two-dimensional motion whose Eulerian description is

$$u_1 = -\kappa(t)x_1, \qquad u_2 = \kappa(t)x_2,$$

where $\kappa(t)$ is an arbitrary piecewise continuous function. The solution is

$$x_1 = a_1 \exp\left(-\int_0^t \kappa(\tau)\,d\tau\right), \qquad x_2 = a_2 \exp\left(\int_0^t \kappa(\tau)\,d\tau\right).$$

Thus $x_1 x_2 = a_1 a_2$, which means that all the particle paths are rectangular hyperbolae.

Now, let f denote any scalar variable (e.g. a component of a vector) with Eulerian representation $f = f(\underset{\sim}{x}, t)$. Its Lagrangian representation is therefore

$$f(\underset{\sim}{x}(\underset{\sim}{a}, t), t) = g(\underset{\sim}{a}, t), \text{ say.}$$

The <u>material</u> or <u>convective derivative</u> of f (or 'derivative following the motion') is defined to be

$$\frac{Df}{Dt} = \frac{\partial g(\underset{\sim}{a}, t)}{\partial t}$$

$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial t}$$ (summation convention applies: sum over $j = 1, 2, 3$)

$$= \frac{\partial f}{\ldots} \quad \ldots \quad \partial f \quad (\ldots$$

In particular, <u>acceleration</u> $\frac{D\underset{\sim}{u}}{Dt} = \frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u}$,

since each component Du_i/Dt $(i = 1,2,3)$ is given by the preceding formula.

<u>Convection Theorem</u>. Let D_t be a fluid domain and $f(\underset{\sim}{x},t) \in C^1(D_t \times \mathbb{R} \to \mathbb{R})$. Then

$$\frac{D}{Dt}\int_{D_t} f \, dx = \int_{D_t}\left(\frac{Df}{Dt} + f\,\nabla\cdot\underset{\sim}{u}\right) dx.$$

Note $\nabla\cdot\underset{\sim}{u} = \partial u_i/\partial x_i$ (summation convention).)

<u>Outline of proof</u>. By assumption, $D_t = T_t D_0$ and so

$$\int_{D_t} f\,dx = \int_{D_0} g(\underset{\sim}{a},t)\, J\, da,$$

where the Jacobian $J = J(\underset{\sim}{a},t) \equiv \partial(x_1,x_2,x_3)/\partial(a_1,a_2,a_3)$ $\equiv \det(\partial x_i/\partial a_j)$. Hence

$$\frac{D}{Dt}\int_{D_t} f\,dx = \frac{\partial}{\partial t}\int_{D_0} g\,J\,da$$

$$= \int_{D_0}\left\{\frac{\partial g}{\partial t} J + g\frac{\partial J}{\partial t}\right\} dx$$ (commutation of $\partial/\partial t$ & $\int_{D_0}$ can be justified).

Since $\partial J/\partial t = J\,\nabla\cdot\underset{\sim}{u}$ (see below), the equality stated by the theorem follows immediately.

Note that, for any D_t including all sub-domains of a particular fluid domain, evidently

$$d(\text{volume})/dt = \int_{\partial D_t}\underset{\sim}{u}\cdot\underset{\sim}{n}\,dx = \int_{D_t}\nabla\cdot\underset{\sim}{u}\,dx = \int_{D_0}(\nabla\cdot\underset{\sim}{u})J\,da$$

and " $= \frac{D}{Dt}\int dx = \frac{\partial}{\partial t}\int J\,da = \int \frac{\partial J}{\partial t}\,da.$

Since D_0 is arbitrary, it follows that $\partial J/\partial t = J\,\nabla\cdot\underset{\sim}{u}$.
(EXERCISE. Confirm this result directly by differentiation.)

COROLLARY TO CONVECTION THEOREM. Let D_1 be a *fixed* domain such that $D_1 = D_t$ at $t = t_1$. Then, at $t = t_1$,

$$\frac{D}{Dt}\int_{D_t} f\,dx = \frac{\partial}{\partial t}\int_{D_1} f\,dx + \int_{\partial D_1} f\,\underset{\sim}{u}\cdot\underset{\sim}{n}\,dx.$$

(EXERCISE. Prove this corollary.) With reference to the theorem, note that, by definition,

$$Df/Dt + f\,\nabla\cdot\underset{\sim}{u} = \partial f/\partial t + \nabla\cdot(f\underset{\sim}{u}).$$

MASS CONSERVATION. Intuitively, this is an easy concept. It may be expressed in mathematical terms as follows. The fundamental hypothesis is that a function $\rho(\underset{\sim}{x},t)$ — the density of the fluid — can be defined on D_t such that

$$\int_{D_t} \rho\,dx = m(D_0) > 0$$

is independent of t; and a similar positive invariant is associated with every sub-domain of D_t. Thus ρ is implied to be a positive function (at least 'almost everywhere' on D_t).

It follows by the convection theorem applied to arbitrary sub-domains that, if ρ and $\underset{\sim}{u}$ are C^1 functions, then

$$\frac{D\rho}{Dt} + \rho\,\nabla\cdot\underset{\sim}{u} = \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underset{\sim}{u}) = 0$$

everywhere in D_t. Also, if f is another C^1 scalar function then $$\frac{D}{Dt}\int_{D_t} \rho f \, dx = \int_{D_t} \rho \frac{Df}{Dt} \, dx .$$

For many applications, an <u>incompressible</u> fluid is defined by the property that $\rho =$ const. throughout the fluid domain. This idealization is an excellent approximation for a liquid such as water at constant temperature, and for air at constant temperature if the local Mach number $M = |\underset{\sim}{u}|/c$, where c is the velocity of sound $[\doteqdot (dp/d\rho)^{\frac{1}{2}}_{\text{ADIA.}} = (\gamma p/\rho)^{\frac{1}{2}}]$, is very small everywhere in D_t. However, some important applications (e.g. to 'internal waves') are concerned with fluids that are incompressible but have densities varying with position in D_0: thus, ρ is a function of $\underset{\sim}{a}$ alone. In either case it follows from the last equations on p. 7 that, for an incompressible fluid,

$$\Delta \cdot \underset{\sim}{u} = 0 .$$

Note that if $\rho(\underset{\sim}{a})$ has isolated discontinuities (i.e. the fluid is discretely 'stratified' or heterogeneous), $\underset{\sim}{u}$ may have discontinuities without invalidating the last equation. (<u>EXERCISE</u>. Examine this proposition.)

<u>MOMENTUM PRINCIPLE</u> (momentum hypothesis; Newton's Second Law). On the closure $\overline{D}_t$ of a fluid domain, and similarly on any sub-domain, a vector field $\underset{\sim}{f}$ and a tensor field with components p_{ij} can be defined such that

$$\frac{D}{Dt}\int \rho u_i \, dx = \int \rho f \, dx + \int p_{ij} n_j \, dx ,$$

where n_j $(j = 1,2,3)$ are the components of the outward unit normal to ∂D_t.

Note: (i) $\rho \underset{\sim}{f}$ is the body force per unit volume acting on the fluid (due to gravity, magnetic or electric fields, etc.).

(ii) p_{ij} on ∂D_t is the stress exerted by the fluid on the boundary of the domain.

(iii) This principle underlies the whole of continuum dynamics, not only fluid dynamics.

If the components of $\underset{\sim}{u}$ and p_{ij} are C' functions, the momentum principle leads to a system of p.d.e.'s. For we then have

$$\int_{\partial D_t} p_{ij} n_j \, dx = \int_{D_t} \frac{\partial p_{ij}}{\partial x_j} \, dx$$

by the divergence theorem for C' functions. Since $\rho > 0$ we hence obtain, using the result noted at the top of p. 8,

$$\int_{D_t} \rho \left\{ \frac{Du_i}{Dt} - f_i - \frac{1}{\rho} \frac{\partial p_{ij}}{\partial x_j} \right\} dx = 0.$$

And since D_t is arbitrary, it follows that

$$\frac{Du_i}{Dt} = f_i + \frac{1}{\rho} \frac{\partial p_{ij}}{\partial x_j}.$$

This is the familiar equation of motion for a continuum. But note that it is less fundamental than the momentum principle from which it has been derived. If we cannot assume the terms of the equation to be continuous functions of $\underset{\sim}{x}$ and t (and there is seldom strict *a priori* justification for this assumption), then the equation may be significant only in some 'weak' sense — e.g. the terms may be meaningful only as distributions, not as ordinary functions.

An idealized case having great theoretical importance will next be reviewed very briefly. An <u>inviscid fluid</u> may be defined to be one in which, whatever the velocity field,

$$p_{ij} = -p(\underset{\sim}{x},t)\,\delta_{ij} \qquad \forall \quad \underset{\sim}{x} \in \bar{D}_t ,$$

where p is a scalar function (the pressure). (Note $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$.) Hence

$$\frac{\partial p_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_j} .$$

Let us further assume that the external-force field $\underset{\sim}{f}$ is irrotational, so that $\underset{\sim}{f} = -\nabla\chi$ (as is very often the case in applications, e.g. when $\underset{\sim}{f}$ is due to gravity). Then the equation of motion becomes

$$\frac{D\underset{\sim}{u}}{Dt} = -\frac{1}{\rho}\nabla p - \nabla\chi$$

<u>OR</u>

$$\frac{\partial \underset{\sim}{u}}{\partial t} - \underset{\sim}{u} \times (\nabla \times \underset{\sim}{u}) = -\frac{1}{\rho}\nabla p - \nabla(\tfrac{1}{2}q^2 + \chi)$$

in view of the vector identity

$$(\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u} = \nabla(\tfrac{1}{2}q^2) - \underset{\sim}{u} \times (\nabla \times \underset{\sim}{u})$$

with $q^2 = |\underset{\sim}{u}|^2 = u_i^2$. Taking the scalar product of $\underset{\sim}{u}$ and the terms of the equation of motion in this last form, we obtain

$$\tfrac{1}{2}\,\partial(q^2)/\partial t + \underset{\sim}{u}\cdot\nabla(\tfrac{1}{2}q^2 + \chi) + \frac{1}{\rho}\underset{\sim}{u}\cdot\nabla p = 0,$$

a generalized form of 'Bernoulli's equation'. In the case of a steady (time-independent) flow, this reduces to

$$\underset{\sim}{u} \cdot \nabla\left(\tfrac{1}{2}q^2 + \chi\right) + \tfrac{1}{\rho}\underset{\sim}{u}\cdot\nabla p = 0,$$

and, if ρ = const.,

$$\underset{\sim}{u} \cdot \nabla\left(\tfrac{1}{2}q^2 + \chi + p/\rho\right) = 0,$$

which means that the quantity

$$\tfrac{1}{2}q^2 + \chi + p/\rho,$$

often called 'total head', is invariant along streamlines in the steady flow.

If $\nabla \times \underset{\sim}{u} = 0$ on D_t (i.e. the flow is irrotational) so that $\underset{\sim}{u} = \nabla\phi$, then we obtain, in the case ρ = const.,

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}q^2 + \chi + \frac{p}{\rho} = \mathcal{H}(t),$$

a form of Bernoulli's equation useful in, for example, water-wave theory.

Taking the curl of the equation of motion for an inviscid fluid gives, in the case ρ = const.,

$$\partial\underset{\sim}{\omega}/\partial t = \nabla \times (\underset{\sim}{u} \times \underset{\sim}{\omega}), \quad \text{where } \underset{\sim}{\omega} = \nabla \times \underset{\sim}{u} \text{ (VORTICITY)}.$$

It follows that if γ_0 is any closed circuit in D_0 and $\gamma_t = T_t\gamma_0$, then

$$\oint_{\gamma_t} \underset{\sim}{u}\cdot d\underset{\sim}{x} \qquad \text{(the 'circulation' around } \gamma_t\text{)}$$

is independent of t. This is 'Kelvin's circulation theorem'. (EXCERCISE. Prove the theorem.) If the fluid is incompressible but $\rho(\underset{\sim}{a})$ is not a constant, the evolution equation for $\underset{\sim}{\omega}$ is more complicated and the circulation theorem does not apply. Consequently, novel and interesting effects appear in the study of inhomogeneous (stratified) inviscid fluids.

<u>Newtonian Viscous Fluids.</u> This theoretical model provides an excellent approximation to the observed behaviour of many fluids (e.g. oil water, air at moderate temperatures and pressures, treacle). The constitutive relation between p_{ij} and the velocity field is assumed to be homogeneous and isotropic (i.e. independent of orientation in D_t and of $\underset{\sim}{a} \in D_0$), and consequently the Eulerian description is particularly convenient. It is found experimentally that in a <u>uniform flow</u> (i.e. where $\partial u_i/\partial x_j \equiv 0$) $p_{ij} = -p(\underset{\sim}{x},t)\,\delta_{ij}$, just as in the idealized inviscid fluid. We accordingly write

$$p_{ij} = -p(\underset{\sim}{x},t)\,\delta_{ij} + \tau_{ij},$$

where τ_{ij}, called the 'deviatoric stress', is supposed to depend on the velocity-derivative tensor $\partial u_i/\partial x_j$. There are various plausible arguments indicating that τ_{ij} should depend only on the symmetric part

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = e_{ji}.$$

of this tensor; but it is simplest merely to consider this as an empirical fact and incorporate the property into the theory as a hypothesis — like the essential hypothesis of linear dependence on e_{ij}.

The following theorem in elementary tensor theory helps to fix ideas:

<u>Theorem.</u> Let p_{ij} be symmetric, linear in $\partial u_i/\partial x_j$, and independent of $\underset{\sim}{u}$, $\partial \underset{\sim}{u}/\partial t$, HIGHER DERIVATIVES, and the antisymmetric part of $\partial u_i/\partial x_j$. Let the relation between the tensors p_{ij} and

e_{ij} be isotropic (i.e. invariant under rotations of the coordinate system. Then

$$p_{ij} = -p\delta_{ij} + 2\mu e_{ij} + \mu'\delta_{ij}e_{kk},$$

where μ and μ' are scalar constants.

(For a proof see, for example, the book by W. Prager 'Introduction to Mechanics of Continua'.)

Henceforth in these lectures we shall consider only incompressible fluids with ρ = const., for which case the trace $e_{kk} = 0$. Also, we omit external-force fields (which, if irrotational, can be included in the scalar p). Thus we consider the Navier-Stokes equations for a Newtonian fluid in the form

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\Delta u_i,$$

where $\nu = \mu/\rho$ (the 'kinematic viscosity') and $\Delta \equiv \partial^2/\partial x_i^2$ is the Laplacian operator.

To express the equations in convenient dimensionless form, put $\underset{\sim}{x} = \underset{\sim}{x}^*/L$, $t = t^*\nu/L^2$, $u_i = u_i^*/U$, $p = p^*L/\mu U$, where the asterisks denote dimensional quantities and L, U are appropriate length and velocity scales. Then we have

$$\frac{\partial u_i}{\partial t} + R\,u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \Delta u_i,$$

in which $R = UL/\nu$ is the Reynolds number, a positive parameter.